Quantum membranes and AdS/CFT duality

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work with: S. Giombi and M. Beccaria
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S. Giombi and S. Kurlyand, arXiv:2408.10070

"Applied" string theory

class of string theories dual to gauge theories

they "exist" and relevant without any doubt

- remarkable progress in last 27 years: gauge-string duality

AdS/CFT: $\mathcal{N} = 4$ SYM $\leftrightarrow$ superstring in AdS$_5 \times S^5$

example of quantum-consistent string theory in curved space

- all-loop finiteness of superstring theory consistent

with $1/N$ expansion of $\mathcal{N} = 4$ SYM theory being well defined

- non-perturbative defn of string theory in terms of gauge theory

describes "quantum gravity" phenomena in certain regime

- string theory is "hidden" in gauge theory

gauge theory at large N and large coupling $\lambda = g^2 N$:

$\rightarrow$ classical string theory and gravity "emerges"

Aims:

- understand quantum gauge theories at any coupling
- understand string theories in non-trivial curved backgrounds

gauge-string duality: uncovers hidden symmetries on both sides

- integrability: at leading order in large N and

any value of string tension $\frac{\sqrt{\lambda}}{2\pi}$ and any $\lambda = g^2_{\rm YM} N$
in string action and also in perturbative SYM theory

- determines spectrum of dimensions of primary operators

= spectrum of string energies in curved AdS background

- beyond large N ?

QFT: major problem – beyond perturbation theory

$SU(N)$ SYM : $\lambda = g_{\rm YM}^2 N$, large N

$$F(\lambda, N) = N^2 F_0(\lambda) + F_1(\lambda) + \frac{1}{N^2} F_2(\lambda) + ..., \qquad F_n(\lambda) = ?$$

remarkable progress in superconformal theories
using combination of different methods

1. Integrability: anomalous dims in 4d and 3d conformal theories as exact functions of $\lambda = g_{\rm YM}^2 N$ at leading order in large N
2. Localization: special supersymmetric observables (free energy on S^d, Wilson loop, some BPS correlators) computed exactly in $g_{\rm YM}$ and N
3. Bootstrap: constraints from symmetries and general principles

Anomalous dimensions in $1/N$ expansion:

$$\Delta(\lambda, N) = \Delta_0(\lambda) + \frac{\Delta_1(\lambda)}{N^2} + \frac{\Delta_2(\lambda)}{N^4} + ...$$

• Δ_0 controlled by integrability

at weak coupling $\Delta_0 = c_1\lambda + c_2\lambda^2 + ...$ while at strong coupling

$\Delta_0 = \sqrt{\lambda}[a_0 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + ...]$ or $\Delta_0 = \sqrt[4]{\lambda}[b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + ...]$

• how to compute non-planar correction $\Delta_1(\lambda)$?

little known even at weak coupling: in $\mathcal{N} = 4$ SYM

non-planar correction to cusp anomalous dimension $f(\lambda, N)$

$\Delta = S + f(\lambda, N) \log S + ..., \quad \mathcal{O} = \text{Tr}\,(\Phi D^S \Phi)$

first appears at 4-loop order [Henn, Korchemsky 2019]

$f = \bar{\lambda} - c_2\bar{\lambda}^2 + c_3\bar{\lambda}^3 - \left(c_{4,0} + \frac{1}{N^2}c_{4,1}\right)\bar{\lambda}^4 + \mathcal{O}(\bar{\lambda}^5), \qquad \bar{\lambda} \equiv \frac{1}{(2\pi)^2}\lambda$

$c_2 = \frac{1}{12}\pi^2,\ c_3 = \frac{11}{720}\pi^4,\ c_{4,0} = \frac{73}{20160}\pi^6 + \frac{1}{8}\zeta^2(3),\ c_{4,1} = \frac{31}{5040}\pi^6 + \frac{9}{4}\zeta^2(3)$

Konishi operator: non-planar correction also appears at 4-loop order

$\Delta_1 \sim \zeta(5)\lambda^4 + ...$ [Velizhanin 2009]

Thus at weak coupling leading non-planar correction

$$\Delta_1\Big|_{\lambda\ll 1} = d_4\lambda^4 + \ldots$$

• what to expect at strong coupling: $\Delta_1\Big|_{\lambda\gg 1} \sim \lambda^p$?

string side: requires 1-loop (torus) computation in $\mathrm{AdS}_5 \times S^5$ appears to be challenging problem

• remarkably, strong-coupling limit of non-planar corrections can be computed in 3d ABJM gauge theory: semiclassical M2 brane quantization in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ captures leading order $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms at each order in $g_s^2 \sim \frac{1}{N^2}$

Review: 11d supergravity and M2 brane

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R - \tfrac{1}{2\cdot 4!} F_{MNKL} F^{MNKL} + \cdots \right), \qquad F_4 = dC_3$$

- M2 brane solution [Duff, Stelle 90]

$$ds^2 = H^{-2/3}(y) dx^m dx_m + H^{1/3}(y) dy^r dy_r, \qquad C_{mnk} = H^{-1} \epsilon_{mnk}$$

$$H = 1 + \frac{Q}{y^6}, \qquad Q \sim N$$

"near-horizon" limit is $\mathrm{AdS}_4 \times S^7$: [Freund, Rubin 80]

$$ds_{11}^2 = \mathrm{L}^2 \left(\tfrac{1}{4} ds^2_{\mathrm{AdS}_4} + ds^2_{S^7} \right), \quad F_4 \sim N, \quad \left(\frac{\mathrm{L}}{\ell_P}\right)^6 = 32\pi^2 N$$

- collective coordinates → M2 action: [Bergshoeff, Sezgin,Townsend 87]

$$S_{\rm M2} = T_2 \int d^3\sigma \Big[\sqrt{-\det g_{ab}} + \hat{C}_3 \Big]$$

$$g_{ab} = G_{MN}(x)\, \Pi_a^M \Pi_b^N + \dots\,, \qquad \hat{C}_3 = \tfrac{1}{6}\epsilon^{abc}\, C_{MNK}(x)\, \Pi_a^M \Pi_b^N \Pi_c^K$$

$$\Pi_a^M = \partial_a x^M - i\bar\theta \Gamma^M \partial_a \theta\,, \qquad x^M = x^M(\sigma)$$

- parameters

$$2\kappa_{11}^2 = (2\pi)^8\, \ell_P^9\,, \qquad T_2 = \Big(\frac{2\pi^2}{\kappa_{11}^2}\Big)^{1/3} = \frac{1}{(2\pi)^2 \ell_P^3}$$

- relation to 10d string: $S = \frac{1}{2\kappa_{10}^2} \int d^{10}x\, \sqrt{G}\, e^{-2\phi}(R + \dots)$

$$ds_{11}^2 = e^{-\frac{2}{3}\phi}\, ds_{10}^2 + e^{\frac{4}{3}\phi}(dx_{11} + e^{-\phi} A)^2, \qquad x_{11} \sim x_{11} + 2\pi R_{11}$$

$$g_s = e^{\phi}; \qquad 2\kappa_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4$$

• "double dimensional reduction": [Duff, Howe, Inami, Stelle 87]
M2 action in 11d backgr. $\rightarrow$
superstring action in IIA 10d backgr.

perturbative string theory – theory of quantum strings
perturbative M-theory – theory of quantum M2 branes?

• Bosonic membrane action [Dirac 1962]

$$S = -T_2 \int d^3\sigma \sqrt{-\det g}\,, \qquad g_{ab} = \eta_{MN}\partial_a X^M \partial_b X^N$$

can gauge-fix only 3 out of 6 components of 3d metric
non-linear action in any gauge (cf. string); non-renormalizable
no mass gap in spectrum of “small" membranes, etc.

• Supermembrane: well defined at quantum level?
1-loop finite as 3d theory; what about 2-loop finiteness?
indications that may be true in flat space and in $AdS_4 \times S^7$

Semiclassical expansion of M2 brane path integral

- M2 action near solution with non-degenerate induced 3d metric: straightforward to quantize in static gauge
- 1-loop partition function is well defined (no log UV ∞)

$$S_B = S_{\rm V} + S_{\rm WZ}, \qquad S_{\rm V} = T_2 \int d^3\xi \, \sqrt{g}, \qquad T_2 = \frac{1}{(2\pi)^2 \ell_P^3}$$

$$S_{\rm WZ} = -i\, T_2 \int d^3\xi \, \tfrac{1}{3!} \epsilon^{abc} C_{MNK}(X)\, \partial_a X^M \partial_b X^N \partial_c X^K,$$

$$S_F = i T_2 \int d^3\xi \Big[\sqrt{g}\, g^{ab} \partial_a X^M \, \bar\theta\, \Gamma_M \hat D_b \theta - \tfrac{1}{2} \epsilon^{abc} \partial_a X^M \partial_b X^N \, \bar\theta\, \Gamma_{MN} \hat D_c \theta + ... \Big]$$

$$g_{ab} = \partial_a X^M \partial_b X^N G_{MN}(X), \qquad G_{MN} = E^A_M E^A_N, \qquad \Gamma_M = E^A_M(X) \Gamma_A$$

$$\hat D_a = \partial_a X^M \hat D_M, \qquad \hat D_M = {\rm D}_M - \tfrac{1}{288} (\Gamma^{PNKL}{}_M - 8 \Gamma^{PNK} \delta^L_M) F_{PNKL}$$

$$Z_{\rm M2} = \int [dX\, d\theta]\, e^{-S[X,\theta]} = \mathcal{Z}_1\, e^{-{\rm T}_2 \bar S_{\rm cl}} \left[1 + \mathcal{O}({\rm T}_2^{-1}) \right], \qquad {\rm T}_2 = L^3 T_2$$

$$\mathcal{Z}_1 = e^{-\Gamma_1}, \qquad \Gamma_1 = \tfrac{1}{2} \sum_k \nu_k \log\det \Delta_k$$

Some general points:

• do not sum over M2 topologies: only over semiclassical saddles
string loop expansion is already encoded in contribution
of expansion near M2 semiclassical saddle:
g_s dependence via R_{11} or G_{11} of 11d background

• assume $\Sigma^2 \times S^1$ membrane topology (to have 10d string limit):
3d action = 2d action for string modes
+ KK tower of massive 2d fields

$$m_n^2 = \frac{n^2}{R_{11}^2} = n^2 k^2 = n^2 \frac{8\pi T}{g_s^2}$$

integrating out massive modes $\to$
effective (non-local) theory for string modes depending on $g_{\rm s}$
[cf. 2d theory with insertions of handle operators on 2-sphere]

• can be seen explicitly on examples of M2 in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$:
1-loop M2 correction sums leading $T^{-1} \sim \alpha'$ terms at each order in $g_{\rm s}$

ABJM theory: [Aharony, Bergman, Jafferis, Maldacena 08]

• one M2-brane: 3d scalar $\mathcal{N}=8$ multiplet (x^i, θ^i)

theory on N coincident M2 branes? $\rightarrow$

3d superconf theory dual to M-theory in $AdS_4 \times S^7$ [Maldacena 97]

• to have perturbative description requires extra parameter k

$k=1$ case as limit of more general 3d theory

dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

• N M2 branes on $M^{11} = R^{1,2} \times \mathbb{R}^8/\mathbb{Z}_k$

described by $U_k(N) \times U_{-k}(N)$

3d Chern-Simons-matter $\mathcal{N}=6$ superconformal theory

($\mathcal{N}=8$ susy restored for $k=1$)

• fields: $A_m,\ \tilde{A}_m$; bi-fundamental 4 scalars Φ^A and 4 fermions ψ_A

$$S = k \int d^3x \left[L_{CS}(A) - L_{CS}(\tilde{A}) + |D\Phi|^2 + V(\Phi) + \bar{\psi} D\psi + \bar{\psi}\psi\Phi^\dagger\Phi \right]$$

$$L_{CS} = \epsilon^{mnk}\mathrm{Tr}\left(A_m\partial_n A_k + \tfrac{2}{3}A_m A_n A_k\right), \qquad V = \mathrm{Tr}\left(\Phi\Phi^\dagger\Phi\Phi^\dagger\Phi\Phi^\dagger\right) + \ldots$$

• parameters N and k ($\sim \frac{1}{g^2_{\rm YM}}$ in YM case)

• large N, large k, $\lambda \equiv \frac{N}{k}$ = fixed ("string regime")

dual to IIA superstring on $\mathrm{AdS}_4 \times \mathrm{CP}^3$

• large N, fixed k ("M-theory regime")

dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$: $\phi \equiv \phi + \frac{2\pi}{k}$

$$ds^2_{11} = \mathrm{L}^2\Big(\tfrac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{S^7/\mathbb{Z}_k}\Big), \qquad \mathrm{L} = (2^5\pi^2 Nk)^{1/6}\ell_P$$

$$ds^2_{S^7/\mathbb{Z}_k} = ds^2_{\mathrm{CP}^3} + \frac{1}{k^2}(d\varphi + k\mathrm{A})^2\,, \qquad \varphi = k\,\phi \equiv \varphi + 2\pi$$

$$ds^2_{\mathrm{CP}^3} = h_{rs}(w)dw^s d\bar{w}^s, \quad d\mathrm{A} = ih_{rs}(w)dw^r \wedge d\bar{w}^s, \quad h_{rs} = \frac{\delta_{sr} - \frac{w_s\bar{w}_r}{1+|w|^2}}{1+|w|^2}$$

• string limit $k \gg 1$: $ds^2_{10} = L^2\Big(\tfrac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{\mathrm{CP}^3}\Big), \qquad L = g_{\mathrm{s}}^{1/3}\,\mathrm{L}$

$$g_{\mathrm{s}} = \Big(\frac{\mathrm{L}}{k\,\ell_P}\Big)^{3/2} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}\,, \quad \lambda = \frac{N}{k}\,, \qquad T = \frac{L^2_{\mathrm{ads}}}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{\sqrt{2}}$$

$$\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{g_{\mathrm{s}}^2}{8\pi\,T}$$

non-planar corrections to planar $\Delta_0(\lambda)$ are $\sim \frac{1}{k^2}$

• ABJM operators dual to M2 brane states in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ (dimensions = energies, etc.)

• M-theory expansion: $\frac{\mathrm{L}}{\ell_P} \gg 1$ is large N for fixed $k = 1, 2, ...$ or large effective membrane tension

$$\mathrm{T}_2 \equiv \mathrm{L}^3 T_2 = \frac{1}{\pi}\sqrt{Nk} \quad \gg 1$$

e.g.

$$F = \mathrm{T}_2 F_0(k) + F_1(k) + (\mathrm{T}_2)^{-1} F_2(k) + ...$$

expansion in large k translates into $1/N$, fixed $\lambda = \frac{N}{k}$ expansion

First examples that semiclassical quantization of M2 is consistent: matching defect anomaly, localization predictions for BPS WL and instanton contribution to free energy on S^3 in ABJM theory and superconformal index computation in (2,0) theory:

- N. Drukker, S. Giombi, AT and X. Zhou, arXiv:2004.04562
"Defect CFT in the 6d (2,0) theory from M2 brane dynamics in $AdS_7 \times S^4$"

- S. Giombi, AT, arXiv:2303.15207
"Wilson loops at large N and the quantum M2-brane"

- M. Beccaria, S. Giombi and AT, arXiv:2307.14112
"Instanton contributions to ABJM free energy from quantum M2 brane"

- M. Beccaria, S. Giombi and AT, arXiv:2309.10786
"(2,0) theory on $S^5 \times S^1$ and quantum M2 branes"

One example that M2 semiclassical expansion is consistent and matches gauge theory side – $\frac{1}{2}$ BPS susy Wilson loop W: special observable controlled by localization for any N and k

strategy: W = M2 brane partition function expanded around minimal $\mathrm{AdS}_2 \times S^1$ 3-surface representing M2 probe intersecting AdS_4 boundary (multiple M2's) over circle

- compute M2 partition function for $\mathrm{T}_2 \gg 1$
- compare to localization result for large N, fixed k expansion of WL

$\frac{1}{2}$ BPS circular WL in SYM and ABJM

• $\mathcal{N} = 4$ $SU(N)$ SYM: $\qquad \mathcal{W} = \text{Tr}\, P e^{\int (iA+\Phi)}$

Localization $\to$ Gaussian matrix model: any N, g^2_{YM}

[Erickson, Semenoff, Zarembo 00; Drukker, Gross 01; Pestun 07]

$$\langle \mathcal{W} \rangle = e^{\frac{(N-1)g^2_{\text{YM}}}{8N}} L^1_{N-1}(-\tfrac{1}{4}g^2_{\text{YM}}) \,, \qquad L^1_n(x) \equiv \frac{1}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$$

Large N, fixed $\lambda = N g^2_{\text{YM}}$: $\quad \langle \mathcal{W} \rangle = N\left[\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + ...\right]$

$$\lambda \gg 1: \qquad \langle \mathcal{W} \rangle = \frac{N}{\lambda^{3/4}}\sqrt{\frac{2}{\pi}}\, e^{\sqrt{\lambda}} + ...$$

• ABJM: similar $\frac{1}{2}$ BPS operator $\quad \mathcal{W} = \text{Tr}\, P e^{\int (iA+\phi^*\phi+...)}$

Localization matrix model (two bi-fund. scalars)

$$Z(N,k) = \int d^N x_i\, d^N y_i\, M(x_i,y_j)\ \exp\left[i\frac{k}{4\pi}\sum_{i=1}^N (x_i^2 - y_i^2)\right]$$

$$M(x_i,y_j) = \prod_{i,j=1}^N \left[\sinh\frac{x_i-x_j}{2}\ \sinh\frac{y_i-y_j}{2}\ (\cosh\frac{x_i-y_j}{2})^{-2}\right]$$

$\langle \mathcal{W} \rangle = \langle \exp \sum_i x_i \rangle$ [Drukker, Marino, Putrov 10; Klemm et al 12]

$$\langle \mathcal{W} \rangle = \frac{1}{2\ \sin\frac{2\pi}{k}}\ \frac{\mathrm{Ai}\left[(\frac{\pi^2}{2}k)^{1/3}\left(N-\frac{k}{24}-\frac{7}{3k}\right)\right]}{\mathrm{Ai}\left[(\frac{\pi^2}{2}k)^{1/3}\left(N-\frac{k}{24}-\frac{1}{3k}\right)\right]}$$

- "M-theory regime": large N at fixed k

$$\mathrm{Ai}(x)\Big|_{x\gg 1} \simeq \frac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}\ \sum_{n=0}^{\infty} \frac{(-\frac{3}{4})^n\,\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{2\pi\,n!\,x^{3n/2}}$$

$$\langle \mathcal{W} \rangle = \frac{1}{2\sin\frac{2\pi}{k}}\ e^{\pi\sqrt{\frac{2N}{k}}}\left[1-\frac{\pi\left(k^2+32\right)}{24\sqrt{2}\,k^{3/2}}\frac{1}{\sqrt{N}}+\mathcal{O}(\frac{1}{N})\right]$$

- "string regime": $N, k \gg 1,\ \ \lambda = \frac{N}{k}$= fixed

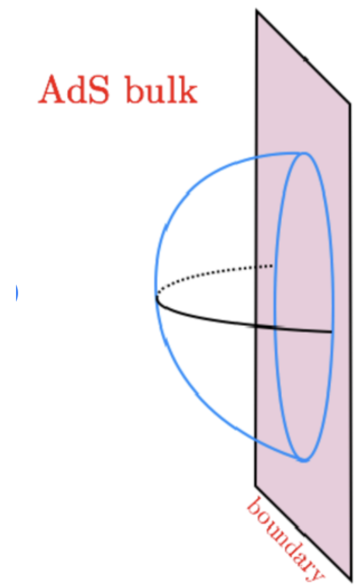
$$\langle \mathcal{W} \rangle = \frac{1}{2\sin\frac{2\pi\lambda}{N}}\ e^{\pi\sqrt{2\lambda}}\left[1-\frac{\pi}{24\sqrt{2}}\frac{1}{\sqrt{\lambda}}+\mathcal{O}(\frac{1}{N})\right] = \frac{N}{4\pi\lambda}\ e^{\pi\sqrt{2\lambda}}\left[1+...\right]$$

• compare to predictions of dual string theories in $\mathrm{AdS}_5 \times S^5$ and in $\mathrm{AdS}_4 \times \mathrm{CP}^3$

$$\mathrm{SYM}: \quad g_{\mathrm{s}} = \frac{g_{\mathrm{YM}}^2}{4\pi} = \frac{\lambda}{4\pi N}\,, \qquad T = \frac{\sqrt{\lambda}}{2\pi}\,, \qquad \lambda = g_{\mathrm{YM}}^2 N$$

$$\mathrm{ABJM}: \quad g_{\mathrm{s}} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}\,, \qquad T = \frac{\sqrt{\lambda}}{\sqrt{2}}\,, \qquad \lambda = \frac{N}{k}$$

$\langle \mathcal{W} \rangle$ = disk part. function expanded near AdS_2 minimal surface



$$\langle \mathcal{W} \rangle = Z_{\rm str} = \frac{1}{g_{\rm s}} Z_1 + \mathcal{O}(g_{\rm s})\,, \qquad Z_1 = \int [dx] ...\, e^{-T\int d^2\sigma\, L}$$

$$\text{SYM}: \quad \langle \mathcal{W} \rangle = \sqrt{\frac{2}{\pi}} \frac{N}{\lambda^{3/4}}\, e^{\sqrt{\lambda}} + ... = \frac{1}{2\pi}\, \frac{\sqrt{T}}{g_{\rm s}}\ \ e^{2\pi T} + ...$$

$$\text{ABJM}: \quad \langle \mathcal{W} \rangle = \frac{N}{4\pi\lambda}\, e^{\pi\sqrt{2\lambda}} + ... = \frac{1}{\sqrt{2\pi}}\, \frac{\sqrt{T}}{g_{\rm s}}\ \ e^{2\pi T} + ...$$

- universal form at strong coupling [Giombi, AT 2020]

$$\langle \mathcal{W} \rangle = c_0\, \frac{\sqrt{T}}{g_{\rm s}}\ \ e^{2\pi T} \Big[1 + \mathcal{O}(T^{-1}) \Big] + \mathcal{O}(g_{\rm s}), \qquad c_0 = \frac{1}{(\sqrt{2\pi})^{d-3}}$$

- reason: dual $\mathrm{AdS}_d \times M^{10-d}$ strings ($d = 4, 5$) – similar structure

$c_0\sqrt{T}$ from 1-loop superstring partition function in $\mathrm{AdS}_d \times M^{10-d}$

det's of fluctuation operators near AdS_2 minimal surface

$$Z_1 \sim \frac{\big[\det(-\nabla^2 + \frac{1}{2})\big]^{\frac{2d-2}{2}}\ \big[\det(-\nabla^2 - \frac{1}{2})\big]^{\frac{10-2d}{2}}}{\big[\det(-\nabla^2 + 2)\big]^{\frac{d-2}{2}}\ \big[\det(-\nabla^2)\big]^{\frac{10-d}{2}}}$$

$$(\mathrm{Z}_1)_\chi \sim (\sqrt{T})^\chi\,, \qquad (\mathrm{Z}_1)_{\rm disk} \sim \sqrt{T}$$

disk with h handles $\chi = 1-2h$: $g_s^{-1} \to g_s^\chi$, $\sqrt{T} \to (\sqrt{T})^\chi$

• thus prediction from string side:

$$\langle \mathcal{W} \rangle = e^{2\pi T} \sum_{h=0}^{\infty} c_h \left(\frac{g_s}{\sqrt{T}} \right)^{2h-1} \left[1 + \mathcal{O}(T^{-1}) \right]$$

remarkably, is consistent with $1/N$ terms in gauge theory:

• SYM: $N \gg 1$, then $\lambda \gg 1$

$$\langle \mathcal{W} \rangle = e^{\frac{(N-1)\lambda}{8N^2}} L^1_{N-1}(-\tfrac{\lambda}{4N}) = e^{\sqrt{\lambda}} \sum_{h=0}^{\infty} \frac{\sqrt{2}}{96^h \sqrt{\pi}\, h!} \frac{\lambda^{\frac{3}{4}(2h-1)}}{N^{2h-1}} \left[1 + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}}) \right]$$

• $\frac{\lambda^{3/4}}{N} \sim \frac{g_s}{\sqrt{T}}$ appears as expansion parameter; $c_h = \frac{1}{2\pi\, h!} \left(\frac{\pi}{12} \right)^h$

from localization expression:

large $T = \frac{\sqrt{\lambda}}{2\pi}$ terms at each order in $g_s = \frac{\lambda}{N}$ exponentiate

$$\langle \mathcal{W} \rangle = W_1 \, e^H \left[1 + \mathcal{O}(T^{-1}) \right] , \qquad W_1 = \frac{1}{2\pi} \frac{\sqrt{T}}{g_s} \, e^{2\pi T}$$

$$H \equiv \frac{\pi}{12} \frac{g_s^2}{T} = \frac{1}{96\pi} \frac{\lambda^{3/2}}{N^2}$$

conjectured interpretation: H = "handle operator"

• computing even 1-loop (torus) $\mathrm{AdS}_5 \times S^5$ string correction directly is an open problem

• remarkably, can derive direct analog $\frac{1}{2 \sin(\sqrt{\frac{\pi}{2}} \frac{g_s}{\sqrt{T}})}$

of all-loop string factor $\exp\left(\frac{\pi}{12} \frac{g_s^2}{T}\right)$ in ABJM case
from semiclassical M2 brane partition function

$1/N$ expansion of $\frac{1}{2}$ BPS circular WL in ABJM

- string side: universal form of expansion in small g_s, large T

$$\langle \mathcal{W} \rangle = e^{2\pi T} \frac{\sqrt{T}}{g_s} \Big(\big[c_0 + \frac{c_{01}}{T} + ...\big] + \frac{g_s^2}{T}\big[c_1 + \frac{c_{11}}{T} + ...\big] + \big(\frac{g_s^2}{T}\big)^2 \big[c_2 + ...\big] + ... \Big)$$

$$\frac{g_s^2}{T} \sim \frac{\lambda^2}{N^2} = \frac{1}{k^2}; \qquad \text{corrections at each } g_s^2\text{:} \quad \sim \frac{1}{T} \sim \frac{1}{\sqrt{\lambda}} = \frac{\sqrt{k}}{\sqrt{N}}$$

- gauge side (localization): exp of leading terms? here not:

summed by $\frac{1}{\sin\frac{2\pi}{k}}$: $\quad \frac{2\pi}{k} = 2\pi\,\frac{\lambda}{N} = \sqrt{\frac{\pi}{2}}\,\frac{g_s}{\sqrt{T}}$ [Beccaria, AT 20]

$$\langle \mathcal{W} \rangle = \frac{1}{2\sin\frac{2\pi}{k}}\, e^{\pi\sqrt{\frac{2N}{k}}} \Big[1 + \mathcal{O}\big(\tfrac{1}{\sqrt{N}}\big)\Big] = \frac{1}{2\sin\Big(\sqrt{\frac{\pi}{2}}\,\frac{g_s}{\sqrt{T}}\Big)}\, e^{2\pi\,T} \Big[1 + O(T^{-1})\Big]$$

$$\frac{1}{2\sin\Big(\sqrt{\frac{\pi}{2}}\,\frac{g_s}{\sqrt{T}}\Big)} = \frac{\sqrt{T}}{\sqrt{2\pi}g_s}\Big[1 + \frac{\pi}{12}\frac{g_s^2}{T} + \frac{7\pi^2}{1440}\big(\frac{g_s^2}{T}\big)^2 + ...\Big]$$

- $\frac{1}{\sin\frac{2\pi}{k}}$ is precisely 1-loop M2 brane contribution [Giombi, AT 2023]:

sums all leading $\frac{1}{T}$ corrections at each order in g_s: $\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{g_s^2}{8\pi T}$

1-loop M2 brane partition function

• $\mathrm{AdS}_2 \times S^1$ membrane solution dual to Wilson loop:
wrapping AdS_2 of AdS_4 and S^1_φ of $S^7/\mathbb{Z}_k$

$$S_{\mathrm{M2}} = \tfrac{1}{4}\mathrm{T}_2\mathrm{vol}(\mathrm{AdS}_2)\,\frac{2\pi}{k} = -\pi\sqrt{\frac{2N}{k}}$$

$e^{-S_{\mathrm{M2}}}$ matches leading factor in $\langle \mathcal{W} \rangle$

• expand M2 brane action near $\mathrm{AdS}_2 \times S^1$ solution
static gauge: M2 coordinates (σ_0, σ_1) = AdS_2;
$\sigma_2 = \varphi$ of radius $R = \frac{1}{k}$:
κ-symmetry gauge: 8+8 3d fluctuations [Sakaguchi, Shin, Yoshida 2010]

• Fourier expansion of 3d fields in $\sigma_2 = (0, 2\pi)$:
towers of bosonic + fermionic 2d fields on AdS_2: KK masses $\frac{n^2}{R^2} = n^2 k^2$

• fluctuations in 2 $\perp$ AdS_4 directions: $m^2 = \frac{1}{4}(kn-2)(kn-4)$

• fluctuations in 6 CP^3 directions: $m^2 = \frac{1}{4}kn(kn+2)$

• fermions: 6+2 towers of 2d spinors: $m = \frac{1}{2}kn \pm 1, \quad m = \frac{1}{2}kn$

• string theory limit $k \to \infty$: $n \neq 0$ modes decouple
$n = 0$: same as 2d fluctuations of string on $\mathrm{AdS}_4 \times \mathrm{CP}^3$

• 1-loop M2 partition function on $\mathrm{AdS}_2 \times S^1$

$$Z_{\rm M2} = Z_1\, e^{-S_{\rm M2}} \Big[1 + \mathcal{O}\big(\mathrm{T}_2^{-1}\big)\Big]\,, \qquad S_{M2} = -\tfrac{\pi}{k}\mathrm{T}_2$$

$$Z_1 = \textstyle\prod_{n=-\infty}^{\infty} \mathcal{Z}_n\,, \qquad \mathcal{Z}_0 = \mathcal{Z}(\mathrm{AdS}_4 \times \mathrm{CP}^3 \text{ string on } \mathrm{AdS}_2)$$

$$\mathcal{Z}_n = \frac{\Big[\det\!\Big(-\nabla^2 - \frac{1}{2} + (\frac{kn}{2}+1)^2\Big)\Big]^{\frac{3}{2}} \Big[\det\!\Big(-\nabla^2 - \frac{1}{2} + (\frac{kn}{2}-1)^2\Big)\Big]^{\frac{3}{2}} \det\!\Big(-\nabla^2 - \frac{1}{2} + (\frac{kn}{2})^2\Big)}{\det\!\Big(-\nabla^2 + \frac{1}{4}(kn-2)(kn-4)\Big) \Big[\det\!\Big(-\nabla^2 + \frac{1}{4}kn(kn+2)\Big)\Big]^3}$$

• no 1-loop log UV div in 3d: $\Gamma_1 = -\log Z_1 = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \zeta'_{\rm tot}(0;n)$

$$\Gamma_1 = \sum_{n=1}^{\infty} \log\big(\frac{k^2n^2}{4} - 1\big) = 2\sum_{n=1}^{\infty} \log\frac{kn}{2} + \sum_{n=1}^{\infty} \log\big(1 - \frac{4}{k^2n^2}\big)$$

• final result for $k > 2$

$$Z_1 = e^{-\Gamma_1} = \frac{1}{2\sin\frac{2\pi}{k}}$$

precise agreement with localization result in gauge theory:
non-trivial test of AdS/CFT to all orders in $1/N$

Subleading correction

• higher loops in M2 semiclassical expansion?

expansion parameter: (M2 tension)$^{-1}$= $(\mathrm{T}_2)^{-1} = \frac{\pi}{\sqrt{2}\,k}\frac{1}{\sqrt{N}}$

$$\langle \mathcal{W} \rangle = \frac{1}{2\sin\frac{2\pi}{k}}\, e^{\pi\sqrt{\frac{2N}{k}}}\Big[1 - \frac{\pi(k^2+32)}{24\sqrt{2}\,k^{3/2}}\frac{1}{\sqrt{N}} + \mathcal{O}\big(\frac{1}{N}\big)\Big]$$

$$= \frac{1}{2\sin\frac{2\pi}{k}}\, e^{\frac{\pi^2}{k}\mathrm{T}_2}\Big[1 - \frac{k^2+32}{24k}\frac{1}{\mathrm{T}_2} + \mathcal{O}(\mathrm{T}_2^{-2})\Big]$$

• compare to string loop expansion (large k limit):

$\frac{k}{T_2} \sim \sqrt{\frac{k}{N}} \sim \frac{1}{\sqrt{\lambda}}$: subleading α' corrections at each order in g_s^2

• subleading $\mathrm{T}_2^{-1} \sim \frac{1}{\sqrt{N}}$ reproduced by 2-loop M2 contribution?

• suggests conjecture: ∞'s cancel also at 2-loop order in M2 theory like that happened in GS action in $AdS_5 \times S^5$ or $AdS_4 \times CP^3$ [Roiban, Tirziu, AT 07; Giombi et al 09; Bianchi et al 14]

• possible reason? hidden symmetry in M2 theory beyond susy?

Non-planar corrections to operator dimensions

• Lesson: take semiclassical M2 brane quantization seriously $\rightarrow$
use semiclassical M2 expansion to get $1/N$ strong coupling
corrections to non-BPS observables in ABJM
which are not determined by localization or integrability

• string theory regime: non-planar corrections to dim of operators
with large spins represented by string loop corrections
to energies of semiclassical strings on $AdS_4 \times CP^3$

• M-theory regime: semiclassical quantization of corresponding
spinning M2 branes with topology $(R_t \times S^1) \times S^1$
dependence on finite $k = \frac{\lambda}{N}$ from
dependence of M2 action on 11d background with $R_{11} \sim \frac{1}{k}$

Example: cusp anomalous dimension in ABJM

$$\Delta = S + f(\lambda, N) \log S + ..., \qquad \mathcal{O} = \mathrm{Tr}\,(\Phi D^S \Phi)$$

how to find $f(\lambda, N)$ at strong coupling beyond planar limit?

• membrane analog of long rotating folded string in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

$$ds^2 = \mathrm{L}^2 \Big(\tfrac{1}{4} [- \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, (d\psi^2 + \cos^2 \psi \, d\phi^2)] + \tfrac{1}{k^2} d\varphi^2 \Big)$$

$$t = \kappa\sigma_0, \qquad \rho = \kappa\sigma_1, \qquad \phi = \kappa\sigma_0, \qquad \psi = 0, \qquad \varphi = \sigma_2 \in (0, 2\pi)$$

$$S_{\mathrm{M2}} = T_2 \int d^3\sigma \sqrt{-\det g} + T_2 \int C_3 \ + \ \text{fermionic terms}$$

$$C_3 = -\tfrac{3}{8} R^3 \cosh\rho \, \sinh^2 \rho \, \sin\psi \, dt \wedge d\rho \wedge d\phi$$

• classical energy as in string case: $\kappa = \frac{1}{\pi} \log S$

$$E - S = 2T \log S, \qquad T = \tfrac{2\pi}{k} T_2 \mathrm{L}^3 = \sqrt{\tfrac{\lambda}{2}}\ , \qquad \mathrm{L}^6 = 32\pi^2 N k \ell_P^6$$

• 1-loop correction: static gauge $t = \kappa\sigma_0, \ \rho = \kappa\sigma_1, \ \varphi = \sigma_2$

8+8 fluctuations in modes in σ_2: $\psi \sim \sum_n e^{in\sigma_2} \psi_n, \ \ n = 0, \pm 1, ...$

• bosons: 6 CP^3 fluctuations with masses: $m^2 = \frac{1}{4}kn(kn+2)$

2 from $\psi, \tilde{\phi}$ mixing: $m_\pm^2 = 3 + \frac{1}{4}k^2n^2 \pm \sqrt{1+\frac{9}{4}k^2n^2}$

• fermions: 6+2 with $m = \frac{kn}{2} \pm 1$, and $m = \frac{kn}{2}$

loop corrections contribute to $f(\lambda, k)$:

$\Gamma = \sum_{r=1}^{\infty} (T_2)^{-r+1}\Gamma_r(k) \sim V \sim \kappa^2 \int d\sigma_0 \quad \rightarrow \quad E_{\text{quant}} \sim \log S$

$$\Gamma_1 = -\log Z_1 = \frac{1}{2}\frac{1}{(2\pi)^2} V \int d^2p \left[X_0(p^2) + 2\sum_{n=1}^{\infty} X_n(p^2) \right]$$

$$X_n(p^2) = \log\left[p^2 + 3 + \tfrac{1}{4}k^2n^2 + \sqrt{1+\tfrac{9}{4}k^2n^2}\right]$$
$$+ \log\left[p^2 + 3 + \tfrac{1}{4}k^2n^2 - \sqrt{1+\tfrac{9}{4}k^2n^2}\right]$$
$$+3\log\left[p^2 + \tfrac{1}{4}(kn)^2 + \tfrac{1}{2}kn\right] + 3\log\left[p^2 + \tfrac{1}{4}(kn)^2 - \tfrac{1}{2}kn\right]$$
$$-3\log\left[p^2 + (1+\tfrac{1}{2}kn)^2\right] - 3\log\left[p^2 + (1-\tfrac{1}{2}kn)^2\right] - 2\log\left[p^2 + (\tfrac{1}{2}kn)^2\right]$$

• string part ($n=0$) [McLoughlin, Roiban; Alday, Arutyunov, Bykov 08]

$$\frac{1}{2}\frac{1}{2\pi}\int_0^\infty dp^2\, X_0(p^2) = \frac{1}{4\pi}\int_0^\infty dp^2 \Big[\log(p^2+4)+\log(p^2+2)$$
$$+4\log p^2 - 6\log(p^2+1)\Big] = -\frac{5}{2\pi}\log 2$$

• $f = \sqrt{2\lambda} - \frac{5}{2\pi}\log 2 + \frac{c_2}{\sqrt{\lambda}} + \frac{c_3}{(\sqrt{\lambda})^2} + ... + \mathcal{O}(\frac{1}{k^2})$

$$\equiv \mathrm{f}_0(\lambda) + \mathrm{f}_1(k) + ...$$

c_n terms come from string loop corrections;
planar part $\mathrm{f}_0(\lambda)$ fixed by integrability
(related to $\mathrm{AdS}_5\times S^5$ up to $\sqrt{\lambda}\to h(\lambda)$ [Gromov, Vieira 08])

• membrane n-mode contribution: $\Upsilon_n \equiv \frac{1}{2\pi}\int_0^\infty dp^2\, X_n(p^2)$

$$= -\frac{1}{8\pi}\Big[-3((kn)^2+8)\log((kn)^2-4) + 18kn\log\frac{kn-2}{kn+2}$$
$$+(kn)^2\log(kn)^2 + ((kn)^2+12)\log((kn)^4-12(kn)^2+128)$$
$$+2\sqrt{9(kn)^2+4}\log\frac{(kn)^2+12+2\sqrt{9(kn)^2+4}}{(kn)^2+12-2\sqrt{9(kn)^2+4}}\Big] = \sum_{m=1}^\infty \frac{d_m}{(kn)^{2m}}$$

• sum is finite and has regular expansion at large k (coeffs. $\sim \zeta(2m)$)

$$\mathrm{f}_1(k) = \sum_{n=1}^{\infty} Y_n = \frac{2\pi}{3k^2} + \frac{2\pi^3}{45k^4} - \frac{1616\pi^5}{14175k^6} + ...$$

$$= \frac{2\pi\lambda^2}{3N^2} + \frac{2\pi^3\lambda^4}{45N^4} - \frac{1616\pi^5\lambda^6}{14175N^6} + ...$$

• at weak coupling non-planar correction $\mathrm{f}\big|_{\lambda\ll 1} \sim \frac{\lambda^4}{N^2} + ...$

prediction is that at strong coupling $\mathrm{f}\big|_{\lambda\ll 1} \sim \frac{\lambda^2}{N^2} + ...$

• Higher loop M2 corrections to $f(\lambda, N) = f(\mathrm{T}_2, k)$

$$f = \frac{4\pi}{k}\mathrm{T}_2 + \mathrm{f}(k) + \frac{q_2(k)}{\mathrm{T}_2} + \frac{q_3(k)}{\mathrm{T}_2^2} + ..., \qquad \mathrm{T}_2 = L^3 T_2 = \frac{1}{\pi}\sqrt{kN}$$

should give subleading strong-coupling corrections
at each order in $1/N^2$: i.e.

$$f(\lambda, N) = \sqrt{2\lambda}\Big[1 - \frac{5\log 2}{2\pi\sqrt{2\lambda}} + ...\Big] + \frac{2\pi\lambda^2}{3N^2}\Big[1 + \frac{a_1}{\sqrt{\lambda}} + ...\Big]$$
$$+ \frac{2\pi^3\lambda^4}{45N^4}\Big[1 + \frac{b_1}{\sqrt{\lambda}} + ...\Big] + ...$$

• similar computations can be done for other
semiclassical M2 branes in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ generalizing
large-spin string solutions in $\mathrm{AdS}_4 \times CP^3$ dual to
particular operators in ABJM
e.g. for M2 brane with 2 angular momenta $J_1 = J_2 \equiv J$ in $S^7/\mathbb{Z}_k$
generalizing string solution
dual to $\mathcal{O} = \mathrm{Tr}\,(\Phi_1\Phi_4^\dagger)^J$
and analogous "small string" solutions dual to "short" operators

• "short" M2 brane solution: $\bar\lambda \equiv 2\pi^2\lambda$ [Giombi, Kurlyand, AT 24])

$$E_{\rm M2} = 2\sqrt{\sqrt{\bar\lambda}J} + \tfrac{1}{2} + \tfrac{1}{2}\bar\lambda^{-1/4}J^{1/2} - \tfrac{9}{4}\zeta(3)\bar\lambda^{-3/4}J^{3/2} + \mathcal{O}\big(\bar\lambda^{-1}J^2\big)$$

$$+\frac{1}{k^2}\Big[8\zeta(2)\big(\bar\lambda^{3/4}J^{-3/2} + 2\bar\lambda^{1/4}J^{-1/2}\big) + \mathcal{O}\big(\bar\lambda^{-1/4}J^{1/2}\big)\Big] + \mathcal{O}\big(\frac{1}{k^4}\big)$$

from string theory point of view $\frac{1}{k^2} = \frac{g_s^2}{4\sqrt{\bar\lambda}}$ is leading
large tension asymptotics of string 1-loop (torus) contribution
on the dual ABJM gauge theory side expansion first in $1/N^2$
and then in large λ for fixed J: $\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{\bar\lambda^2}{(2\pi^2)N^2}$
is prediction for non-planar correction to dim of "short" operator

• "long" M2 brane case ($c_1 \approx -0.33$)

$$E_{\rm M2} = 2J + \tfrac{\bar\lambda}{4J}\big(1 - 2\log 2\,\bar\lambda^{-1/2} + ...\big) + c_1\tfrac{\bar\lambda}{2J^2}\big(1 + ...\big) + ...$$

$$+\tfrac{1}{k^2}\Big(- 2\bar\lambda^{-1/2}J - \tfrac{\bar\lambda^{1/2}}{2J} + \tfrac{3\bar\lambda^{3/2}}{64J^3} + ...\Big) + \mathcal{O}\big(\tfrac{1}{k^4}\big)$$

$\frac{1}{k^2} = \frac{\bar\lambda^2}{(2\pi^2)^2N^2}$ is prediction for dim of operator with the large spin J

Conclusions

• evidence that quantum M2 brane theory is well defined
at least in semiclassical expansion

• susy observables (BPS WL and instanton part of S^3 free energy):
non-trivial precision tests of $\mathrm{AdS}_4/\mathrm{CFT}_3$ beyond planar limit
• similar results for $\mathrm{AdS}_7/\mathrm{CFT}_6$:
in 6d (2,0) theory dual to M-theory on $\mathrm{AdS}_7 \times S^4$
quantum effects in susy free energy on $S^5 \times S^1$ (index) captured by
semiclassical M2 branes in $\mathrm{AdS}_7 \times S^4$ [Beccaria, Giombi, AT 23]

• beyond localization: dimensions of operators in ABJM dual to
spinning strings/membranes: prediction of the structure of
non-planar corrections at strong coupling
$\frac{1}{k^2} = \frac{\lambda^2}{N^2}$ correction: prediction for string 1-loop (torus) contribution

Some open questions

• hidden symmetry in M2 theory that constrains loop corrections?
[cf. integrability in GS string or $T\bar{T}$: constraints on UV counterterms]
• way to detect it? study S-matrix of brane excitations [Seibold, AT 24]
(cf. pure phase factorizable S-matrix on long string in flat space)

• M2 world-volume theory with $\Sigma^2 \times S^1$ topology:
effective 2d theory with massive KK towers with $m^2 \sim k^2 \sim g_s^{-2}$
related to effective string with non-local terms
due to small handle resummation?
integrable subsectors in this 2d theory?

• if one could capture non-planar corrections at $\lambda \ll 1$
by perturbation theory around integrable (planar) theory
how then to take $\lambda \gg 1$ to compare to M2 brane results?
• is there a way to do similar computations of non-planar strong
coupling corrections in $\mathcal{N} = 4$ SYM dual to $AdS_5 \times S^5$ string theory?