Linearised Analysis of the Coxeter Higher Spin Theory I

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Table of content

Origins

- Relationship between HS and Strings
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  - M.A. Vasiliev, Class.Quant.Grav. 30 (2013), 104006; M.A. Vasiliev, JHEP 08 (2018), 051
- Tensor $O(N)$ models and its HS dual
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Coxeter groups

Rank-p Coxeter group $\mathcal{C}$ is generated by reflections with respect to a system of root vectors $\{v_a\}$ in a p-dimensional Euclidean vector space V. An elementary reflection associated with the root vector v_a:

$$R_{v_a} x^i = x^i - 2\frac{(v_a, x)}{(v_a, v_a)} v_a^i\,, \quad R_{v_a}^2 = Id\,.$$

Coxeter groups: examples

- The root system of $A_p \simeq S_{p+1}$: $v^{ij} = e^i - e^j$, with orthonormal frame e^i in $\mathbb{R}^{p+1}$. V is the p-dimensional subspace in $\mathbb{R}^{p+1}$ spanned by v^{ij}.
- The root system of $B_p \simeq S_p \ltimes (\mathbb{Z}_2)^p$ consists of two conjugacy classes

$$\mathcal{R}_1 = \{\pm e^i, 1 \leq i \leq p\}\,, \quad \mathcal{R}_2 = \{\pm e^i \pm e^j, 1 \leq i < j \leq p\}\,.$$

Framed Cherednik algebra

Generators I_n, q^n_α and $\hat{K}_v$ for each root vector v (here $\alpha \in \{1,2\}, n \in \{1,...,p\}$):

$$I_n I_m = I_m I_n\,, \quad I_n I_n = I_n\,, \quad I_n q^n_\alpha = q^n_\alpha I_n = q^n_\alpha\,, \quad I_m q^n_\alpha = q^n_\alpha I_m\,,$$

$$\hat{K}_v q^n_\alpha = R_v{}^n{}_m q^m_\alpha \hat{K}_v\,, \quad \hat{K}_v \hat{K}_u = \hat{K}_u \hat{K}_{R_u(v)} = \hat{K}_{R_v(u)} \hat{K}_v\,,$$

$$\hat{K}_v \hat{K}_v = \prod I_{i_1(v)} ... I_{i_k(v)}\,, \quad \hat{K}_v = \hat{K}_{-v}\,,$$

Labels $i_1(v), ..., i_k(v)$ enumerate idempotents I_n affected by the reflection R_v.

Framed Cherednik algebra

$$[q^n_\alpha, q^m_\beta] = -i\varepsilon_{\alpha\beta}\left(2\delta^{nm} I_n + \sum_{v\in\mathcal{R}} \nu(v)\frac{v^n v^m}{(v,v)}\hat{K}_v\right),$$

where $\mathcal{R}$ is a set of conjugacy classes of root vectors, $\nu(v)$ is a function of conjugacy classes.
Klein operators $\hat{K}_v$ obey

$$I_n\hat{K}_v = \hat{K}_v I_n\,, \forall n \in \{1, ..., p\}\,,$$

$$I_n\hat{K}_v = \hat{K}_v I_n = \hat{K}_v\,, \forall n \in \{i_1(v), ..., i_k(v)\}\,.$$

Following M.A. Vasiliev, JHEP 08 (2018), 051

CHS model setup

Fields $W(Y,Z;\hat{K}|x)$, $S(Y,Z;\hat{K}|x)$ and $B(Y,Z;\hat{K}|x)$ which depend on Y^n_A, Z^n_A $(A \in \{1,...,4\}, n \in \{1,...,p\})$, idempotents I_n, differentials dZ^A_n and operators $\hat{K}_v$ associated with $\mathcal{C}$.
The star product

$$(f * g)(Y,Z,I) = \frac{1}{(2\pi)^{4p}} \int d^{4p}S d^{4p}T exp\Big(iS^A_n T^B_m C_{AB} \delta^{nm} \Big) \\ f(Y_i + I_i S_i, Z_i + I_i S_i, I) g(Y+T, Z-T, I)\,,$$

where

$$C_{AB} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

CHS algebra

It is demanded that

$$Y^m_A * I_n = I_n * Y^m_A\,, \quad Y^n_A * I_n = I_n * Y^n_A = Y^n_A\,,$$
$$I_n * I_n = I_n\,, \quad I_n * I_m = I_m * I_n\,.$$

Z^n_A obeys analogous conditions.
From the star product and properties of I_n it follows

$$[Y^n_A, Y^m_B]_* = -[Z^n_A, Z^m_B]_* = 2iC_{AB}\delta^{nm}I_n\,, \quad [Y^n_A, Z^m_B]_* = 0\,.$$

The star product admits inner Klein operators $\varkappa_v$, $\bar{\varkappa}_v$ associated with the root vectors v

$$\varkappa_v = exp\left(i\frac{v^n v^m}{(v,v)}z_{\alpha n}y^\alpha_m\right), \quad \bar{\varkappa}_v = exp\left(i\frac{v^n v^m}{(v,v)}\bar{z}_{\dot\alpha n}\bar{y}^{\dot\alpha}_m\right),$$

$$\varkappa_v * q^n_\alpha = R_v{}^n{}_m q^m_\alpha * \varkappa_v\,, \quad q^n_\alpha = y^n_\alpha, z^n_\alpha\,.$$

CHS equations

Nonlinear equations for the generalized CHS theory are

$$\mathrm{d}_x W + W * W = 0\,, \tag{1}$$

$$\mathrm{d}_x B + W * B - B * W = 0\,, \tag{2}$$

$$\mathrm{d}_x S + W * S + W * S = 0\,, \tag{3}$$

$$S * B = B * S\,, \tag{4}$$

$$S * S = i\Big(dZ^{An} dZ_{An} + \sum_i \sum_{v \in \mathcal{R}_i} \Big[F_{i*}(B) \frac{v^n v^m}{(v,v)} dz_n^{\alpha} dz_{\alpha m} * \varkappa_v \hat{k}_v + \\ + \bar{F}_{i*}(B) \frac{v^n v^m}{(v,v)} d\bar{z}_n^{\dot\alpha} d\bar{z}_{\dot\alpha m} * \bar{\varkappa}_v \hat{\bar{k}}_v \Big]\Big)\,, \tag{5}$$

where $\varkappa_v \hat{k}_v$ acts on dz_n^{α} as

$$\varkappa_v \hat{k}_v * dz_n^{\alpha} = R_{v\,n}{}^{m} dz_m^{\alpha} * \varkappa_v \hat{k}_v\,,$$

$F_{i*}(B)$ is any star-product function of B on the conjugacy classes $\mathcal{R}_i$ of $\mathcal{C}$. We set $F_{i*}(B) = \eta_i B$.

Following M.A. Vasiliev, JHEP 08 (2018), 051

AdS_4 solution

The vacuum solution

$$B_0 = 0\,, \quad S_0 = dZ^{An} Z_{An}\,, \quad W = W_0(Y, I|x)$$

with flat $W_0(Y, I|x)$

$$\mathrm{d}_x W_0(Y, I|x) + W_0(Y, I|x) * W_0(Y, I|x) = 0\,.$$

In a general CHS theory, AdS_4 is represented by a dx one-form

$$\Omega_{AdS}(Y|x) = -\frac{i}{4}\delta^{nm}\left(\omega_{\alpha\beta}(x) y_n^{\alpha} y_m^{\beta} + \bar{\omega}_{\dot{\alpha}\dot{\beta}}(x) \bar{y}_n^{\dot{\alpha}} \bar{y}_m^{\dot{\beta}} + 2h_{\alpha\dot{\alpha}}(x) y_n^{\alpha} \bar{y}_m^{\dot{\alpha}}\right).$$

The AdS_4 connection has no explicit dependence on idempotents I_n.

Covariant derivatives

The covariant derivative is

$$
\begin{aligned}
&D_\Omega f(Y;\hat{k},\hat{\bar{k}}|x) = D_L f(Y;\hat{k},\hat{\bar{k}}|x)+\\
&+\frac{1}{2}\delta^{nm}h^{\alpha\dot{\alpha}}\left(\mathbb{1}_n^k\bar{\mathbb{1}}_m^l + R(k)_n^k\bar{R}(\bar{k})_m^l\right)(y_{\alpha k}l_l\bar{\partial}_{\dot{\alpha}l} + \bar{y}_{\dot{\alpha}l}l_k\partial_{\alpha k})f(Y;\hat{k},\hat{\bar{k}}|x)-\\
&-\frac{i}{2}\delta^{nm}h^{\alpha\dot{\alpha}}\left(\mathbb{1}_n^k\bar{\mathbb{1}}_m^l - R(k)_n^k\bar{R}(\bar{k})_m^l\right)(y_{\alpha k}\bar{y}_{\dot{\alpha}l} - l_k l_l\partial_{\alpha k}\bar{\partial}_{\dot{\alpha}l})f(Y;\hat{k},\hat{\bar{k}}|x)\,,
\end{aligned}
$$

where $\mathbb{1}_n^k$ and $\bar{\mathbb{1}}_m^l$ are identity matrices, $\hat{k}$ and $\hat{\bar{k}}$ are products of $\hat{k}_v$ and $\hat{\bar{k}}_v$, matrices $R(k)_n^k$ and $\bar{R}(\bar{k})_m^l$ are corresponding reflections in the root space.

General observation about modules

- If $P^{kl}_{\pm} = \frac{1}{2}\delta^{nm}\left(\mathbb{1}^k_n\bar{\mathbb{1}}^l_m \pm R(k)^k_n\bar{R}(\bar{k})^l_m\right)$ is a set of orthogonal projectors, i.e., $(R\bar{R})^2 = \mathbb{1}$, then the CHS module is a product of adjoint and twisted-adjoint modules of standard HS.
- If $P^{kl}_{\pm}$ are not projectors, i.e., $(R\bar{R})^2 \neq \mathbb{1}$, then the CHS module is a product of standard HS modules and infinite-dimensional entangled modules of the new type.
- Both cases appear for any nontrivial (not $\mathbb{Z}_2$) group $\mathcal{C}$.

Group B_2

The root system of B_2 has two conjugacy classes

$$\mathcal{R}_1 = \{\pm e^1, \pm e^2\}, \quad \mathcal{R}_2 = \{\pm e^1 \pm e^2\}.$$

The group B_2 is generated by

$$\left\{ k_i, k_{12} | k_i^2 = 1, k_{12}^2 = 1, k_1 k_{12} = k_{12} k_2, k_2 k_{12} = k_{12} k_1, i \in \{1, 2\} \right\}.$$

Group B_2

For reflection matrices $R(k)$ we have

$$R(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R(k_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, R(k_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$R(k_{12}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R(k_1 k_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, R(k_1 k_{12}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$R(k_2 k_{12}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R(k_1 k_2 k_{12}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Analogous matching occurs for $\bar{R}(\bar{k})$.

B_2 equations

All possible matrix products $R(k)\bar{R}(\bar{k})$ group into the 8 categories

$$R(k)\bar{R}(\bar{k}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

inducing 8 types of covariant constancy equations on the component fields

$$C(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x) = \sum_{a,b,c,\bar{a},\bar{b},\bar{c}=0}^{1} C_{abc\bar{a}\bar{b}\bar{c}}(Y_1, Y_2|x)\hat{k}_1^a \hat{k}_2^b \hat{k}_{12}^c \hat{\bar{k}}_1^{\bar{a}} \hat{\bar{k}}_2^{\bar{b}} \hat{\bar{k}}_{12}^{\bar{c}}\,.$$

B_2 equations

For instance,

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\left(D_L + h^{\alpha\dot{\alpha}} \sum_{i=1}^{2} (y_{\alpha i}\bar{\partial}_{\dot{\alpha}i} + \bar{y}_{\dot{\alpha}i}\partial_{\alpha i})\right) C(Y;\hat{K}|x) = 0\,,$$

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Updownarrow$$

$$\left(D_L - ih^{\alpha\dot{\alpha}} \sum_{i=1}^{2} (y_{\alpha i}\bar{y}_{\dot{\alpha}i} - \partial_{\alpha i}\bar{\partial}_{\dot{\alpha}i})\right) C(Y;\hat{K}|x) = 0\,,$$

B_2 equations

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\left(D_L - ih^{\alpha\dot{\alpha}}(y_{\alpha 1}\bar{y}_{\dot{\alpha}1} - \partial_{\alpha 1}\bar{\partial}_{\dot{\alpha}1}) + h^{\alpha\dot{\alpha}}(y_{\alpha 2}\bar{\partial}_{\dot{\alpha}2} + \bar{y}_{\dot{\alpha}2}\partial_{\alpha 2})\right)C(Y;\hat{K}|x) = 0\,,$$

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\Big(D_L + \frac{1}{2}h^{\alpha\dot{\alpha}}\Big[(y_{\alpha 1} + y_{\alpha 2})(\bar{\partial}_{\dot{\alpha}1} + \bar{\partial}_{\dot{\alpha}2}) + (\bar{y}_{\dot{\alpha}1} + \bar{y}_{\dot{\alpha}2})(\partial_{\alpha 1} + \partial_{\alpha 2})\Big] -$$
$$-\frac{i}{2}h^{\alpha\dot{\alpha}}\Big[(y_{\alpha 1} - y_{\alpha 2})(\bar{y}_{\dot{\alpha}1} - \bar{y}_{\dot{\alpha}2}) - (\partial_{\alpha 1} - \partial_{\alpha 2})(\bar{\partial}_{\dot{\alpha}1} - \bar{\partial}_{\dot{\alpha}2})\Big]\Big)C(Y;\hat{K}|x) = 0\,,$$

B_2 equations

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\Big(D_L + \frac{1}{2}h^{\alpha\dot{\alpha}}\Big[y_{\alpha 1}(\bar{\partial}_{\dot{\alpha}1} - \bar{\partial}_{\dot{\alpha}2}) + y_{\alpha 2}(\bar{\partial}_{\dot{\alpha}1} + \bar{\partial}_{\dot{\alpha}2}) + \bar{y}_{\dot{\alpha}1}(\partial_{\alpha 1} + \partial_{\alpha 2} - $$
$$-\bar{y}_{\dot{\alpha}2}(\partial_{\alpha 1} - \partial_{\alpha 2})\Big] - \frac{i}{2}h^{\alpha\dot{\alpha}}\Big[y_{\alpha 1}(\bar{y}_{\dot{\alpha}1} + \bar{y}_{\dot{\alpha}2}) - y_{\alpha 2}(\bar{y}_{\dot{\alpha}1} - \bar{y}_{\dot{\alpha}2}) -$$
$$-\partial_{\alpha 1}(\bar{\partial}_{\dot{\alpha}1} + \bar{\partial}_{\dot{\alpha}2}) + \partial_{\alpha 2}(\bar{\partial}_{\dot{\alpha}1} - \bar{\partial}_{\dot{\alpha}2})\Big]\Big) C(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x) = 0\,.$$

Entangled module

$$R(k)\bar{R}(\bar{k}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Updownarrow$$

Ansatz $C(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x) = e^{-iy_{1\alpha}y_2{}^{\alpha} + i\bar{y}_{1\dot{\alpha}}\bar{y}_2{}^{\dot{\alpha}}}\tilde{C}(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x)$

$$\Big[D_L - ih^{\alpha\dot{\alpha}}\Big(y_{1\alpha}\bar{y}_{1\dot{\alpha}} + y_{2\alpha}\bar{y}_{2\dot{\alpha}} + y_{1\alpha}\bar{y}_{2\dot{\alpha}} - y_{2\alpha}\bar{y}_{1\dot{\alpha}}\Big) +$$
$$+\frac{i}{2}h^{\alpha\dot{\alpha}}\Big(\partial_{1\alpha}\bar{\partial}_{1\dot{\alpha}} + \partial_{2\alpha}\bar{\partial}_{2\dot{\alpha}} + \partial_{1\alpha}\bar{\partial}_{2\dot{\alpha}} - \partial_{2\alpha}\bar{\partial}_{1\dot{\alpha}}\Big)\Big]\tilde{C}(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x) = 0\,.$$

Unitarity and boundary conditions

- CHS modules with adjoint part can be restricted to the unitary submodules.
- Restriction in terms of the boundary condition in the stereographic coordinates of AdS_4

$$\lim_{z\to 0}\frac{1}{\sqrt{z}}C(Y_1, Y_2; \hat{K}|x) = 0\,,$$

where

$$x_{\alpha\dot{\beta}} = \sigma^a_{\alpha\dot{\beta}}x_a, \quad x^2 = x_a x^a = \frac{1}{2}x_{\alpha\dot{\beta}}x^{\alpha\dot{\beta}}, \quad z = 1 + x^2\,.$$