On the Fradkin-Vasiliev formalism in $d = 4$

Yu. M. Zinoviev

Institute for High Energy Physics, Protvino, Russia

Lebedev Institute, Moscow September 2–6, 2024

centennial conference **Efim FRADKIN 100**

Outlook

1. Massless fields
2. One massless and two massive fields
3. Three massive fields
4. Partially massless fields

Frame-like formalism

- Set of one-forms and gauge invariant two-forms (curvatures)

$$\Omega^{\alpha(s-1+m)\dot\alpha(s-1-m)}, \quad \mathcal{R}^{\alpha(s-1+m)\dot\alpha(s-1-m)}, \quad 0 \le |m| \le s-1$$

- What is "on-shell"?

$$\begin{aligned}
0 &\approx D\Omega^{\alpha(s-1+m)\dot\alpha(s-1-m)} + e_\beta{}^{\dot\alpha}\Omega^{\alpha(s-1+m)\beta\dot\alpha(s-2-m)} + O(\lambda^2) \\
0 &\approx \mathcal{R}^{\alpha(2s-2)} - E_{\beta(2)} W^{\alpha(2s-2)\beta(2)} \\
0 &\approx DW^{\alpha(2s+k)\dot\alpha(k)} + e_{\beta\dot\beta} W^{\alpha(2s+k)\beta\dot\alpha(k)\dot\beta} + \lambda^2 e^{\alpha\dot\alpha} W^{\alpha(2s+k-1)\dot\alpha(k-1)}
\end{aligned}$$

- Free Lagrangian in terms of curvatures

$$\mathcal{L}_0 \sim \sum_{m=1}^{s-1} c_m \mathcal{R}_{\alpha(s-1+m)\dot\alpha(s-1-m)} \mathcal{R}^{\alpha(s-1-m)\dot\alpha(s-1-m)} + h.c.$$

Cubic vertices

- Metsaev's classification $d = 4$ $s_1 \geq s_2 \geq s_3$
  - Type I: $n = s_1 + s_2 + s_3$
  - Type II: $n = s_1 + s_2 - s_3$
- Ansatz for type I:

$$\mathcal{L}_1 \sim W^{\alpha(\hat{s}_2)\beta(\hat{s}_3)} W^{\gamma(\hat{s}_1)}{}_{\beta(\hat{s}_3)} W_{\alpha(\hat{s}_2)\gamma(\hat{s}_1)}$$

We must have

$$\hat{s}_2 + \hat{s}_3 = 2s_1, \qquad \hat{s}_1 + \hat{s}_3 = 2s_2, \qquad \hat{s}_1 + \hat{s}_2 = 2s_3$$

This gives

$$\hat{s}_1 = s_2 + s_3 - s_1, \qquad \hat{s}_2 = s_1 + s_3 - s_2, \qquad \hat{s}_3 = s_1 + s_2 - s_3$$

Fradkin-Vasiliev formalism

- Constructive approach

$$\delta_0 \mathcal{L}_1|_{e.o.m.} = 0 \quad \Rightarrow \quad \delta_1 \Omega = \dots$$

- Consistent deformations of curvatures $\hat{\mathcal{R}} = \mathcal{R} + \Delta\mathcal{R}$

$$\Delta\mathcal{R} \sim \Omega\Omega \quad \Leftrightarrow \quad \delta_1 \Omega \sim \Omega\xi$$

- Consistency means

$$\delta\hat{\mathcal{R}} \sim \mathcal{R}\xi$$

- Interacting Lagrangian

$$\mathcal{L} \sim \sum \hat{\mathcal{R}}\hat{\mathcal{R}} \quad (+\mathcal{R}\mathcal{R}\Omega)$$

Fradkin-Vasiliev formalism

- Constructive approach

$$\delta_0 \mathcal{L}_1 |_{e.o.m.} = 0 \quad \Rightarrow \quad \delta_1 \Omega = \dots$$

- Consistent deformations of curvatures $\hat{\mathcal{R}} = \mathcal{R} + \Delta\mathcal{R}$

$$\Delta\mathcal{R} \sim \Omega\Omega \quad \Leftrightarrow \quad \delta_1 \Omega \sim \Omega\xi$$

- Consistency means

$$\delta\hat{\mathcal{R}} \sim \mathcal{R}\xi$$

- Interacting Lagrangian

$$\mathcal{L} \sim \sum \hat{\mathcal{R}}\hat{\mathcal{R}} \quad (+\mathcal{R}\mathcal{R}\Omega)$$

Cubic vertices, type II

- Ansatz

$$\Delta\mathcal{R}^{\alpha(2s_1-2)} \sim \Omega^{\alpha(\hat{s}_3)\beta(\hat{s}_1)}\Omega^{\alpha(\hat{s}_2)}{}_{\beta(\hat{s}_1)} + \dots$$

$$\hat{s}_1 = s_2 + s_3 - s_1 - 1, \quad \hat{s}_2 = s_1 + s_3 - s_2 - 1, \quad \hat{s}_3 = s_1 + s_2 - s_3 - 1$$

- Number of derivatives

$$s_1 + s_2 + s_3 - 2$$
$$s_1 + s_2 + s_3 - 3$$
$$\dots$$
$$s_1 + s_2 - s_3 + 1$$
$$s_1 + s_2 - s_3$$
$$\dots$$

- Flat vertex

$$\mathcal{L}_1 \sim \Omega_1^{\alpha(\hat{s}_2)\dot{\alpha}(\hat{s}_3)}\Omega_2^{\beta(\hat{s}_1)}{}_{\dot{\alpha}(\hat{s}_3)} D\Omega_{3,\alpha(\hat{s}_2)\beta(\hat{s}_1)} + h.c.$$

Cubic vertices, type II

- Ansatz

$$\Delta\mathcal{R}^{\alpha(2s_1-2)} \sim \Omega^{\alpha(\hat{s}_3)\beta(\hat{s}_1)}\Omega^{\alpha(\hat{s}_2)}{}_{\beta(\hat{s}_1)} + \dots$$

$$\hat{s}_1 = s_2 + s_3 - s_1 - 1, \quad \hat{s}_2 = s_1 + s_3 - s_2 - 1, \quad \hat{s}_3 = s_1 + s_2 - s_3 - 1$$

- Number of derivatives

$$s_1 + s_2 + s_3 - 2$$
$$s_1 + s_2 + s_3 - 3$$
$$\dots$$
$$s_1 + s_2 - s_3 + 1$$
$$s_1 + s_2 - s_3$$
$$\dots$$

- Flat vertex

$$\mathcal{L}_1 \sim \Omega_1^{\alpha(\hat{s}_2)\dot{\alpha}(\hat{s}_3)}\Omega_2^{\beta(\hat{s}_1)}{}_{\dot{\alpha}(\hat{s}_3)} D\Omega_{3,\alpha(\hat{s}_2)\beta(\hat{s}_1)} + h.c.$$

Massless supermultiplets

- For three supermultiplets (B_i, F_i), $i = 1, 2, 3$ we can construct four elementary vertices

$$V_0(B_1, B_2, B_3), \quad V_1(F_2, B_1, F_3), \quad V_2(F_1, B_2, F_3), \quad V_3(F_1, F_2, B_3).$$

In general $\delta B \sim F\zeta \quad , \delta F \sim dB\zeta \quad \Rightarrow \quad N_{BBB} = N_{BFF} + 1$

- Consider curvature deformations for the first supermultiplet

$$\begin{aligned} \Delta\mathcal{R}_1 &= a_0 \Delta\mathcal{R}_1(\Omega_2, \Omega_3) + a_1 \Delta\mathcal{R}_1(\Phi_2, \Phi_3), \\ \Delta\mathcal{F}_1 &= a_2 \Delta\mathcal{F}_1(\Omega_2, \Phi_3) + a_3 \Delta\mathcal{F}_1(\Phi_2, \Omega_3) \end{aligned}$$

and require that deformed curvatures transform under the supertransformations as the undeformed ones.

- In AdS_4 all four elementary vertices present but in the flat limit one of the coupling constants goes to zero in agreement with Metsaev's classification

Massless supermultiplets

- For three supermultiplets (B_i, F_i), $i = 1, 2, 3$ we can construct four elementary vertices

$$V_0(B_1, B_2, B_3), \quad V_1(F_2, B_1, F_3), \quad V_2(F_1, B_2, F_3), \quad V_3(F_1, F_2, B_3).$$

  In general $\delta B \sim F\zeta \quad , \delta F \sim dB\zeta \quad \Rightarrow \quad N_{BBB} = N_{BFF} + 1$

- Consider curvature deformations for the first supermultiplet

$$\begin{aligned} \Delta\mathcal{R}_1 &= a_0 \Delta\mathcal{R}_1(\Omega_2, \Omega_3) + a_1 \Delta\mathcal{R}_1(\Phi_2, \Phi_3), \\ \Delta\mathcal{F}_1 &= a_2 \Delta\mathcal{F}_1(\Omega_2, \Phi_3) + a_3 \Delta\mathcal{F}_1(\Phi_2, \Omega_3) \end{aligned}$$

  and require that deformed curvatures transform under the supertransformations as the undeformed ones.

- In AdS_4 all four elementary vertices present but in the flat limit one of the coupling constants goes to zero in agreement with Metsaev's classification

Gauge invariance for massive fields

Collection of massless fields $0 \le k \le s$:

$$0 \longleftrightarrow \cdots \longleftrightarrow k-1 \longleftrightarrow k \longleftrightarrow \cdots \longleftrightarrow s$$

Figure: Massless vs massive case.

Gauge invariance for massive fields

Collection of massless fields $0 \le k \le s$:

$$0 \longleftrightarrow \cdots \longleftrightarrow k-1 \longleftrightarrow k \longleftrightarrow \cdots \longleftrightarrow s$$

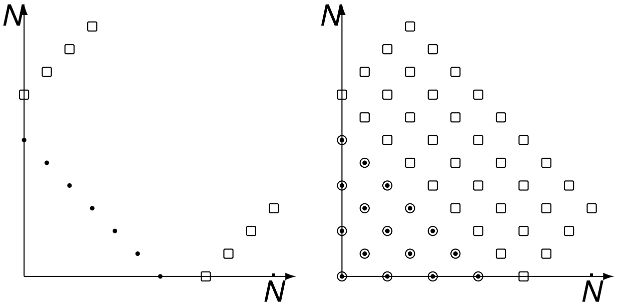

Figure: Massless vs massive case.

Cubic vertices

- Non-zero on-shell

$$\mathcal{R}^{\alpha(2s-2)} + h.c., \qquad \mathcal{B}^{\alpha(2s-2-k),\dot{\alpha}(k)}, \qquad 0 \le k \le 2s-2$$

  so that abelian vertices do exist even in $d = 4$

- Two types: $M_1 = M_2$ and $M_1 \neq M_2$
- Field redefinitions due to Stueckelberg fields

$$\Delta\mathcal{R} \sim \mathcal{B}\Phi \Rightarrow \delta\Phi \sim \mathcal{B}\xi$$

  so that any vertex can be reduced to the abelian form

- This can be used for the classification of vertices
- Note that results in the unitary gauge do not depend on field redefinitions

Cubic vertices

- Non-zero on-shell

$$\mathcal{R}^{\alpha(2s-2)} + h.c., \qquad \mathcal{B}^{\alpha(2s-2-k),\dot{\alpha}(k)}, \qquad 0 \le k \le 2s-2$$

  so that abelian vertices do exist even in $d = 4$
- Two types: $M_1 = M_2$ and $M_1 \neq M_2$
- Field redefinitions due to Stueckelberg fields

$$\Delta\mathcal{R} \sim \mathcal{B}\Phi \Rightarrow \delta\Phi \sim \mathcal{B}\xi$$

  so that any vertex can be reduced to the abelian form
- This can be used for the classification of vertices
- Note that results in the unitary gauge do not depend on field redefinitions

Cubic vertices

- Non-zero on-shell

$$\mathcal{R}^{\alpha(2s-2)} + h.c., \qquad \mathcal{B}^{\alpha(2s-2-k),\dot{\alpha}(k)}, \qquad 0 \le k \le 2s-2$$

  so that abelian vertices do exist even in $d = 4$

- Two types: $M_1 = M_2$ and $M_1 \neq M_2$
- Field redefinitions due to Stueckelberg fields

$$\Delta\mathcal{R} \sim \mathcal{B}\Phi \Rightarrow \delta\Phi \sim \mathcal{B}\xi$$

  so that any vertex can be reduced to the abelian form

- This can be used for the classification of vertices
- Note that results in the unitary gauge do not depend on field redefinitions

Massless spin 3/2

- Massive superblock $(2, 3/2)$ Ansatz for abelian vertices

$$\begin{aligned} \mathcal{L}_a &= g_1 \mathcal{R}^{\alpha\beta} \mathcal{C}_\alpha \Psi_\beta + g_2 \mathcal{B}^{\alpha\beta} \mathcal{F}_\alpha \Psi_\beta + g_3 \Pi^{\alpha\dot\alpha} \mathcal{F}_{\dot\alpha} \Psi_\alpha \\ &+ f_1 e_\alpha{}^{\dot\alpha} \mathcal{B}^{\alpha\beta} \mathcal{C}_{\dot\alpha} \Psi_\beta + f_2 e^\alpha{}_{\dot\alpha} \mathcal{B}^{\dot\alpha\dot\beta} \mathcal{C}_{\dot\beta} \Psi_\alpha + f_3 e^\alpha{}_{\dot\alpha} \Pi^{\beta\dot\alpha} \mathcal{C}_\alpha \Psi_\beta + h.c. \end{aligned}$$

- Invariance under the local supertransformations gives
  - Two solutions which exist for arbitrary masses M, $\tilde{M}$ and are equivalent to trivially gauge invariant ones
  - One additional solution for $M^2 = \tilde{M}^2$ only
- Some combination of these vertices reproduces minimal (with no more than one derivative) vertex
- Similarly
  - Superblock $(5/2, 2)$: $3 + 1$
  - Superblock $(3, 5/2)$: $4 + 1$

Massless spin 3/2

- Massive superblock $(2, 3/2)$ Ansatz for abelian vertices

$$\mathcal{L}_a = g_1 \mathcal{R}^{\alpha\beta} \mathcal{C}_\alpha \Psi_\beta + g_2 \mathcal{B}^{\alpha\beta} \mathcal{F}_\alpha \Psi_\beta + g_3 \Pi^{\alpha\dot\alpha} \mathcal{F}_{\dot\alpha} \Psi_\alpha + f_1 e_\alpha{}^{\dot\alpha} \mathcal{B}^{\alpha\beta} \mathcal{C}_{\dot\alpha} \Psi_\beta + f_2 e^\alpha{}_{\dot\alpha} \mathcal{B}^{\dot\alpha\dot\beta} \mathcal{C}_{\dot\beta} \Psi_\alpha + f_3 e^\alpha{}_{\dot\alpha} \Pi^{\beta\dot\alpha} \mathcal{C}_\alpha \Psi_\beta + h.c.$$

- Invariance under the local supertransformations gives
  - Two solutions which exist for arbitrary masses M, $\tilde{M}$ and are equivalent to trivially gauge invariant ones
  - One additional solution for $M^2 = \tilde{M}^2$ only
- Some combination of these vertices reproduces minimal (with no more than one derivative) vertex
- Similarly
  - Superblock $(5/2, 2)$: $3 + 1$
  - Superblock $(3, 5/2)$: $4 + 1$

Massless spin 2

- Massive spin 3/2
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There are three linearly independent abelian vertices and only one is equivalent to the trivially gauge invariant vertex
  - Some combination of these vertices reproduces minimal gravitational interaction which corresponds to the spontaneously broken $N = 1$ supergravity
- Massive spin 2
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There exist three independent trivially gauge invariant vertices and two abelian vertices which can not be reduced to the trivially gauge invariant ones.
  - Some particular combination of these vertices reproduces minimal (with no more than two derivatives) gravitational interaction which corresponds to the (linearized) bigravity

Massless spin 2

- Massive spin 3/2
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There are three linearly independent abelian vertices and only one is equivalent to the trivially gauge invariant vertex
  - Some combination of these vertices reproduces minimal gravitational interaction which corresponds to the spontaneously broken $N = 1$ supergravity
- Massive spin 2
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There exist three independent trivially gauge invariant vertices and two abelian vertices which can not be reduced to the trivially gauge invariant ones.
  - Some particular combination of these vertices reproduces minimal (with no more than two derivatives) gravitational interaction which corresponds to the (linearized) bigravity

General analysis

- Classification in $d = 4$ contains two different cases
  - critical: $M_1 = M_2 + M_3$
  - non-critical: $M_1 \neq M_2 + M_3$
- Boulanger e.a. 2018: we always have enough field redefinitions to bring any such vertex into trivially gauge invariant form
- But in this case the general structure for such vertices

$$\mathcal{L}_1 \sim \mathcal{RBB} + \mathcal{BBB}$$

does not depend on masses?

Examples

- Massive spin 2 and two massive spin 3/2
  - There exist six trivially gauge invariant vertices
  - Minimal vertex exists for arbitrary masses M_2, M, $\tilde{M}$
  - Limit $M_2 \to 0 \Rightarrow M = \tilde{M}$
  - Limit $\tilde{M} \to 0 \Rightarrow M_2 = M$
- Massive spin 2 (selfinteraction)
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There exist four independent abelian vertices which appear to be equivalent to trivially gauge invariant ones
  - Some particular combination of these vertices reproduces minimal (having no more that two derivatives) one

Examples

- Massive spin 2 and two massive spin 3/2
  - There exist six trivially gauge invariant vertices
  - Minimal vertex exists for arbitrary masses M_2, M, $\tilde{M}$
  - Limit $M_2 \to 0 \Rightarrow M = \tilde{M}$
  - Limit $\tilde{M} \to 0 \Rightarrow M_2 = M$
- Massive spin 2 (selfinteraction)
  - By field redefinitions any such vertex can be reduced to the abelian form
  - There exist four independent abelian vertices which appear to be equivalent to trivially gauge invariant ones
  - Some particular combination of these vertices reproduces minimal (having no more that two derivatives) one

Partially massless fields

Exist for special values of $M^2 \sim \Lambda$

$$0 \longleftrightarrow \cdots \longleftrightarrow k-1 \qquad k \longleftrightarrow \cdots \longleftrightarrow s$$

Figure: Partially massless limit before vs after gauge fixing

Partially massless fields

Exist for special values of $M^2 \sim \Lambda$

$$0 \longleftrightarrow \cdots \longleftrightarrow k-1 \qquad k \longleftrightarrow \cdots \longleftrightarrow s$$

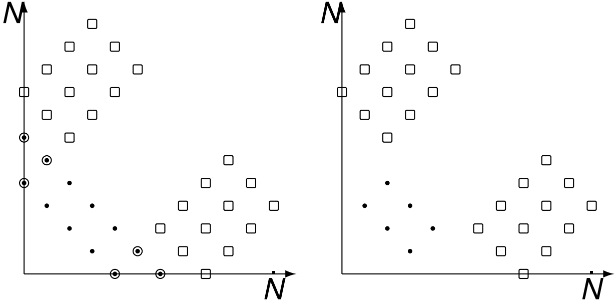

Figure: Partially massless limit before vs after gauge fixing

Cubic vertices

- Till now there exist just a few explicit examples, more will appear soon.
- Field redefinitions
  - Before gauge fixing: in general we do not have enough to bring the vertex into abelian form.
  - After gauge fixing: there are no any ambiguities, formalism works as in the massless case.
- There exist some candidates for the infinite dimensional algebra corresponding to the collections of massless and partially massless fields
- "Triangular inequality" $l = s - k$

$$\begin{aligned} s_1 - l_1 &< s_2 - l_2 + s_3 - l_3 \\ l_1 &\leq l_2 + l_3 \end{aligned}$$

Cubic vertices

- Till now there exist just a few explicit examples, more will appear soon.
- Field redefinitions
  - Before gauge fixing: in general we do not have enough to bring the vertex into abelian form.
  - After gauge fixing: there are no any ambiguities, formalism works as in the massless case.
- There exist some candidates for the infinite dimensional algebra corresponding to the collections of massless and partially massless fields
- "Triangular inequality" $l = s - k$

$$\begin{aligned} s_1 - l_1 &< s_2 - l_2 + s_3 - l_3 \\ l_1 &\leq l_2 + l_3 \end{aligned}$$

Cubic vertices

- Till now there exist just a few explicit examples, more will appear soon.
- Field redefinitions
  - Before gauge fixing: in general we do not have enough to bring the vertex into abelian form.
  - After gauge fixing: there are no any ambiguities, formalism works as in the massless case.
- There exist some candidates for the infinite dimensional algebra corresponding to the collections of massless and partially massless fields
- ”Triangular inequality” $l = s - k$

$$\begin{aligned} s_1 - l_1 &< s_2 - l_2 + s_3 - l_3 \\ l_1 &\leq l_2 + l_3 \end{aligned}$$