Flowing Between String Vacua for the Critical Non-Abelian Vortex with Deformation of $\mathcal{N} = 2$ Liouville theory

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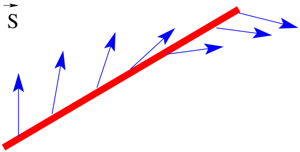
Introduction

Seiberg and Witten 1994 : Confinement in the monopole vacuum of $\mathcal{N} = 2$ supersymmetric QCD
Abelian confinement

In the search for a non-Abelian confinement
Non-Abelian vortex strings
were found in $\mathcal{N} = 2$ U(N) QCD
Hanany, Tong 2003
Auzzi, Bolognesi, Evslin, Konishi, Yung 2003
Shifman Yung 2004
Hanany Tong 2004
Non-Abelian string : Orientational zero modes
Rotation of color flux inside SU(N).
Non-Abelian vortex string is BPS and preserves $\mathcal{N} = (2, 2)$ supersymmetry on its world sheet.



Shifman and Yung, 2015: Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD is a critical superstring

Idea:

Non-Abelian string has more moduli then Abrikosov-Nielsen-Olesen string.

It has translational + orientaional moduli

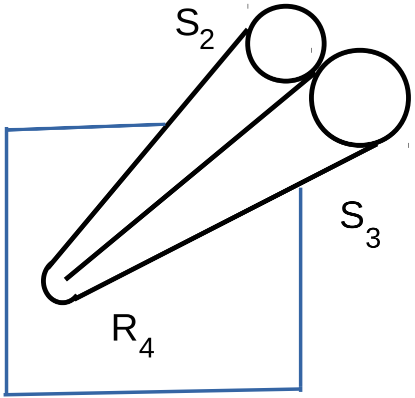
We can fulfill the criticality condition: In $\mathcal{N} = 2$ QCD with $U(N = 2)$ gauge group and $N_f = 4$ quark flavors.

- The solitonic non-Abelian vortex have six orientational and size moduli, which, together with four translational moduli, form a ten-dimensional space.
- For $N_f = 2N$ 2D world sheet theory on the string is conformal.

For $N = 2$ and $N_f = 4$ the target space of the 2D sigma model on the string world sheet is

$$R^4 \times Y_6,$$

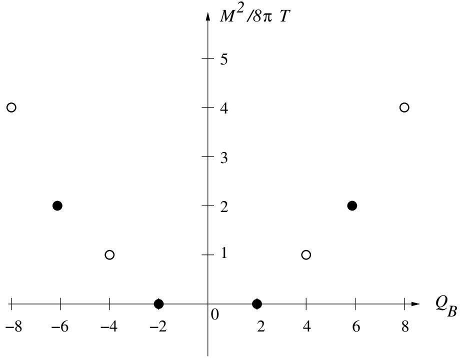
where Y_6 is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely **conifold.**



We studied states of closed type IIA string propagating on $R^4 \times Y_6$ and interpreted them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Shifman and Yung, 2017

spectrum of low lying string states = hadrons of $\mathcal{N} = 2$ QCD



To find the spectrum of string states we used Little String Theory approach

Ghoshal, Vafa, 1995; Giveon Kutasov 1999 proposed that

Critical string on a conifold at strong coupling is equivalent to non-critical string on

$$\mathcal{R}^4 \times \mathcal{R}_\phi \times S^1,$$

with linear in the Liouville field ϕ dilaton

$\mathcal{N} = 2$ supersymmetric Liouville theory

Recently it was proven in a direct way

Gavrilenko, Ievlev, Marshakov, Monastyrskii, Yung 2023

Coulomb branches of world sheet weighted $CP(N-1)$ ($\mathbb{WCP}(N, N)$) models on non-compact CY manifolds are described by $\mathcal{N} = 2$ Liouville theory with background charge depending on N.

Now using the $\mathcal{N}=2$ Liouville theory approach we make a step towards broadening the class of 4D $\mathcal{N}=2$ SQCDs where the solitonic string-gauge duality can be applied.

We introduce quark masses in $\mathcal{N}=2$ SQCD and changing values of mass parameters interpolate between SQCDs with different gauge groups and numbers of quark flavors.

$\mathbb{WCP}(N, N)$ models

World sheet sigma models on non-Abelian strings in $\mathcal{N} = 2$ SQCD with $N_f = 2N$ are $\mathbb{WCP}(N, N)$ models. Can be understood as Higgs branches of U(1) gauge theory, $e_0 \to \infty$ (*Witten, 1993*). Conformal in the massless limit.

$$S = \int d^2x \left\{ \left|\nabla_\alpha n^i\right|^2 + \left|\widetilde{\nabla}_\alpha \rho^j\right|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} \left|\partial_\alpha \sigma\right|^2 \right.$$
$$+ \frac{1}{2e_0^2} D^2 - \left|\sqrt{2}\sigma + m_i\right|^2 \left|n^i\right|^2 + \left|\sqrt{2}\sigma + m_j\right|^2 \left|\rho^j\right|^2$$
$$\left. + D\left(\left|n^i\right|^2 - \left|\rho^j\right|^2 - \mathrm{Re}\,\beta\right) - \frac{\vartheta}{2\pi} F_{01} \right\},$$

where $i = 1, ..., N$, $j = (N+1), ..., 2N$ and the complex scalar fields n^i and ρ^j have charges $\mathtt{Q} = +1$ and $\mathtt{Q} = -1$

$$\nabla_\alpha = \partial_\alpha - iA_\alpha\,, \qquad \widetilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha\,,$$

$$\Sigma = \sigma + \sqrt{2}\theta_R \bar{\lambda}_L - \sqrt{2}\bar{\theta}_L \lambda_R + \sqrt{2}\theta_R \bar{\theta}_L (D - iF_{01})$$

$$-\frac{\beta}{2}\int d^2\tilde{\theta}\sqrt{2}\,\Sigma = -\frac{\beta}{2}\,(D - iF_{01}), \qquad \beta = \mathrm{Re}\,\beta + i\,\frac{\vartheta}{2\pi}$$

Twisted masses m_i and m_j coincide with quark masses of $2N$ flavors in 4D SQCD.

Dimension of the Higgs branch in the $m_i = m_j = 0$ limit

$$\dim_R \mathcal{H} = 4N - 1 - 1 = 2\,(2N - 1)$$

The model is conformal and $\mathcal{N} = (2,2)$ supersymmetric $\Rightarrow$ target space is Ricci-flat and Kähler $\Rightarrow$ Calabi-Yau
Central charge

$$\hat{c}_{CY} \equiv \frac{c_{CY}}{3} = \dim_C \mathcal{H} = 2N - 1$$

For $N = 2$ $\dim_R \mathcal{H} = 6$ – conifold   6+4=10 – critical non-Abelian string

We consider all N, moreover use large N approximation as a first step.

Interpolation procedure

Classical vacuum structure (at $\mathrm{Re}\,\beta > 0$)

$$\sqrt{2}\sigma = -m_{i_0}, \qquad |n^{i_0}|^2 = \mathrm{Re}\,\beta, \qquad i_0 = 1, ..., N.$$

Fields n^i , $i \neq i_0$ and fields ρ^j have masses $|m_i - m_{i_0}|$ and $|m_j - m_{i_0}|$ respectively.
Take $N = 2K$

$$m_i = (0, ..., 0, M, ..., M), \qquad m_i = m_{i+N}$$
$$\leftarrow K \rightarrow \leftarrow K \rightarrow \tag{1}$$

Fields ρ^j has the same masses as fields n^i, $j = i + N$, while half of n^i fields acquire masses M.

Starting point: $M \to \infty$

Half of n and ρ fields decouple. We have $\mathbb{WCP}(K, K)$ model

Final point: $M \to 0$

We have $\mathbb{WCP}(N, N)$ model $N = 2K$.

In 4D SQCD:

Starting point: $M \to \infty$
Two non-interacting copies of $\mathcal{N} = 2$ U(K) SQCD with $N_f = 2K$

Final point: $M \to 0$ $\mathcal{N} = 2$ U(N) SQCD with $N_f = 2N$

We will take the limit $K = 2$ so the initial point is the critical non-Abelian string on the conifold

Coulomb branch

D'Adda, Davis, DiVeccia, Salamonson, 1983; Witten, 1993 ...

Critical points of the exact twisted superpotential for Σ are given by the vacuum equation

$$\prod_{i=1}^{N}\left(\sqrt{2}\,\sigma + m_i\right) = e^{-2\pi\beta} \prod_{j=N+1}^{2N}\left(\sqrt{2}\,\sigma + m_j\right)$$

In the limit $m_i = m_j = 0$

$$\sigma^N = e^{-2\pi\beta}\,\sigma^N, \qquad \sigma = 0 \quad \text{for} \quad \beta \neq 0$$

For $\beta = 0$ σ is arbitrary – Coulomb branch

We will see that Coulomb branch is described by $\mathcal{N} = 2$ Liouville theory

$\mathcal{N}=2$ Liouville theory from $\mathbb{WCP}(N,N)$ model

Massless theory

Take $\mathbb{WCP}(N,N)$ at large N and $\beta = 0$. Fields n and ρ become "massive" at $\sigma \neq 0$ and can be integrated out.
Witten, 1979 for CP$(N-1)$ model.

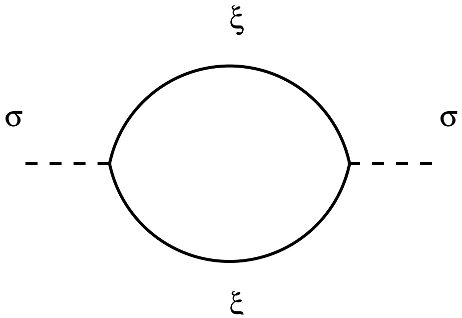
Similar calculation for $\mathbb{WCP}(N,N)$ model.
Gavrilenko, Ievlev, Marshakov, Monastyrskii, Yung 2023

Consider the most important kinetic term for σ

$$S_\sigma^{\rm kin} = \int d^2x \frac{1}{e^2} |\partial_\alpha \sigma|^2$$

where

$$\frac{1}{e^2} = \left(\frac{1}{e_0^2} + \frac{2N}{4\pi}\frac{1}{2|\sigma|^2}\right)\Bigg|_{e_0^2\to\infty} = \frac{2N}{4\pi}\frac{1}{2|\sigma|^2} \qquad \sum_1^{2N} \mathtt{Q}^2 = 2N$$



Thus we get

$$S_{\text{eff}}^{\sigma} = \frac{2N}{4\pi} \int d^2x \, \frac{1}{2} \, \frac{|\partial_\alpha \sigma|^2}{|\sigma|^2}$$

Change of variables

$$\sigma = e^{-\frac{\phi + iY}{Q}}$$

gives

$$S_{\text{eff}}^{\sigma} = \frac{1}{4\pi} \int d^2x \, \left(\frac{1}{2} (\partial_\alpha \phi)^2 + \frac{1}{2} (\partial_\alpha Y)^2 \right)$$

Here Y is a compact variable, $Y + 2\pi Q \sim Y$, where

$$Q|_{N\to\infty} \approx \sqrt{2N}$$

To get background charge we need to introduce world sheet metric $h_{\alpha\beta}$. Similar calculation gives

$$S^{\sigma}_{\rm eff} = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\left(\frac{1}{2}\,h^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) - \frac{Q}{2}\phi\,R^{(2)}\right),$$

where 2D Ricci scalar $R^{(2)} = -\frac{1}{\sqrt{h}}\partial^2_\alpha \log\sqrt{h}$ in the conformal gauge.

This is the bosonic part of the $\mathcal{N} = 2$ Liouville action with linear dilaton

$$\Phi(\phi) = -\frac{Q}{2}\phi$$

The holomorphic stress tensor is

$$T = -\frac{1}{2}\left[(\partial_z\phi)^2 + Q\,\partial_z^2\phi + (\partial_z Y)^2\right], \qquad Y \sim Y + 2\pi Q$$

Central charge

$$c_L = 3 + 3Q^2, \qquad \hat{c}_L \equiv \frac{c_L}{3} = 1 + Q^2.$$

The $\mathcal{N} = 2$ Liouville interaction superpotential comes from the 2D FI term in the $\mathbb{WCP}(N, N)$ model

$$S_{\rm FI} = \mu \int d^2x d^2\tilde{\theta}\, \Sigma + c.c. = \mu \frac{D - iF_{01}}{\sqrt{2}} + c.c.$$

This superpotential is a marginal deformation of $\mathcal{N} = 2$ Liouville theory.
The conformal dimension of σ is

$$\Delta(\sigma = e^{-\frac{\phi + iY}{Q}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Exact equivalence

Relax large N condition.

The form of the action after integrating out fields n and ρ is fixed on dimensional grounds and by supersymmetry. We need only to find $Q(N) \approx \sqrt{2N}$. Require that two central charges should be the same.

$$\hat{c}_{CY} = 2N - 1 = 1 + Q^2 = \hat{c}_L$$

gives

$$Q = \sqrt{2(N-1)}$$

Mass deformation

Now consider $\mathbb{WCP}(N,N)$ model with nonzero twisted masses starting with the large N approximation.

$$S^{\sigma}_{\text{eff}} = \frac{1}{4\pi}\int d^2x \sum_{A=1}^{2N} \frac{|\partial_\alpha \sigma|^2}{|\sqrt{2}\sigma + m_A|^2}$$

$$= \frac{1}{4\pi}\int d^2x \, \frac{1}{2}\, \frac{|\partial_\alpha \sigma|^2}{|\sigma|^2} \sum_{A=1}^{2N} \frac{1}{\left|1 + \frac{m_A}{\sqrt{2}\sigma}\right|^2} \qquad (2)$$

Take $\sigma = e^{-\frac{\phi + iY}{Q}}$, $N = 2K$ and $m_i = (0, \ldots, 0, M, \ldots, M)$. We get

$$S_{\text{eff}} = \frac{1}{4\pi}\int d^2x \, g_{cl}(\phi, Y) \left(\frac{1}{2}(\partial_\alpha \phi)^2 + \frac{1}{2}(\partial_\alpha Y)^2\right)$$

$$g_{cl}(\phi, Y) = 1 + \frac{1}{\left|1 + \frac{M}{\sqrt{2}}\, e^{\frac{\phi + iY}{Q}}\right|^2}, \qquad Q^2 \approx 2K$$

We use this just as initial conditions, namely

$$g_{cl}(\phi, Y) \approx 1 + \frac{2}{|M|^2}\, e^{-\frac{2\phi}{Q}}$$

for the metric warp factor and

$$\Phi \approx -\frac{Q}{2}\,\phi$$

for the dilaton.

True metric and dilaton will be found by solving the gravity equations of motion

Relax large K approximation

$$Q = \sqrt{2(K-1)}, \qquad \text{for} \;\; K = 2 \qquad Q = \sqrt{2}$$

Gravity equations

The bosonic part of the action of the type-II supergravity in the string frame is given by

$$S = \frac{1}{2\kappa^2}\int d^D x \sqrt{-G}\, e^{-2\Phi}\, \left\{R + 4G^{MN}\partial_M\Phi\partial_N\Phi_N + \cdots\right\}$$

Einstein's equations:

$$R_{MN} + 2D_M D_N \Phi = 0$$

Dilaton equation:

$$R = 4G^{MN}\partial_M\Phi\partial_N\Phi - 4G^{MN}D_M D_N\Phi + p,$$

where $p = \frac{D-10}{2}$.
Minkowski 4D $\times$ deformed Liouville theory. $D = 6,\ p = -2$
Ansatz for the internal metric:

$$ds^2_{\mathrm{int}} = g(\phi, Y)\left\{d^2\phi + d^2 Y\right\}$$

Solutions to gravity equations

Solution for for the dilaton:

$$\Phi(\phi) = -\frac{Q}{2}\,\phi + \frac{1}{2}\,\ln g$$

and for the metric warp factor:

$$g(\phi) = \frac{1}{1 - \frac{1}{A}\,e^{-Q\phi}} = \frac{1}{1 - e^{-Q(\phi-\phi_0)}},$$

where A is a constant and $\phi_0 = -\frac{1}{Q}\ln A$.
We see that these solutions satisfy initial conditions for the mass-deformed metric and dilaton with

$$A = \frac{M^2}{2}, \qquad \phi_0 = -\frac{1}{Q}\ln\left(\frac{M^2}{2}\right)$$

only if $Q = \sqrt{2}$

The metric warp factor develop a naked singularity at $\phi = \phi_0$

$$g|_{\phi\to\phi_0} \approx \frac{1}{Q(\phi - \phi_0)}$$

where the curvature is singular. Thus, the geometry is defined only at $\phi > \phi_0$.

Turns out that the Liouville superpotential (Liouville wall) is not modified and is still a marginal deformation of the theory.

Liouville wall prevents field ϕ from penetrating to the region of large negative values.

$$\phi_{\rm wall} \sim -\, Q \ln \frac{1}{|\mu|}$$

- At $\phi_0 \ll \phi_{\rm wall}$ string theory describe hadrons of slightly deformed $\mathcal{N} = 2$ SQCD with U(2) gauge group and $N_f = 4$ quark flavors
- At $\phi_0 \gg \phi_{\rm wall}$ string theory describe hadrons of $\mathcal{N} = 2$ SQCD with U(4) gauge group and $N_f = 8$ quark flavors

Conclusions

- We show that non-Abelian critical string supported in mass-deformed $\mathcal{N} = 2$ SQCD interpolating between theory with U(2) gauge group and $N_f = 4$ quarks and theory with U(4) gauge group and $N_f = 8$ quarks is associated with mass deformation of $\mathcal{N} = 2$ Liouville world sheet theory.
- To find the true string vacuum we solve the effective gravity equation of motion.
- The solution shows the presence of a naked singularity of the metric.