Linearised Analysis of the Coxeter Higher Spin Theory II

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Entangled Module

One module is missing from previous considerations as it is not a product of two standard modules for $R(k)\bar{R}(\bar{k}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ The covariant consistency equation takes form

$$\Big(D_L + \frac{1}{2}h^{\alpha\dot{\alpha}}\Big[y_{\alpha 1}(\bar{\partial}_{\dot{\alpha}1} + \bar{\partial}_{\dot{\alpha}2}) + \bar{y}_{\dot{\alpha}1}(\partial_{\alpha 1} + \partial_{\alpha 2})...\Big] -$$

$$- \frac{i}{2}h^{\alpha\dot{\alpha}}\Big[y_{\alpha 1}(\bar{y}_{\dot{\alpha}1} - \bar{y}_{\dot{\alpha}2}) - \partial_{\alpha 1}(\bar{\partial}_{\dot{\alpha}1} - \bar{\partial}_{\dot{\alpha}2}) + ...\Big]\Big) C(Y_1, Y_2; \hat{k}, \hat{\bar{k}}|x) = 0$$

Fock Module

As the first step, we move from Y^i_A to a doubled set of oscillators $a_{1,2A}$, ${b_{1,2}}^B$ with the star product

$$(f*g)(a,b) = \frac{1}{\pi^8}\int d^4u_{1,2}d^4v_{1,2}d^4s_{1,2}d^4t_{1,2}f(a+u,b+t)g(a+s,b+v) \\ \times exp\left(2s_{1A}t_1^A - 2u_{1A}v_1^A + 2s_{2A}t_2^A - 2u_{2A}v_2^A\right).$$

$$[a_{iA}, {b_j}^B]_* = \delta_{ij}{\delta_A}^B\,, \quad [a_{iA}, a_{iB}]_* = 0\,, \quad [{b_i}^A, {b_j}^B]_* = 0$$

following M.A. Vasiliev, Phys. Rev. D **66** (2002), 066006

Fock Module

The bilinears spanning $gl(4,\mathbb{C})\oplus gl(4,\mathbb{C})$

$$T_{iA}{}^{B}=a_{iA}b_i{}^{B}\equiv\frac{1}{2}(a_{iA}*b_i{}^{B}+b_i{}^{B}*a_{iA})\,.\quad i=1,2$$

can be quotioned by the central element

$$N_i=a_{iA}b_i{}^{A}\equiv\frac{1}{2}(a_{iA}*b_i{}^{A}+b_i{}^{A}*a_{iA})\,.$$

resulting in generators of $sl(4,\mathbb{C})\oplus sl(4,\mathbb{C})$

$$t_{iA}{}^{B}=(a_{iA}b_i{}^{B}-\frac{1}{4}\delta_A{}^{B}N_i)\,.$$

Fock Module

By imposing reality conditions we single out the $su(2,2) \oplus su(2,2)$ real part, which splits our oscillators into pairs of two-component spinors

$$[a_{i\alpha}, b_j^\beta]_* = \delta_{ij}\delta_\alpha^\beta\,, \quad [\tilde{a}_{i\dot{\alpha}}, \tilde{b}_j{}^{\dot{\beta}}]_* = \delta_{ij}\delta_{\dot{\alpha}}^{\dot{\beta}}\,,$$

$$\bar{a}_{i\alpha} = \tilde{b}_{i\dot{\alpha}}\,, \quad \bar{b}_i^\alpha = \tilde{a}_i^{\dot{\alpha}}\,, \quad \bar{\tilde{a}}_{i\dot{\alpha}} = b_{i\alpha}\,, \quad \bar{\tilde{b}}_i^{\dot{\alpha}} = a_i^\alpha\,.$$

Fock Module

One can introduce vacua for each set of the oscillators

$$
\begin{aligned}
&a_{i\alpha} * \pi^i{}_1 = 0\,, \quad b_i^{\alpha} * \pi^i{}_2 = 0\,, \quad a_{i\alpha} * \pi^i{}_3 = 0\,, \quad b_i^{\alpha} * \pi^i{}_4 = 0\,,\\
&\tilde{b}_i^{\dot\alpha} * \pi^i{}_1 = 0\,, \quad \tilde{a}_{i\dot\alpha} * \pi^i{}_2 = 0\,, \quad \tilde{a}_{i\dot\alpha} * \pi^i{}_3 = 0\,, \quad \tilde{b}_i^{\dot\alpha} * \pi^i{}_4 = 0\,.
\end{aligned}
$$

realised as

$$
\begin{aligned}
\pi^i{}_1 &= \exp\Big\{-2a_{i\alpha}b_i{}^{\alpha} + 2\tilde{a}_{i\dot\alpha}\tilde{b}_i{}^{\dot\alpha}\Big\}\,, \quad \pi^i{}_2 = \exp\Big\{2a_{i\alpha}b_i{}^{\alpha} - 2\tilde{a}_{i\dot\alpha}\tilde{b}_i{}^{\dot\alpha}\Big\}\,,\\
\pi^i{}_3 &= \exp\Big\{-2a_{i\alpha}b_i{}^{\alpha} - 2\tilde{a}_{i\dot\alpha}\tilde{b}_i{}^{\dot\alpha}\Big\}\,, \quad \pi^i{}_4 = \exp\Big\{2a_{i\alpha}b_i{}^{\alpha} + 2\tilde{a}_{i\dot\alpha}\tilde{b}_i{}^{\dot\alpha}\Big\}\,.
\end{aligned}
$$

The space of states is

$$
\big|C^{11}\big\rangle = C^{11}(2b_1, 2\tilde{a}_1, 2b_2, 2\tilde{a}_2)\pi^1{}_1\pi^2{}_1\,.
$$

Fock Module

Introducing the flat $su(2,2) \oplus su(2,2)$ connection

$$\omega_0 = {\omega_0}^{\alpha}{}_{\beta}({L^1}_{\alpha}{}^{\beta} + {L^2}_{\alpha}{}^{\beta}) + {\bar{\omega}_0}{}^{\dot{\alpha}}{}_{\dot{\beta}}({\bar{L}^1}{}_{\dot{\alpha}}{}^{\dot{\beta}} + {\bar{L}^2}{}_{\dot{\alpha}}{}^{\dot{\beta}}) + \\ + {h_0}^{\alpha}{}_{\dot{\beta}}({P^1}_{\alpha}{}^{\dot{\beta}} + {P^2}_{\alpha}{}^{\dot{\beta}} + {K^1}{}^{\dot{\beta}}{}_{\alpha} + {K^2}{}^{\dot{\beta}}{}_{\alpha}),$$

where the $su(2,2)$ generators are realised in a canonical way

$$\begin{aligned} {L^i}_{\alpha}{}^{\beta} &= a_{i\alpha} {b_i}^{\beta} - \frac{1}{2}\delta_{\alpha}{}^{\beta} a_{i\gamma} {b_i}^{\gamma}, \quad {P^i}_{\alpha}{}^{\dot{\beta}} = a_{i\alpha} {\tilde{b}_i}{}^{\dot{\beta}}, \\ {\bar{L}^i}{}_{\dot{\alpha}}{}^{\dot{\beta}} &= \tilde{a}_{i\dot{\alpha}} {\tilde{b}_i}{}^{\dot{\beta}} - \frac{1}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tilde{a}_{i\dot{\gamma}} {\tilde{b}_i}{}^{\dot{\gamma}}, \quad {K^i}_{\dot{\alpha}}{}^{\beta} = \tilde{a}_{i\dot{\alpha}} {b_i}^{\beta}. \\ D^i &= \frac{1}{2}(a_{i\alpha} {b_i}^{\alpha} - \tilde{a}_{i\dot{\alpha}} {\tilde{b}_i}{}^{\dot{\alpha}}), \end{aligned}$$

the twisted-adjoint modules can be constructed by restricting the Fock modules with an equation

$$\mathrm{d}_x \left|C^{11}\right\rangle + \omega_0 * \left|C^{11}\right\rangle = 0.$$

Automorphism of the oscillator algebra

Since there exist automorphisms on the oscillator algebra, there is no need to consider each module independently. The following two are of a particular interest to us

$$\left\{\rho_i(a_{i\alpha}) = b_{i\alpha}\,,\quad \rho_i(b_i^\alpha) = a_i^\alpha\,,\quad \bar{\rho}_i(\tilde{a}_{i\dot{\alpha}}) = \tilde{b}_{i\dot{\alpha}}\quad \bar{\rho}_i(\tilde{b}_i^{\dot{\alpha}}) = \tilde{a}_i^{\dot{\alpha}}\right\} \Leftrightarrow \hat{k}_i\,, \hat{\bar{k}}_i\,,$$

$$\left\{\psi_+(a_{1\alpha}) = \frac{1}{2}(b_1 + b_2 + a_1 - a_2)_\alpha\,,\ \psi_+(a_{2\alpha}) = \frac{1}{2}(b_1 + b_2 + a_2 - a_1)_\alpha\,,\right.$$

$$\left.\psi_+(b_1^\alpha) = \frac{1}{2}(a_1 + a_2 + b_1 - b_2)^\alpha\,,\ \psi_+(b_2^\alpha) = \frac{1}{2}(a_1 + a_2 + b_2 - b_1)^\alpha\right\} \Leftrightarrow \hat{k}_{12}^+$$

Transitioning between modules

The composition of these automorphism allows us to arrive at the equations for the entangled module

$$\mathrm{d}_x \left|C^{11}\right\rangle + \rho_i(\psi_+(\omega_0)) * \left|C^{11}\right\rangle = 0\,.$$

To construct the unitary module we introduce a new set of oscillators $e^i_{\nu A}$ and $f^i{}_A{}^\nu$ such that

$$[e^i_{\nu A}, e^j_{\mu B}]_* = 0\,, \quad [f^i{}_A{}^\nu, f^j{}_B{}^\mu]_* = 0\,, \quad [e^i_{\nu A}, f^j{}_B{}^\mu]_* = \delta^{ij}\delta^\mu_\nu K_{AB}\,,$$

where $K_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Transitioning between modules

- Bilinears of these operators realise $sp(8) \oplus sp(8) \supset su(2,2) \oplus su(2,2)$
- Total energy operator

$$E = \sum_{i=1}^{2} \left(f^{i}{}_{1}{}^{\lambda} e^{i}_{\lambda 1} + f^{i}{}_{2}{}^{\lambda} e^{i}_{\lambda 2} \right).$$

- A Fock module associated with $e^{i}_{\nu A}$ and $f^{i}{}_{A}{}^{\nu}$

$$e^{i}_{\nu 1} * \Pi = 0\,, \quad f^{i}{}_{2}{}^{\mu} * \Pi = 0\,, \quad \Pi * e^{i}_{\nu 2} = 0\,, \quad \Pi * f^{i}{}_{1}{}^{\mu} = 0\,.$$

Transitioning between modules

New oscillators are related to old ones via a Bogoliubov transform

$$e^i_{11} = \frac{1}{\sqrt{2}}(a_{i1} + i\tilde{a}_{i\dot{2}})\,, \quad e^i_{12} = \frac{1}{\sqrt{2}}(a_{i1} - i\tilde{a}_{i\dot{2}})\,, \quad \ldots$$

$$f^{i1}{}_1 = \frac{1}{\sqrt{2}}(b_{i2} + i\tilde{b}_{i\dot{1}})\,, \quad f^{i1}{}_2 = \frac{1}{\sqrt{2}}(-b_{i2} + i\tilde{b}_{i\dot{1}})\,, \quad \ldots$$

Fock module F is suitable for the description of physical states when it satisfies two conditions

- F is a highest/lowest weight module meaning the energy E is bounded from above/from below.
- F admits an invariant positive-definite Hermitian form, i.e. F is unitary.

Transitioning between modules

These conditions are true for the twisted-adjoint module, but for the entagled module, a check shows that they cannot be both satisfied. To achieve the lowest-weight representation, the energy needs to be diagonal:

$$\rho_1\psi_+(E) = \sum_{a=1}^{8} v_a^- v_a^+ \,.$$

where

$$v_a^- = \{\frac{1}{\sqrt{2}}(ie^2_{21} + f^2{}_2{}^1), \frac{1}{\sqrt{2}}(ie^2_{11} + f^2{}_2{}^2), \frac{1}{2}(ie^1_{22} - f^1{}_1{}^1 + f^2{}_2{}^1 - ie^2_{21}),$$
$$\frac{1}{2}(ie^1_{12} - f^1{}_1{}^2 + ie^2_{11} - f^2{}_2{}^2), \frac{1}{\sqrt{2}}(ie^1_{11} + f^1{}_2{}^2), \frac{1}{\sqrt{2}}(ie^1_{21} + f^1{}_2{}^1),$$
$$\frac{1}{2}(ie^2_{12} - ie^1_{11} - f^2{}_1{}^2 + f^1{}_2{}^2), \frac{1}{2}(f^2{}_1{}^1 + f^1{}_2{}^1 - ie^1_{21} - ie^2_{22})\}\,,$$

$$v_a^+ = \{\ldots\}\,,$$

Transitioning between modules

There exists no positive-definite Hermitian form that leads to $\rho_1\psi_+(E) = \sum_a (v_a^+)^\dagger v_a^+$. Therefore, this module cannot represent physical states.

In the flat limit:

$$\left(\mathrm{d}_x + \frac{i}{2}h^{\alpha\dot\alpha}\left(\partial_{\alpha 1}\bar\partial_{\dot\alpha 1} + \partial_{\alpha 1}\bar\partial_{\dot\alpha 2} + \partial_{\alpha 2}\bar\partial_{\dot\alpha 2} - \partial_{\alpha 2}\bar\partial_{\dot\alpha 1}\right)\right) C(Y_1, Y_2|x) = 0\,.$$

This equation admits plane wave solutions

$$C(Y_1, Y_2|x) = \exp\left\{ i\left(A^{IJ}\xi_{I\alpha}\bar\xi_{J\dot\alpha}x^{\alpha\dot\alpha} + \delta^{IJ}\xi_{I\alpha}y^\alpha_J + \delta^{IJ}\bar\xi_{I\dot\alpha}\bar y^{\dot\alpha}_J\right)\right\},$$

where ξ, $\bar\xi$ are the Fourier partners for y and $\bar y$ and

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Consistent truncation to the physical sector

- Since this module is entirely non-unitary, it thus needs to be excluded from the complete non-linear system
- A consistent truncation of the unfolded system can be achieved by restricting the system to an invariant space of some automorphism. We consider total Klein parity.
- $S * S(-k) = S * S(k) \implies B(-k) = -B(k)$ The only such fields belong to the product of adjoint and twisted-adjoint modules

$$S * S = i\left(dZ^{An}dZ_{An} + \sum_{i}\sum_{v\in\mathcal{R}_i}\left[\eta_i B\frac{v^n v^m}{(v,v)}dz_n^{\alpha}dz_{\alpha m} * \varkappa_v \hat{k}_v + \bar{\eta}_i B\frac{v^n v^m}{(v,v)}d\bar{z}_n^{\dot{\alpha}}d\bar{z}_{\dot{\alpha}m} * \bar{\varkappa}_v \hat{\bar{k}}_v\right]\right)$$

Modified Shifted Homotopy Technique

To reconstruct the free equations (Central On-Shell Theorem) we modify the shifted homotopy technique [1]

$$\Delta_Q g(Y, Z; dZ) = \left(Z_n^A + I_n Q_n^A\right) \frac{\partial}{\partial dZ_n^A} \int_0^1 \frac{dt}{t} g(Y, tZ_i - (1-t) I_i Q_i; t dZ)$$

Along with a cohomology projector

$$h_Q f(Z; dZ) = f(-I_n Q_n; 0)\,.$$

satisfy usual identities

$$\Delta_Q \Delta_P = -\Delta_P \Delta_Q\,.$$

$$h_P \Delta_Q = -h_Q \Delta_P\,.$$

[1] V.E.Didenko, O.A.Gelfond, A.V.Korybut and M.A.Vasiliev, 2018

Modified Shifted Homotopy Technique

$$h_c\Delta_b\Delta_a f(y,z)dz^{n\mu}dz_{n\mu} =$$
$$= 2\int_{[0,1]^3} d^3\tau\delta(1-\tau_1-\tau_2-\tau_3)I_nI_m(b-c)_{n\nu}(a-c)^{m\nu}f(y,-\tau_1c-\tau_3b-\tau_2a)\,.$$

$$h_c\Delta_b\Delta_a\hat{\gamma}_v = 2\int_{[0,1]^3} d^3\tau\delta(1-\tau_1-\tau_2-\tau_3)(b-c)_{n\nu}(a-c)^{\nu}_{m}\frac{v^nv^m}{(v,v)}$$
$$\times\, exp\Big\{-i\frac{v^pv^q}{(v,v)}(\tau_1c+\tau_2a+\tau_3b)_{p\alpha}y^{\alpha}_{q}\Big\}\hat{k}_v\,.$$

$$\hat{\gamma}_v = exp\left(i\frac{v^pv^q}{(v,v)}z_{\alpha p}y^{\alpha}_{q}\right)\frac{v^nv^m}{(v,v)}dz^{\alpha}_{n}dz_{\alpha m}\hat{k}_v$$

Central On-Shell Theorem (general Coxeter group)

The obvious difference comes in the star-exchange properties:

$$\Delta_{q+\alpha y}\hat{\gamma}_v * C(y) = C(y) * \Delta_{q+\alpha y+(1-\alpha)Ip-(1+\alpha)IR_v(p)}\hat{\gamma}_v \,.$$

And thus the vertices themselves are somewhat modified

$$\Upsilon^\eta(\omega, C, \omega) = -\frac{1}{4i}\sum_k \eta_k \sum_{v\in\mathcal{R}_k} \omega * C * \omega *$$
$$\Big(h_{I(t_1+t_2+p)}\Delta_{I(p+t_1+t_2-R_v(t_2))}\Delta_{I(p+t_2)}\hat{\gamma}_v +$$
$$+ h_{I(p+t_1+t_2-R_v(t_2))}\Delta_{I(p+t_2-R_v(t_2))}\Delta_{I(p+t_2)}\hat{\gamma}_v \Big) \,,$$

Central On-Shell Theorem (B_2 group)

The Central On-Shell theorem in the $R(\hat{K}_C)\bar{R}(\hat{K}_C) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Upsilon^{\eta_1}_{tot}(\Omega_{AdS}, \Omega_{AdS}, C) = -\frac{i\eta_1}{2}\bar{H}_{\dot{\alpha}\dot{\beta}}\bar{\partial}_1^{\dot{\alpha}}\bar{\partial}_1^{\dot{\beta}} C(0, y_2, \bar{y}_1, \bar{y}_2; \hat{K}_C|x) * \hat{k}_1 + \frac{i\eta_1}{2}\bar{H}_{\dot{\alpha}\dot{\beta}}\bar{y}_2^{\dot{\alpha}}\bar{y}_2^{\dot{\beta}} C(y_1, 0, \bar{y}_1, \bar{y}_2; \hat{K}_C|x) * \hat{k}_2\,,$$

$$\Upsilon^{\eta_2}_{tot}(\Omega_{AdS}, \Omega_{AdS}, C) =$$

$$\frac{i\eta_2}{4}\bar{H}_{\dot{\alpha}\dot{\beta}}(\bar{y}_2 - i\bar{\partial}_1)^{\dot{\alpha}}(\bar{y}_2 - i\bar{\partial}_1)^{\dot{\beta}} C\left(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_1 + y_2), \bar{y}_1, \bar{y}_2; \hat{K}_C|x\right) * \hat{k}_{12} +$$

$$\frac{i\eta_2}{4}\bar{H}_{\dot{\alpha}\dot{\beta}}(\bar{y}_2 + i\bar{\partial}_1)^{\dot{\alpha}}(\bar{y}_2 + i\bar{\partial}_1)^{\dot{\beta}} C\left(\frac{1}{2}(y_1 - y_2), -\frac{1}{2}(y_1 - y_2), \bar{y}_1, \bar{y}_2; \hat{K}_C|x\right) * \hat{k}^+_{12}$$

Results

- An embedding of the AdS_4 solution into a Coxeter model demanding equating idempontents I and $\bar{I}$
- For the case of the B_2 model, all possible covariant derivatives and the resulting modules have been found
- All modules are either tensor products of modules in the standard theory or belong to a new class of infinite-dimensional non-unitarizable modules
- A consistent truncation of non-linear systems which eliminates the entagled moduleshas been found
- A restriction to unitary submodules of C field interpreted as boundary conditions
- Generalization of the Central On-Shell theory has been found for a general Coxeter group by an uplifted shifted homotopy technique. In the case of B_2 all linear vertices have been presented

Questions

Question Time