Non-ultralocal classical r-matrix structures in field analogues of integrable systems

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Plan of the talk:

- Integrable systems and field generalizations
- Yang-Baxter equations
- Integrable tops and 1+1 Landau-Lifshitz equations from R-matrices
- Many-body systems and field generalizations
- IRF-Vertex correspondence: gauge equivalence

Integrable systems

Mechanics (ODE)	Field theory (PDE)
1-body closed Toda model: $q = q(t),\ \ddot{q} = \sinh(q)$	Sine-Gordon equation: $q = q(t,x),\ \partial_t^2 q - \partial_x^2 q = \sinh(q)$
1-body open Toda model: $q = q(t),\ \ddot{q} = -e^q$	Liouville filed theory: $q = q(t,x),\ \partial_t^2 q - \partial_x^2 q = -e^q$
Euler top in 3d: $\vec{S} = \vec{S}(t),\ \dot{\vec{S}} = \vec{S} \times J(\vec{S})$ $\vec{S}(t)$ – angular momentum vector	Landau-Lifshitz equation: $\vec{S} = \vec{S}(t,x),\ \partial_t \vec{S} = \vec{S} \times \partial_x^2 \vec{S} + \vec{S} \times J(\vec{S})$ $\vec{S}(t,x)$ – magnetization vector in magnet
Free (isotropic) top: $\vec{S} = \vec{S}(t),\ \dot{\vec{S}} = 0$	Heisenberg magnet: $\vec{S} = \vec{S}(t,x),\ \partial_t \vec{S} = \vec{S} \times \partial_x^2 \vec{S}$

Integrable systems in classical mechanics

Hamiltonian mechanics: Hamiltonian function H and Poisson brackets $\{\ ,\ \}$ provide equations of motion $\dot{f} = \{H, f\}$.

Many-body systems: N particles with positions $q_i \in \mathbb{C}$, momenta $p_i \in \mathbb{C}$ and the Hamiltonian function of the form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j}^{N} U(q_i - q_j)\,.$$

Canonical Poisson structure:

$$\{p_i, q_j\} = \delta_{ij}\,, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0\,, \quad i, j = 1...N\,.$$

We need: N independent integrals of motion $H_k(p, q)$ with involution property

$$\{H_i, H_j\} = 0\,.$$

Euler-Arnold tops

Dynamical variables:

$$S = \sum_{i,j=1}^{N} S_{ij} E_{ij} \in \mathrm{Mat}(N)\,,$$

where E_{ij} is a standard basis in $\mathrm{Mat}(N, \mathbb{C})$.
The Euler-Arnold equations:

$$\dot{S} = [S, J(S)]\,,$$

where

$$J(S) = \sum J_{ij,kl} S_{kl} E_{ij}$$

is a linear functional, and $J_{ij,kl}$ – some set of constants. From the point of view of rotation in multidimensional space J – inverse inertia tensor.

The Hamiltonian

$$H = \frac{1}{2}\,\mathrm{tr}(SJ(S))$$

The Poisson brackets are given by the linear Poisson-Lie structure on gl_N^*:

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}\,.$$

Lax equations and classical r-matrix

Let L and M are matrices (matrix-valued functions on the phase space), and equations of motion are represented in the form of Lax equation:

$$\dot{L} = [L, M]\,, \quad \dot{L} = \{H, L\} = \sum_{i,j} E_{ij}\{H, L_{ij}\}\,, \qquad L, M \in \mathrm{Mat}(N, \mathbb{C})$$

Then $H_k = \mathrm{tr}(L^k)$ are integrals of motion:

$$\dot{L^k} = [L^k, M]\,, \quad \frac{d}{dt}\,\mathrm{tr}(L^k) = \mathrm{tr}(L^k M - M L^k) = 0\,.$$

Introduce notation $\{L_1\,, L_2\} = \sum\limits_{ijkl} E_{ij} \otimes E_{kl}\,\{L_{ij}, L_{kl}\} \in \mathrm{Mat}_N^{\otimes 2}$. If there exists such $r_{12} \in \mathrm{Mat}_N^{\otimes 2}$ (classical r-matrix) that

$$\{L_1\,, L_2\} = [L_1, r_{12}] - [L_2, r_{21}]\,, \quad L_1 = L \otimes 1\,, \quad L_2 = 1 \otimes L$$

$$r_{12} = \sum_{ijkl} r_{ij,kl}\, E_{ij} \otimes E_{kl}\,, \qquad r_{21} = \sum_{ijkl} r_{ij,kl}\, E_{kl} \otimes E_{ij}$$

then the conservation laws are in involution $\{H_i, H_j\} = 0$.

The same with spectral parameter

Let $L(z)$ and $M(z)$ are matrices (matrix-valued functions on the phase space), z – auxiliary parameter and equations of motion are represented in the form

$$\dot{L}(z) = [L(z), M(z)]\,, \quad \forall z \quad \dot{L}(z) = \{H, L(z)\} = \sum_{ij} E_{ij}\{H, L_{ij}(z)\}$$

Then $H_k(z) = \mathrm{tr}(L^k(z))$ are generating functions of integrals of motion:

$$\mathrm{tr}(L^k(z)) = \sum_m (z - z_0)^m H_{k,m}\,, \qquad \frac{d}{dt} H_{k,m} = 0 \;\forall k, m$$

Introduce notation $\{L_1(z)\,, L_2(w)\} = \sum_{ijkl} E_{ij} \otimes E_{kl}\,\{L_{ij}(z), L_{kl}(w)\}$. If there exists such $r_{12} \in \mathrm{Mat}^{\otimes 2}$ (classical r-matrix) that

$$\{L_1(z)\,, L_2(w)\} = [L_1(z), r_{12}(z,w)] - [L_2(w), r_{21}(w,z)]\,,$$

where $L_1(z) = L(z) \otimes 1$ and $L_2(w) = 1 \otimes L(w)$. Then $\{H_{i,m}, H_{j,n}\} = 0$.

The Jacobi identity in $\mathrm{Mat}^{\otimes 3}$ for the Poisson brackets

$$\{\{L_1(z_1), L_2(z_2)\}\, L_3(z_3)\} + \text{cycl.} = 0$$

is fulfilled if the classical Yang-Baxter equation holds true

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\,, \quad r_{ij} = r_{ij}(z_i - z_j)\,, \quad r_{12}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes E_{kl} \otimes 1_N\,,$$

$$r_{23}(z) = \sum_{ijkl} r_{ij,kl}(z)\, 1_N \otimes E_{ij} \otimes E_{kl}\,, \quad r_{13}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes 1_N \otimes E_{kl}\,.$$

In quantum case it is generalized to the quantum Yang-Baxter equation for quantum R-matrix:

$$R^\hbar_{12}\, R^\hbar_{13}\, R^\hbar_{23} = R^\hbar_{23}\, R^\hbar_{13}\, R^\hbar_{12}\,, \qquad R^\hbar_{ij} = R^\hbar_{ij}(z_i - z_j)$$

(here R-matrix is non-dynamical). In the quasi-classical limit

$$R^\hbar_{12} = 1 \otimes 1 + \hbar r_{12} + O(\hbar^2)$$

the latter reproduces the classical Yang-Baxter equation.

The Calogero-Moser model:

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i<j}^{N} \wp(q_i - q_j)\,,$$

where ν – coupling constant, $\wp(q)$ – Weierstrass $\wp$-function. Its equations of motion are written in the Lax form with

$$L(z) = \begin{pmatrix} p_1 & \nu\phi(z, q_1 - q_2) & \nu\phi(z, q_1 - q_3) & \dots & \nu\phi(z, q_1 - q_N) \\ \nu\phi(z, q_2 - q_1) & p_2 & \nu\phi(z, q_2 - q_3) & \dots & \nu\phi(z, q_2 - q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu\phi(z, q_N - q_1) & \nu\phi(z, q_N - q_2) & \nu\phi(z, q_N - q_3) & \dots & p_N \end{pmatrix}$$

where z – spectral parameter (Krichever 1980). It is a local coordinate on (elliptic) curve.

$$L_{ij}(z) = \delta_{ij} p_i + \nu(1-\delta_{ij})\,\phi(z, q_{ij})\,, \quad q_{ij} = q_i - q_j\,, \quad \phi(z,q) = \frac{\vartheta'(0)\vartheta(q+z)}{\vartheta(q)\,\vartheta(z)}\,,$$

$$M_{ij}(z) = \nu d_i \delta_{ij} + \nu(1-\delta_{ij}) f(z, q_{ij})\,, \quad d_i = -\sum_{k\neq i}^{N} f(0, q_{ik})\,,$$

where $f(z,q) = \partial_q \phi(z,q)$.

$\phi(z,q)$ is the elliptic Kronecker function. It satisfies the Fay identity (summation formula):

$$\phi(\hbar, q_1-q_2)\phi(\eta, q_2-q_3) = \phi(\hbar-\eta, q_1-q_2)\phi(\eta, q_1-q_3) + \phi(\eta-\hbar, q_2-q_3)\phi(\hbar, q_1-q_3)$$

for the Kronecker function:

$$\phi(\eta,z) = \begin{cases} 1/\eta + 1/z\,, \\ \coth(\eta) + \coth(z)\,, \\ \dfrac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)}\,. \end{cases} \qquad E_1(z) = \begin{cases} 1/z\,, \\ \coth(z)\,, \\ \dfrac{\vartheta'(z)}{\vartheta(z)} \end{cases} \qquad \wp(z) = \begin{cases} 1/z^2\,, \\ 1/\sinh^2(z)\,, \\ -E_1'(z) + const \end{cases}$$

Degeneration:

$$\phi(z, q_{ab}) f(z, q_{bc}) - f(z, q_{ab})\phi(z, q_{bc}) = \phi(z, q_{ac})(\wp(q_{ab}) - \wp(q_{bc}))\,,$$

$$f(z,q) = \partial_q \phi(z,q)\,, \qquad f(0,q) = -\wp(q) + const$$

These identities are widely used in integrable systems (Lax equations, R-matrix structures,...)

Quantum Yang-Baxter equation:

$$R^{\hbar}_{12}(q_1,q_2)R^{\hbar}_{13}(q_1,q_3)R^{\hbar}_{23}(q_2,q_3) = R^{\hbar}_{23}(q_2,q_3)R^{\hbar}_{13}(q_1,q_3)R^{\hbar}_{12}(q_1,q_2)$$

We assume $R^{\hbar}_{12}(q_1,q_2) = R^{\hbar}_{12}(q_1-q_2) \equiv R^{\hbar}_{12}(q_{12}) \equiv R^{\hbar}_{12}$

Associative Yang-Baxter equation (A. Kirillov, A. Polishchuk):

$$R^{\hbar}_{12}(q_{12})R^{\eta}_{23}(q_{23}) = R^{\eta}_{13}(q_{13})R^{\hbar-\eta}_{12}(q_{12}) + R^{\eta-\hbar}_{23}(q_{23})R^{\hbar}_{13}(q_{13})$$

Example of common solution: Yang's R-matrix

$$R^{\mathrm{Yang}}_{12}(q_1,q_2) = \frac{1\otimes 1}{\hbar} + \frac{NP_{12}}{q_1-q_2}\,, \quad P_{12} = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji}\,,$$

where P_{12} is the matric permutation operator: $P_{12}(a\otimes b) = b\otimes a$, $a,b\in\mathbb{C}^N$.

...,trigonometric (XXZ),..., elliptic Baxter-Belavin (XYZ).

In scalar case YB is empty condition while the AYBE

$$R^{\hbar}_{12}(q_{12})R^{\eta}_{23}(q_{23}) = R^{\eta}_{13}(q_{13})R^{\hbar-\eta}_{12}(q_{12}) + R^{\eta-\hbar}_{23}(q_{23})R^{\hbar}_{13}(q_{13})$$

is the Fay identity (can be considered as functional equation for the Kronecker elliptic function)

$$\phi(\hbar, q_{12})\phi(\eta, q_{23}) = \phi(\hbar-\eta, q_{12})\phi(\eta, q_{13}) + \phi(\eta-\hbar, q_{23})\phi(\hbar, q_{13})\,, \quad q_{ij} = q_i - q_j$$

The quasi-classical limit

$$R^{\hbar}_{12}(q) = \frac{1}{\hbar}\,1\otimes 1 + r_{12}(q) + \hbar\, m_{12}(q) + O(\hbar^2)\,.$$

is similar to expansion of $\phi(\hbar, q)$ near $\hbar = 0$:

$$\phi(\hbar, q) = \hbar^{-1} + E_1(q) + \hbar\,(E_1^2(q) - \wp(q))/2 + O(\hbar^2)\,, \quad E_1(q) = \vartheta'(q)/\vartheta(q)$$

$R^{\hbar}_{12}(q)$ – matrix analogue of $\phi(\hbar, q)$ and $r_{12}(q)$ – matrix analogue of $E_1(q) = \vartheta'(q)/\vartheta(q)$.

Euler-Arnold tops

Dynamical variables:

$$S = \sum_{i,j=1}^{N} S_{ij} E_{ij} \in \mathrm{Mat}(N)\,,$$

where E_{ij} is a standard basis in $\mathrm{Mat}(N, \mathbb{C})$.
The Euler-Arnold equations:

$$\dot{S} = [S, J(S)]\,,$$

where

$$J(S) = \sum J_{ij,kl} S_{kl} E_{ij}$$

is a linear functional, and $J_{ij,kl}$ – some set of constants. From the point of view of rotation in multidimensional space J – inverse inertia tensor.

The Hamiltonian

$$H = \frac{1}{2}\,\mathrm{tr}(SJ(S))$$

The Poisson brackets are given by the linear Poisson-Lie structure on gl_N^*:

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}\,.$$

If R is a solution of associative Yang-Baxter equation (AYBE) with skew-symmetry and unitarity properties and the quasi-classical expansion

$$R^{\hbar}_{12}(z) = \frac{1}{\hbar}\, 1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2)$$

then the Lax equation

$$\dot{L}(z,S) = [L(z,S), M(z,S)]$$

holds true identically in z and is equivalent to the Euler-Arnold top equations with the following data:

$$L(z,S) = \mathrm{tr}_2(r_{12}(z)S_2)\,, \quad M(z,S) = \mathrm{tr}_2(m_{12}(z)S_2)\,, \quad S_2 = 1 \otimes S$$

$$r_{12}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes E_{lk} \quad \mathrm{tr}_2(r_{12}(z)S_2) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij}\, \mathrm{tr}(E_{kl}S) = \sum_{ijkl} r_{ij,kl}(z)\, S_{lk}\, E_{ij}\,,$$

$$J(S) = \mathrm{tr}_2(m_{12}(0)S_2)$$

where $r_{12}(z)$ and $m_{12}(z)$ comes from the quasi-classical expansion.

Example: Cherednik's 7-vertex trigonometric R-matrix

$$R^{\hbar}(z) = \begin{pmatrix} \coth(z) + \coth(\hbar) & 0 & 0 & 0 \\ 0 & \sinh^{-1}(\hbar) & \sinh^{-1}(z) & 0 \\ 0 & \sinh^{-1}(z) & \sinh^{-1}(\hbar) & 0 \\ -4\, e^{-2\Lambda} \sinh(z+\hbar) & 0 & 0 & \coth(z) + \coth(\hbar) \end{pmatrix}$$

provides

$$J(S) = \frac{1}{6} \begin{pmatrix} 2S_{11} - S_{22} & 0 \\ -24\, e^{-2\Lambda} S_{12} & -S_{11} + 2S_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

1+1 Integrable Field Theories

Instead of Lax equation we deal with the zero curvature equation known as the Zakharov-Shabat equation:

$$\partial_t U(z) - k\partial_x V(z) + [U(z), V(z)] = 0\,, \qquad U(z), V(z) \in \mathrm{Mat}(N, \mathbb{C})\,,$$

In the limit to the finite-dimensional case the term $\partial_x V(z)$ vanishes, and the Zakharov-Shabat equation becomes the Lax equation, where $L(z)$ is the limit of $U(z)$, while $M(z)$ coincides with the limit of $V(z)$ up to some expression commuting with $L(z)$.

The conservation laws in the field case are generated by $\mathrm{tr}(T^k(z,2\pi))$, where $T(z,x) \in \mathrm{Mat}(N,\mathbb{C})$ is the monodromy matrix

$$T(z,x) = \mathrm{Pexp}\Big(\int\limits_0^x dy\, U(z,y) \Big)\,.$$

The sufficient condition for the Poisson commutativity

$$\{\mathrm{tr}(T^k(z,2\pi)), \mathrm{tr}(T^m(w,2\pi))\} = 0$$

was suggested by J.-M. Maillet and is know as the non-ultralocal Maillet bracket:

$$\{U_1(z,x), U_2(w,y)\} =$$

$$\Big(- k\partial_x \mathbf{r}_{12}(z,w|x) + [U_1(z,x), \mathbf{r}_{12}(z,w|x)] - [U_2(w,y), \mathbf{r}_{21}(w,z|x)] \Big)\delta(x-y) -$$

$$-\Big(\mathbf{r}_{12}(z,w|x) + \mathbf{r}_{21}(w,z|x)\Big)\delta'(x-y)\,.$$

If $\mathbf{r}_{12}(z,w|x) + \mathbf{r}_{21}(w,z|x) \neq 0$ then the bracket is called non-ultralocal.

For construction of the U-V pair for Landau-Lifshitz gl_N magnet we use degenerations of AYBE

$$R^{\hbar}_{12}R^{\eta}_{23} = R^{\eta}_{13}R^{\hbar-\eta}_{12} + R^{\eta-\hbar}_{23}R^{\hbar}_{13}\,, \qquad R^{x}_{ab} = R^{x}_{ab}(z_a - z_b)$$

In particular, we have

$$[m_{13}(z_1) + m_{23}(z_2), r_{12}(z_1 - z_2)] = [m_{12}(z_1 - z_2) + m_{13}(z_1), r_{23}(z_2)]$$

$$[m_{13}(z), r_{12}(z)] = [r_{12}(z), m_{23}(0)] - [\partial_z m_{12}(z), NP_{23}] + [m_{12}(z), r^{(0)}_{23}] + [m_{13}(z), r^{(0)}_{23}]\,,$$

where $r^{(0)}_{12}$ comes from the expansion

$$r_{12}(z) = \frac{1}{z}\,NP_{12} + r^{(0)}_{12} + O(z)\,, \quad P_{12} = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji}\,,$$

where P_{12} is the matric permutation operator: $P_{12}(a\otimes b) = b\otimes a$, $a, b\in\mathbb{C}^N$. Also

$$r_{12}(z)r_{13}(z{+}w) - r_{23}(w)r_{12}(z) + r_{13}(z{+}w)r_{23}(w) = m_{12}(z) + m_{23}(w) + m_{13}(z{+}w)\,.$$

$$r_{12}(z)r_{13}(z) = r^{(0)}_{23}r_{12}(z) - r_{13}(z)r^{(0)}_{23} - N\partial_z r_{13}(z)P_{23} + m_{12}(z) + m_{23}(0) + m_{13}(z)\,.$$

Consider the special case:

$$S^2 = cS\,,$$

where $c \in \mathbb{C}$ is some constant. This condition means that the eigenvalues of the matrix S are equal to either 0 or c.

Consider the following ansatz for U-V pair:

$$U(z) = L(S,z) = \frac{1}{N}\operatorname*{tr}_2\Big(r_{12}(z)S_2\Big)\,, \qquad S_2 = 1_N \otimes S\,,$$

and

$$V(z) = -c\partial_z L(S,z) + L(SE(S),z) + L(E(S)S,z) - cL(T,z)$$

with the definition of $E(A)$, $A \in \mathrm{Mat}(N,\mathbb{C})$ given by the following linear map:

$$A \to E(A) = \frac{1}{N}\mathrm{tr}_2\Big(r_{12}^{(0)}A_2\Big)\,, \quad A_2 = 1_N \otimes A\,, \quad r_{12}(z) = \frac{1}{z}NP_{12} + r_{12}^{(0)} + O(z)$$

and $T \in \mathrm{Mat}(N,\mathbb{C})$ is some dynamical matrix to be defined.

Then the zero-curvature equation is equivalent to

$$\partial_t S + c\partial_x T - \partial_x(SE(S) + E(S)S) = 2s_0[S, J(S)] + c[S, E(T)] + c[E(S), T]\,,$$

where $s_0 = \mathrm{tr}(S)/N$ and

$$-\partial_x S = [S, T]\,.$$

The latter equation can be solved (when $S^2 = cS$) as

$$T = -c^{-2}[S, \partial_x S]\,.$$

Finally, the gl_N Landau-Lifshitz equation takes the form:

$$\partial_t S - \frac{1}{c}\,[S, \partial_x^2 S] - \partial_x\Big(SE(S){+}E(S)S\Big){=}2s_0[S, J(S)]{-}\frac{1}{c}\,[S, E([S, \partial_x S])]{-}\frac{1}{c}\,[E(S), [S, \partial_x S]]$$

This is the higher rank Landau-Lifshitz equation.

The most general is the elliptic case. It comes from the elliptic Baxter-Belavin R-matrix in the fundamental representation of GL_N.
Introduce the special matrix basis in $\mathrm{Mat}(N,\mathbb{C})$: $T_a = T_{a_1a_2} = \exp\left(\frac{\pi\imath}{N}\, a_1a_2\right) Q_1^{a_1} Q_2^{a_2}$, where $a=(a_1,a_2)\in\mathbb{Z}_N\times\mathbb{Z}_N$, and $(Q_1)_{kl}=\delta_{kl}\exp(\frac{2\pi\imath}{N}k)$, $(Q_2)_{kl}=\delta_{k-l+1=0 \bmod N}$. In particular, $T_{(0,0)}=1_N$. The basis has the property $\mathrm{tr}(T_\alpha T_\beta)=N\delta_{\alpha+\beta,(0,0)}$. The quantum R-matrix is as follows:

$$R^\hbar_{12}(z)=\sum_{a\in\,\mathbb{Z}_N\times\mathbb{Z}_N} T_a\otimes T_{-a}\exp(2\pi\imath\,\frac{a_2z}{N})\,\phi(z,\frac{a_1+a_2\tau}{N}+\hbar)\in\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}\,.$$

The quasi-classical limit expansion provides the classical (Belavin-Drinfeld) r-matrix

$$r_{12}(z)=E_1(z)1_N\otimes 1_N+\sum_{a\neq(0,0)} T_a\otimes T_{-a}\exp(2\pi\imath\,\frac{a_2z}{N})\,\phi(z,\frac{a_1+a_2\tau}{N})$$

and the next coefficient

$$m_{12}(z)=\rho(z)1_N\otimes 1_N+\sum_{a\neq(0,0)} T_a\otimes T_{-a}\exp(2\pi\imath\,\frac{a_2z}{N})\,f(z,\frac{a_1+a_2\tau}{N})\,,$$

where $f(z,u)=\partial_w\phi(z,w)|_{w=u}$ and $\rho(z)=(E_1^2(z)-\wp(z))/2$.

Then one finds

$$r_{12}^{(0)} = \sum_{a\neq(0,0)} T_a \otimes T_{-a}\Big(2\pi\imath\,\frac{a_2}{N} + E_1(\frac{a_1+a_2\tau}{N})\Big)\,,$$

so that

$$E(A) = \sum_{a\neq(0,0)} T_a A_a\Big(2\pi\imath\,\frac{a_2}{N} + E_1(\frac{a_1+a_2\tau}{N})\Big)$$

And

$$m_{12}(0) = \frac{\vartheta'''(0)}{3\vartheta'(0)} 1_N \otimes 1_N - \sum_{a\neq(0,0)} T_a \otimes T_{-a} E_2(\frac{a_1+a_2\tau}{N})\,, \quad E_2(x) = -E_1'(z) = -\partial_z^2 \log\vartheta(z)\,,$$

that is the inverse inertia tensor

$$J(S) = \frac{\vartheta'''(0)}{3\vartheta'(0)} 1_N S_{0,0} - \sum_{a\neq(0,0)} T_a S_a E_2(\frac{a_1+a_2\tau}{N}) = \frac{\vartheta'''(0)}{3\vartheta'(0)}\, S - \sum_{a\neq(0,0)} T_a S_a \wp(\frac{a_1+a_2\tau}{N})\,.$$

In the last line we also used relation $E_2(z) = \wp(z) - \vartheta'''(0)/(3\vartheta'(0))$.

In particular case, when $N-1$ eigenvalues coincide, we come to the matrix S of rank 1. In this case the obtained equation is simplified.

The case $\mathrm{rank}(S)=1$

If $\mathrm{rank}(S)=1$ then the matrix S is represented as $S=\xi\otimes\psi$, where ξ and ψ are N-dimensional vector column and row respectively. Then

$$S^2=\mathrm{tr}(S)S,$$

so that

$$c=\mathrm{tr}(S)=Ns_0\,.$$

The Landau-Lifshitz equation takes the form:

$$\partial_t S=\frac{1}{c}\,[S,\partial_x^2 S]+\frac{2c}{N}\,[S,J(S)]-2[S,E(\partial_x S)]\,.$$

Hamiltonian description. The equation

$$\partial_t S = \frac{1}{c}\,[S, \partial_x^2 S] + \frac{2c}{N}\,[S, J(S)] - 2[S, E(\partial_x S)]\,.$$

can be easily described in the Hamiltonian formalism. The Poisson structure is given by

$$\{S_{ij}(x), S_{kl}(y)\} = (S_{kj}(x)\delta_{il} - S_{il}(x)\delta_{kj})\delta(x-y)\,.$$

The equations of motion

$$\partial_t S(x) = \{H, S(x)\}$$

are reproduced by the following Hamiltonian:

$$H = \oint dy \Big(\frac{c}{N}\,\mathrm{tr}\,\Big(S\,J(S)\Big) - \frac{1}{2c}\,\mathrm{tr}\,\Big(\partial_y S\,\partial_y S\Big) + \mathrm{tr}\,\Big(\partial_y S\,E(S)\Big)\Big)\,, \qquad S = S(y)\,.$$

The last term vanishes in $N = 2$ case.

Calogero-Moser model and its field generalization

Mechanics.

At the level of classical mechanics the Calogero-Moser model describes interaction of N particles on a complex plane generated by the following Hamiltonian:

$$H^{\mathrm{CM}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - c^2 \sum_{i>j}^{N} \wp(q_i - q_j)\,,$$

where $c \in \mathbb{C}$ is a coupling constant and $\wp$ is the Weierstrass $\wp$-function. The positions of particles $q_i \in \mathbb{C}$ and momenta $p_i \in \mathbb{C}$ are canonically conjugated:

$$\{p_i, q_j\} = \delta_{ij}\,, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0\,.$$

The Hamiltonian provide equations of motion $\dot{f} = \{H, f\}$, i.e.

$$\dot{q}_i = p_i \,, \qquad \dot{p}_i = c^2 \sum_{k:k\neq i}^{N} \wp'(q_{ik}) \,, \quad i = 1, ..., N$$

which are represented in the Lax form

$$\dot{L}(z) + [L(z), M(z)] = 0 \,, \qquad L(z), M(z) \in \mathrm{Mat}(N, \mathbb{C})$$

with spectral parameter z, where

$$L^{\mathrm{CM}}_{ij}(z) = \delta_{ij}(p_i + cE_1(z)) + c(1 - \delta_{ij})\phi(z, q_{ij}) \,, \quad q_{ij} = q_i - q_j \,,$$

$$M^{\mathrm{CM}}_{ij}(z) = -c\delta_{ij} d_i - c(1 - \delta_{ij}) f(z, q_{ij}) \,, \quad d_i = \sum_{k:k\neq i}^{N} \wp(q_{ik}) \,,$$

The classical r-matrix structure:

$$\{L_1(z), L_2(w)\} = [L_1(z), r_{12}(z,w)] - [L_2(w), r_{21}(w,z)]\,.$$

The permutation of r-matrix indices 12 or 21 means the permutation of the tensor components.

$$r_{12}(z,w) = \sum_{i,j,k,l=1}^{N} E_{ij} \otimes E_{kl}\, r_{ij,kl}(z,w)\,, \qquad r_{21}(z,w) = \sum_{i,j,k,l=1}^{N} E_{kl} \otimes E_{ij}\, r_{ij,kl}(z,w)\,,$$

where E_{ij} is the basis matrix units in $\mathrm{Mat}(N,\mathbb{C})$.

The classical r-matrix:

$$r_{12}^{\mathrm{CM}}(z,w) =$$

$$= (E_1(z-w) + E_1(w)) \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \sum_{i\neq j}^{N} \phi(z-w, q_{ij})\, E_{ij} \otimes E_{ji} - \sum_{i\neq j}^{N} \phi(-w, q_{ij})\, E_{ii} \otimes E_{ji}\,.$$

The field generalization is highly nontrivial.

The Hamiltonian

$$\mathcal{H}^{\rm 2dCM} = \int_0^{2\pi} dx H^{\rm 2dCM}(x)$$

is defined by the density

$$H^{\rm 2dCM}(x) = -\sum_{j=1}^{N} p_j^2 (kq_{j,x} + c) + \frac{1}{Nc}\Big(\sum_{j=1}^{N} p_j (kq_{j,x} + c)\Big)^2 + \frac{1}{4}\sum_{j=1}^{N} \frac{k^4 q_{j,xx}^2}{kq_{j,x} + c} +$$
$$+\frac{1}{2}\sum_{i\neq j}^{N}\Big((kq_{i,x}+c)^2(kq_{j,x}+c) + (kq_{i,x}+c)(kq_{j,x}+c)^2 - ck^2(q_{i,x}-q_{j,x})^2\Big)\wp(q_i - q_j)+$$
$$+\frac{k^3}{2}\sum_{i\neq j}^{N}\Big(q_{i,x}q_{j,xx} - q_{i,xx}q_{j,x}\Big)\zeta(q_i - q_j)\,,$$

where $q_{i,x} = \partial_x q_i(x)$ and $q_{i,xx} = \partial_x^2 q_i(x)$.

The canonical Poisson brackets

$$\{p_i(x), q_j(y)\} = \delta_{ij}\delta(x-y)\,, \qquad \{p_i(x), p_j(y)\} = \{q_i(x), q_j(y)\} = 0\,.$$

and the Hamiltonian provide equations of motion

$$\dot{q}_i \equiv \{H^{\text{2dCM}}, q_i\} = -2(kq_{i,x}+c)\Big(p_i - \frac{1}{Nc}\sum_{j=1}^{N} p_j(kq_{j,x}+c)\Big)\,,$$

$$\dot{p}_i \equiv \{H^{\text{2dCM}}, p_i\} = -k\partial_x\left(p_i^2 - 2p_i\frac{1}{Nc}\sum_{j=1}^{N} p_j(kq_{j,x}+c) + \frac{1}{2}\frac{k^3 q_{i,xxx}}{kq_{i,x}+c} - \frac{1}{4}\frac{k^4 q_{i,xx}^2}{(kq_{i,x}+c)^2}\right) -$$
$$-2\sum_{j:j\neq i}^{N}\Big((kq_{j,x}+c)^3\wp'(q_{ij}) - 3k^2(kq_{j,x}+c)q_{j,xx}\wp(q_{ij}) - k^3 q_{j,xxx}\zeta(q_{ij})\Big)\,.$$

Introduce notations

$$\alpha_i = (kq_{i,x} + c)^{1/2}\,, \quad i = 1, ..., N$$

and

$$\kappa = -\frac{1}{Nc}\sum_{j=1}^{N} p_j(kq_{j,x} + c) = -\frac{1}{Nc}\sum_{j=1}^{N} p_j\alpha_j^2\,.$$

Then the Hamiltonian and the equations of motion are written in a slightly more compact form:

$$H^{\mathrm{2dCM}}(x) = -\sum_{i=1}^{N} p_i^2\alpha_i^2 + Nc\kappa^2 + \sum_{i=1}^{N} k^2\alpha_{i,x}^2 +$$

$$+\frac{k}{2}\sum_{i\neq j}^{N}\Big(\alpha_i\alpha_{j,x} - \alpha_j\alpha_{i,x} + c(\alpha_{i,x} - \alpha_{j,x})\Big)\zeta(q_{ij}) + \frac{1}{2}\sum_{i\neq j}^{N}\Big(\alpha_i^4\alpha_j^2 + \alpha_i^2\alpha_j^4 - c(\alpha_i^2 - \alpha_j^2)^2\Big)\wp(q_{ij})$$

and

$$\dot{q}_i = -2\alpha_i^2(p_i + \kappa)\,,$$

$$\dot{p}_i = -k\partial_x\Big(p_i^2 + 2\kappa p_i + k^2\frac{\alpha_{i,xx}}{\alpha_i}\Big) - 2\sum_{j:j\neq i}^{N}\Big(\alpha_j^6\wp'(q_{ij}) - 6\alpha_j^3\alpha_{j,x}\wp(q_{ij}) - k^2\partial_x^2(\alpha_j^2)\zeta(q_{ij})\Big)\,.$$

It was shown by Akhmetshin, Krichever and Volvovski that the equations of motion are represented in the form of the Zakharov-Shabat equations

$$\partial_t U(z) - k\partial_x V(z) + [U(z), V(z)] = 0\,, \qquad U(z), V(z) \in \mathrm{Mat}(N, \mathbb{C})\,,$$

with the U-V pair:

$$U_{ij}^{\mathrm{2dCM}}(z) = \delta_{ij}\Big(p_i + \alpha_i^2 E_1(z) - k\frac{\alpha_{i,x}}{\alpha_i}\Big) + (1-\delta_{ij})\phi(z, q_i - q_j)\alpha_j^2\,,$$

$$V_{ij}^{\mathrm{2dCM}}(z) = \delta_{ij}\Big(q_{i,t}E_1(z) - Nc\alpha_i^2\wp(z) - m_i^0 - \frac{\alpha_{i,t}}{\alpha_i}\Big) +$$

$$+(1-\delta_{ij})\Big(Ncf(z, q_i - q_j) - NcE_1(z)\phi(z, q_i - q_j) - m_{ij}\phi(z, q_i - q_j)\Big)\alpha_j^2\,,$$

where

$$m_i^0 = p_i^2 + 2\kappa p_i + k^2\frac{\alpha_{i,xx}}{\alpha_i} - \sum_{j:j\neq i}^{N}\Big((2\alpha_j^4 + \alpha_i^2\alpha_j^2)\wp(q_i - q_j) + 4k\alpha_j\alpha_{j,x}\zeta(q_i - q_j)\Big)\,,$$

and

$$m_{ij} = p_i + p_j + 2\kappa - k\frac{\alpha_{i,x}}{\alpha_i} + k\frac{\alpha_{j,x}}{\alpha_j} + \sum_{l:l\neq i,j}^{N}\alpha_l^2\eta(q_i, q_l, q_j)\,, \quad i\neq j\,,$$

$$\eta(z_1, z_2, z_3) = E_1(z_1 - z_2) + E_1(z_2 - z_3) + E_1(z_3 - z_1)\,.$$

Classical r-matrix. Main statement is that the U-matrix satisfies the Maillet r-matrix structure

$$\{U_1(z,x),U_2(w,y)\} =$$

$$\Big(-k\partial_x \mathbf{r}_{12}(z,w|x) + [U_1(z,x),\mathbf{r}_{12}(z,w|x)] - [U_2(w,y),\mathbf{r}_{21}(w,z|x)]\Big)\delta(x-y)-$$

$$-\Big(\mathbf{r}_{12}(z,w|x) + \mathbf{r}_{21}(w,z|x)\Big)\delta'(x-y)\,.$$

with the r-matrix for the finite-dimensional Calogero-Moser model, where positions of particles q_i are replaced with the fields $q_i(x)$:

$$\mathbf{r}_{12}^{\text{2dCM}}(z,w|x) = (E_1(z-w)+E_1(w))\sum_{i=1}^{N} E_{ii}\otimes E_{ii}+$$

$$+\sum_{i\neq j}^{N}\phi(z-w,q_i(x)-q_j(x))\,E_{ij}\otimes E_{ji} - \sum_{i\neq j}^{N}\phi(-w,q_i(x)-q_j(x))\,E_{ii}\otimes E_{ji}\,.$$

It is non-ultralocal (the coefficient behind $\delta'(x-y)$ is not zero).

Gauge equivalence between 1+1 Calogero-Moser model and 1+1 Landau-Lifshitz model.

There exists $G(z,x) \in \mathrm{Mat}(N,\mathbb{C})$ that the gauge transformation

$$U(z,x) \to G(z,x)U(z,x)G^{-1}(z,x) + k\partial_x G(z,x)G^{-1}(z,x)$$

which relates these two models in the case $\mathrm{rank}(S) = 1$.
For the Landau-Lifshitz model the classical r-matrix is the non-dynamical elliptic Belavin-Drinfeld r-matrix:

$$r_{12}^{\mathrm{BD}}(z) = E_1(z) 1_N \otimes 1_N + \sum_{\substack{a \in \mathbb{Z}_N \times \mathbb{Z}_N \\ a \neq (0,0)}} T_a \otimes T_{-a} \exp(2\pi\imath \frac{a_2 z}{N})\, \phi(z, \frac{a_1 + a_2\tau}{N}) \,,$$

being written in a special matrix basis (in $\mathrm{Mat}(N,\mathbb{C})$)

$$T_a = T_{a_1 a_2} = \exp\left(\frac{\pi\imath}{N} a_1 a_2\right) Q_1^{a_1} Q_2^{a_2} \in \mathrm{Mat}(N,\mathbb{C})\,, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N\,,$$

defined in terms of the pair of matrices

$$(Q_1)_{kl} = \delta_{kl} \exp(\frac{2\pi\imath}{N} k)\,, \qquad (Q_2)_{kl} = \delta_{k-l+1=0 \bmod N}\,.$$

The corresponding U-matrix is the one of the Sklyanin type, i.e.

$$U^{\mathrm{Skl}}(z,x) = \sum_{\substack{a\in\mathbb{Z}_N\times\mathbb{Z}_N\\ a\neq(0,0)}} T_a\mathcal{S}_a(x)\exp(2\pi\imath\,\frac{a_2 z}{N})\,\phi(z,\frac{a_1+a_2\tau}{N})\,.$$

In the case $N=2$ it is the U-matrix of the original Landau-Lifshitz model. The relation between r-matrices is given by the intertwining matrix

$$g(z,q) = \Xi(z,q)\left(d^0\right)^{-1}(q)\,,$$

where $q=\{q_1,...,q_N\}$ and

$$\Xi_{ij}(z,q) = \vartheta\left[\begin{array}{c}\frac{1}{2}-\frac{i}{N}\\ \frac{N}{2}\end{array}\right]\left(z-Nq_j+\sum_{m=1}^{N}q_m\,\middle|\,N\tau\right)\,,$$

and the diagonal matrix

$$d^0_{ij}(q) = \delta_{ij}d^0_j = \delta_{ij}\prod_{k:k\neq j}^{N}\vartheta(q_j-q_k)\,.$$

Here q_i are some parameters.

Namely, consider

$$G(z,x) = g(z, q_1(x), ..., q_N(x))$$

Then it can be show that

$$r_{12}^{\mathrm{BD}}(z,w) = \mathbf{r}_{12}^{\mathrm{BD}}(z,w|x) = G_1(z,x)G_2(w,x)\Big(\mathbf{r}_{12}^{\mathrm{2dCM}}(z,w|x) -$$

$$- G_1^{-1}(z,x)\{G_1(z,x), U_2^{\mathrm{2dCM}}(w,x)\}\Big) G_1^{-1}(z,x) G_2^{-1}(w,x)\,,$$

This gauge transformation relates ultralocal and non-ultralocal models.

Then the gauge equivalence relating the U–V pairs provides explicit change of variables. For example, in the trigonometric case we have

$$S_{ij}(x) = \frac{(-1)^j \sigma_j(e^q)}{N} \sum_{m=1}^{N} \frac{P_m \left(e^{(i-1)q_m} + (-1)^N \delta_{iN} e^{-q_m} \right) + N\alpha_m^2 (-1)^N \delta_{iN} e^{-q_m}}{\prod\limits_{l:\, l\neq m}^{N} (e^{q_m} - e^{q_l})} ,$$

where $\sigma_j(e^q)$ – notation for the elementary symmetric functions and

$$P_m = -p_m - \frac{k\alpha_{mx}}{\alpha_m} - (i-1)\alpha_m^2 + \frac{(N-2)}{2} k q_{m,x} + \frac{\alpha_m^2}{2} \sum_{l\,:l\neq m}^{N} \coth\left(\frac{q_m - q_l}{2} \right)$$

for $m = 1, ..., N$. Here

$$Spec(S) = (0, ..., 0, c)$$

and

$$\mathrm{tr}(S) = c, \quad S^2 = cS\,.$$

The Poisson brackets $\{S_{ij}(x), S_{kl}(y)\}$ being computed for the upper expressions by means of the canonical brackets, provide the linear Poisson brackets. That is the map between two models is a Poisson map.

The talk is based on papers:

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K. Atalikov, A. Zotov, *Field theory generalizations of two-body Calogero-Moser models in the form of Landau-Lifshitz equations*, J. Geom. Phys., 164 (2021) 104161; arXiv:2010.14297 [hep-th].

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A. Zotov, *Non-ultralocal classical r-matrix structure for 1+1 field analogue of elliptic Calogero-Moser model*, (2024), J. Phys. A: Math. Theor., 57 (2024) 315201.

Thank you!