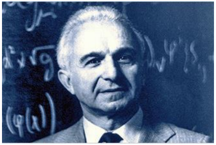

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All-loop results in supersymmetric theories for various renormalization prescriptions

Based on

A.L.Kataev, K.S., INR-TH-2024-010.

The simplest example of a theory with two gauge coupling constants $\alpha_s \equiv g^2/4\pi$ and $\alpha = e^2/4\pi$ is QCD+QED. In the massless limit this theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}^2 - \frac{1}{4e^2}\boldsymbol{F}_{\mu\nu}^2 + \sum_{\mathsf{a}=1}^{N_f} i\bar\psi_{\mathsf{a}}\gamma^\mu \mathcal{D}_\mu \psi_{\mathsf{a}},$$

which is invariant under the transformations of the gauge group $G\times U(1)$. The Dirac spinors ψ_{a} (where the subscript a numerates flavors) lie in a certain irreducible representation R of the group G and have the electromagnetic charges q_{a}. In this case the covariant derivatives are written in the form

$$\mathcal{D}_\mu\psi_{\mathsf{a}} = \partial_\mu\psi_{\mathsf{a}} + A_\mu\psi_{\mathsf{a}} + i q_{\mathsf{a}}\boldsymbol{A}_\mu\psi_{\mathsf{a}},$$

where A_μ and $\boldsymbol{A}_\mu$ are the non-Abelian and Abelian gauge fields, respectively. The corresponding gauge field strengths are given by the expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \qquad \boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu.$$

In quantum field theory the couplings α_s and α depend on scale,

$$\frac{d\alpha}{d\ln\mu} = \beta(\alpha,\alpha_s); \qquad \frac{d\alpha_s}{d\ln\mu} = \beta_s(\alpha_s,\alpha).$$

It is convenient to formulate the supersymmetric version of the above model in terms of superfields

$$S=\frac{1}{2g^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4e^2}\,\mbox{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^+ e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}}+\widetilde\phi_{\mathsf{a}}^+ e^{-2V^T-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde\phi_{\mathsf{a}}\Big),$$

because in this case $\mathcal{N}=1$ supersymmetry is manifest.

Here V and $\boldsymbol{V}$ are the gauge superfields corresponding to the subgroups G and $U(1)$, respectively. The chiral matter superfields ϕ_{a} and $\widetilde\phi_{\mathsf{a}}$ belong to the (conjugated) representations R and $\overline{R}$, respectively, and have opposite $U(1)$ charges.

Two supersymmetric gauge superfield strengths are written in the form

$$W_a=\frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big);\qquad \boldsymbol{W}_a=\frac{1}{4}\bar D^2 D_a\boldsymbol{V}.$$

Is it possible to relate running of two gauge coupling constants in this model?

The exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456.

relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories.

For a general $\mathcal{N}=1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

Investigating the exact results in supersymmetric theories (and, in particular, the NSVZ equation) it is important to remember that the renormalization procedure in quantum field theory is not uniquely defined.

For instance, it is possible to remove divergences in the two-point Green function of the background gauge field by the splitting of the bare coupling constant to the renormalized coupling constant and counterterm either as

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \beta_1 \ln\frac{\Lambda}{\mu} + O(\alpha), \qquad \text{or as} \qquad \frac{1}{\alpha_0} = \frac{1}{\alpha'} - \beta_1\Big(\ln\frac{\Lambda}{\mu} + c_1\Big) + O(\alpha'),$$

where c_1 is a finite constant. These two definitions of the renormalized coupling constant are related by the equation

$$\frac{1}{\alpha'} = \frac{1}{\alpha} + \beta_1 c_1 + O(\alpha)$$

or, in other words, by the finite renormalization

$$\alpha' = \alpha - \alpha^2\beta_1 c_1 + O(\alpha^3).$$

In general, it is possible to make the finite renormalizations

$$\alpha' = \alpha'(\alpha); \qquad Z'(\alpha', \ln\Lambda/\mu) = z(\alpha) Z(\alpha, \ln\Lambda/\mu).$$

Under the finite renormalizations

$$\alpha' = \alpha'(\alpha); \qquad Z'(\alpha', \ln\Lambda/\mu) = z(\alpha) Z(\alpha, \ln\Lambda/\mu).$$

the renormalization group functions (RGFs) change nontrivially

A. A. Vladimirov, Teor. Mat. Fiz. **25** (1975), 335;
Sov. J. Nucl. Phys. **31** (1980), 558.

$$\beta'(\alpha') = \frac{d\alpha'}{d\alpha}\beta(\alpha); \qquad \gamma'(\alpha') = \frac{d\ln z}{d\alpha}\beta(\alpha) + \gamma(\alpha).$$

The perturbative expansions of RGFs can be written as

$$\beta(\alpha) = \sum_{n=1}^{\infty} \beta_n \alpha^{n+1}; \qquad \gamma(\alpha) = \sum_{n=1}^{\infty} \gamma_n \alpha^n,$$

where the index n numerates an order of the perturbation theory.

In the β-function the coefficients β_1 and β_2 are scheme independent. In the anamolous dimension only the first coefficient γ_1 is scheme independent.

Scheme dependence of the NSVZ equation

Note that in the $\overline{\text{DR}}$-scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the β-function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138;
Nucl.Phys. B **486** (1997) 479;
R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP **0612** (2006) 024.

However, in this case it is possible to make a special redefinition of the coupling constant which restore the NSVZ relation, because some scheme-independent consequences of the NSVZ equation are satisfied

A. L. Kataev and K.S., Phys. Lett. B **730** (2014), 184;
Theor. Math. Phys. **181** (2014), 1531.

All-loop NSVZ schemes have been constructed with the help of the higher covariant derivative regularization

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

The HD+MSL scheme

In the supersymmetric case the higher covariant derivative regularization can be formulated in terms of superfields and, therefore, does not break supersymmetry

V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978), 745;
P. C. West, Nucl. Phys. B **268** (1986), 113.

In this case (logarithmic) divergences are given by powers of $\ln \Lambda/\mu$, where Λ is a dimensionful regularization parameter, and μ is the renormalization point.

The NSVZ β-function is valid in all loops if a supersymmetric theory is regularized by Higher covariant Derivatives and renormalization is made by Minimal Subtraction of Logarithms (the so-called HD+MSL scheme), see

K.S., Eur. Phys. J. C **80** (2020) no.10, 911.

and references therein.

The whole class of the NSVZ renormalization prescriptions can be obtained from the HD+MSL scheme by making finite renormalizations which satisfy a special constraint

I. O. Goriachuk, A. L. Kataev and K.S., Phys. Lett. B **785** (2018), 561;
I. O. Goriachuk and A. L. Kataev, arXiv:1912.11700 [hep-th].

The NSVZ relation for $\mathcal{N}=1$ SQED

Let us investigate the scheme dependence of the NSVZ equation. In the simplest case of $\mathcal{N}=1$ SQED with N_f flavors

$$S=\frac{1}{4e^2}\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_\alpha^*e^{2V}\phi_\alpha+\widetilde\phi_\alpha^*e^{-2V}\widetilde\phi_\alpha\Big)$$

the NSVZ β-function takes the form

$$\beta(\alpha)=\frac{\alpha^2N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M. A. Shifman, A. I. Vainshtein and V. I. Zakharov,
JETP Lett. **42** (1985) 224; Phys. Lett. B **166** (1986) 334.

Expressions for the three-loop β-function and the two-loop anomalous dimension of the matter superfields for $\mathcal{N}=1$ SQED can be found in

A. L. Kataev and K.S., Phys. Lett. B **730** (2014) 184;
Theor. Math. Phys. **181** (2014) 1531.

Three-loop NSVZ relation in $\mathcal{N}=1$ SQED

The HD+MSL-scheme

$$\widetilde{\gamma}_{\text{HD+MSL}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{HD+MSL}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3)\Big).$$

The MOM-scheme (The result is the same of dimensional reduction and the higher derivative regularization.)

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

The $\overline{\text{DR}}$-scheme

I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. **B386** (1996) 138.

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

A class of the NSVZ schemes in $\mathcal{N}=1$ SQED

In general, under the finite renormalizations

$$\alpha \to \alpha'(\alpha); \qquad Z'(\alpha',\Lambda/\mu) = z(\alpha) Z(\alpha,\Lambda/\mu)$$

(in the supersymmetric case in our notation $\phi_R \equiv Z^{-1/2}\phi$) the NSVZ relation changes as

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2 N_f}{\pi}\cdot\frac{1-\widetilde{\gamma}'(\alpha')}{1-\alpha^2 N_f (d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}.$$

D. Kutasov and A. Schwimmer, Nucl. Phys. **B 702** (2004) 369.

However, there is an infinite set of finite renormalizations under which the NSVZ relation is invariant.

I.O.Goriachuk, A.L.Kataev and K.S., Phys. Lett. B **785** (2018), 561.

For $\mathcal{N}=1$ SQED they are obtained if the finite functions $\alpha'(\alpha)$ and $z(\alpha)$ satisfy the equation

$$\frac{1}{\alpha'(\alpha)} - \frac{1}{\alpha} - \frac{N_f}{\pi}\ln z(\alpha) = B,$$

where B is a constant.

The NSVZ equations can also be written for theories with multiple gauge couplings,

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046.

In the particular case $q_{\mathrm{a}} = 1$ for $\mathcal{N} = 1$ SQCD+SQED they take the form

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s,\alpha)\Big)\Big];$$

$$\frac{\beta(\alpha,\alpha_s)}{\alpha^2} = \frac{1}{\pi}\,\mathrm{dim}\,R\,N_f\Big(1-\gamma(\alpha_s,\alpha)\Big),$$

where we took into account that the representation for the matter superfields is irreducible, so that in the case under consideration

$$\gamma(\alpha_s,\alpha)_i{}^j = \gamma(\alpha_s,\alpha)\cdot\delta_i^j,$$

where i and j include both the indices numerating chiral matter superfields ϕ_{a} and $\widetilde{\phi}_{\mathrm{a}}$ and the indices corresponding to the representation R (or $\overline{R}$).

Comparing the above expressions for the β-functions we see that the anomalous dimension of the matter superfeilds can be excluded.

The equation relating the renormalization group functions

After excluding the anomalous dimension of the matter superfields we obtain that the β-functions satisfy the all-order exact equation

$$\Big(1-\frac{C_2\alpha_s}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{\beta(\alpha,\alpha_s)}{\alpha^2}.$$

Evidently, this equation is valid in the HD+MSL scheme, because the original NSVZ equations are satisfied for this renormalization prescription.

Taking into account the boundary conditions for the HD+MSL scheme it is possible to integrate the relation between the β-functions over μ. Then we obtain the equation which relates running of strong and electromagnetic couplings in the theory under consideration.

$$\frac{1}{\alpha_s} - \frac{1}{\alpha_{s0}} + \frac{C_2}{2\pi}\ln\frac{\alpha_s}{\alpha_{0s}} = -\frac{3C_2}{2\pi}\ln\frac{\Lambda}{\mu} + \frac{T(R)}{\mbox{dim}\, R}\Big(\frac{1}{\alpha}-\frac{1}{\alpha_0}\Big).$$

This in particular implies that the expression

$$\Big(\frac{\alpha_s}{\mu^3}\Big)^{C_2}\exp\Big(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{2\pi}{\alpha}\Big) = \mbox{RGI},$$

is the renormalization group invariant, i.e. the expression which vanishes afer differentiating with respect to $\ln\mu$.

$\mathcal{N}=1$ SQCD+SQED with different $U(1)$ changes

Let us again consider the theory in which the matter superfields have different $U(1)$ charges q_{a},

$$S = \frac{1}{2g^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4e^2}\,\mathrm{Re}\int d^4x\, d^2\theta\, \boldsymbol{W}^a \boldsymbol{W}_a$$
$$+ \sum_{\mathsf{a}=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\phi_{\mathsf{a}}^+ e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}} + \widetilde{\phi}_{\mathsf{a}}^+ e^{-2V-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde{\phi}_{\mathsf{a}}\Big)$$

and investigate the limit $\alpha = e^2/4\pi \to 0$. In this case the renormalization group running of the strong coupling constant α_s is exactly the same as in usual $\mathcal{N}=1$ SQCD with the gauge group G and N_f flavors. The running of the electromagnetic coupling constant is described by the Adler D-function

S. L. Adler, Phys. Rev. D **10** (1974), 3714.

which is related to the β-function for the coupling constant α in the limit $\alpha\to 0$,

$$D(\alpha_s) = \frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

In the limit $\alpha \to 0$ the anomalous dimensions of the matter superfields do not depend on α and, therefore, on q_{a}. This implies that in this case all anomalous dimensions of chiral matter superfields are the same,

$$\lim_{\alpha\to 0} \gamma_{\mathsf{a}}(\alpha_s,\alpha) = \gamma(\alpha_s).$$

Then the NSVZ β-function for $\mathcal{N}=1$ SQCD takes the form

$$\frac{\beta_s(\alpha_s)}{\alpha_s^2} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s)\Big)\Big].$$

The exact NSVZ-like expression for the Adler D-function in the theory under consideration has been derived in

M. Shifman and K.S., Phys. Rev. Lett. **114** (2015) 051601; Phys. Rev. D **91** (2015), 105008.

$$D(\alpha_s) = \frac{3}{2}\,\mbox{dim}\,R\sum_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2\Big(1-\gamma(\alpha_s)\Big) \equiv \frac{3}{2}\,\boldsymbol{q^2}\,\mbox{dim}\,R\Big(1-\gamma(\alpha_s)\Big),$$

where $\boldsymbol{q^2} \equiv \sum\limits_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2.$

$\mathcal{N}=1$ SQCD+SQED with different $U(1)$ changes

Therefore, the β-function of $\mathcal{N}=1$ SQCD can be related to the Adler D-function by the all-loop equation

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\left[3C_2 - \frac{4\,T(R)N_f D(\alpha_s)}{3\,\boldsymbol{q^2}\dim R}\right],$$

which connects the renormalization group running of the strong and electromagnetic coupling constants in the limit $\alpha\to 0$. Evidently this equation is valid in the HD+MSL scheme in all orders.

Thus, from the NSVZ equation we see that

1. If all $U(1)$ charges q_{a} are the same, then in the $\mathcal{N}=1$ SQCD+SQED, which is a theory with two gauge couplings, it is possible to relate their running.

2. If the charges q_{a} are different, then it is possible to relate the β-function of $\mathcal{N}=1$ SQCD and the Adler D-function. Actually, in this case the exact relation exists only in the limit $\alpha\to 0$.

Perturbative verification, dimensional technique in the supersymmetric case

It is desirable to verify the above exact equation by explicit perturbative calculations. The most popular renormalization prescription in the supersymmetric case is the $\overline{\text{DR}}$ scheme. The matter is that dimensional regularization

G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972), 189;
C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12** (1972), 20;
J. F. Ashmore, Lett. Nuovo Cim. **4** (1972), 289;
G. M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972), 329.

explicitly breaks supersymmetry, because the numbers of boson and fermion degrees of freedom differently depend on the space-time dimension.
That is why in the supersymmetric case it is more convenient to use its modification called dimensional reduction

W. Siegel, Phys. Lett. B **84** (1979), 193.

In this case the γ-matrices are taken in the integer dimension (usually, $D = 4$), while the loop integrals are calculated in the dimension $D = 4 - \varepsilon$.

$\overline{\text{DR}}$ scheme is obtained if the dimensional reduction is supplemented by modified minimal subtraction.

The equation relating the β-function of $\mathcal{N}=1$ SQCD to the Adler D-function can be verified in the three-loop approximation (where the scheme dependence is essential) using the results known in the literature. The three-loop β-function for $\mathcal{N}=1$ SQCD in the $\overline{\text{DR}}$ scheme can be found from the results of

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138; Nucl.Phys. B **486** (1997) 479.

and is given by the expression

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha_s^3}{4\pi^2}\Big(-3(C_2)^2 + 2N_f C_2 T(R) + 4N_f C(R)\,T(R)\Big) + \frac{\alpha_s^4}{8\pi^3}\bigg(-3(C_2)^2 + 2N_f (C_2)^2\,T(R) - 4N_f C(R)^2 \times T(R) + 13 N_f C_2\,C(R)\,T(R) - 6(N_f)^2 C(R) T(R)^2\bigg) + O(\alpha_s^5).$$

For the theory under consideration the Adler D-function in the $\overline{\text{DR}}$ scheme (for both α and α_s) can be constructed from the expression derived in

S. S. Aleshin, A. L. Kataev and K.S., JHEP **03** (2019), 196

The result can be presented in the form

$$D(\alpha_s) = \frac{3}{2}\boldsymbol{q^2}\,\text{dim}\,R\,\Big\{1 + \frac{\alpha_s}{\pi}C(R) - \frac{\alpha_s^2}{2\pi^2}C(R)^2 + \frac{9\alpha_s^2}{8\pi^2}C_2C(R)$$
$$-\frac{3\alpha_s^2N_f}{4\pi^2}C(R)\,T(R) + O(\alpha_s^3)\Big\}.$$

In this case

$$-\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\left[3C_2 - \frac{4\,T(R)N_f\,D(\alpha_s)}{3\,\boldsymbol{q^2}\,\text{dim}\,R}\right] =$$
$$-\frac{\alpha_s^2}{2\pi}\Big(3C_2 - 2N_fT(R)\Big) + \frac{\alpha_s^3}{4\pi^2}\Big(-3(C_2)^2 + 2N_fC_2T(R)$$
$$+4N_fC(R)\,T(R)\Big) + \frac{\alpha_s^4}{8\pi^3}\bigg(-3(C_2)^2 + 2N_f(C_2)^2\,T(R) - 4N_fC(R)^2$$
$$\times T(R) + 13N_fC_2\,C(R)\,T(R) - 6(N_f)^2C(R)T(R)^2\bigg) + O(\alpha_s^5)$$

and we see that the exact relation under consideration is unexpectedly satisfied.

In the HD+MSL scheme the β-function for $\mathcal{N}=1$ SQCD can be constructed from the general result obtained in

A. Kazantsev and K.S., JHEP **06** (2020), 108.

and is given by the expression

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha_s^3}{4\pi^2}\Big(-3(C_2)^2 + 2N_f C_2 T(R)$$
$$+4N_f C(R)\,T(R)\Big) + \frac{\alpha_s^4}{8\pi^3}\bigg(-3(C_2)^2 + 2N_f (C_2)^2\,T(R) - 4N_f C(R)^2$$
$$\times T(R) + 4N_f C_2\,C(R)\,T(R)\Big(3\ln a_\varphi + 4 + \frac{3A}{2}\Big) - 8(N_f)^2 C(R)T(R)^2$$
$$\times\Big(\ln a + 1 + \frac{A}{2}\Big)\bigg) + O(\alpha_s^5),$$

where A, a_φ, and a are the regularization parameters which depend on the higher derivative regulator and the Pauli–Villars masses.

Three-loop verification, HD+MSL scheme

For the same renormalization prescription the Adler D-function has been calculated in

A. L. Kataev, A. E. Kazantsev and K.S., Nucl. Phys. B **926** (2018), 295.

and for an irreducible representation R can be presented in the form

$$D(\alpha_s) = \frac{3}{2}\boldsymbol{q^2}\,\mbox{dim}\,R\,\Big\{1+\frac{\alpha_s}{\pi}C(R) - \frac{\alpha_s^2}{2\pi^2}C(R)^2 + \frac{3\alpha_s^2}{2\pi^2}C_2 C(R)$$

$$\times\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) - \frac{\alpha_s^2 N_f}{\pi^2} C(R)\,T(R)\,\Big(\ln a + 1 + \frac{A}{2}\Big) + O(\alpha_s^3)\Big\}.$$

Comparig this expression with the above β-function of $\mathcal{N}=1$ SQCD we see that the relation

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - \frac{4\,T(R) N_f D(\alpha_s)}{3\,\boldsymbol{q^2}\,\mbox{dim}\,R}\Big],$$

is satisfied in the considered approximation.

Minimal subtractions of logarithms can supplement various versions of the higher covariant derivative regularization, so that there is a class of HD+MSL schemes. In general, various renormalization prescriptions of this class can be related by finite renormalizations

A. A. Vladimirov and D. V. Shirkov, Sov. Phys. Usp. **22** (1979), 860.

Let us reveal under which finite renormalizations the relation between the β-function of $\mathcal{N}=1$ SQCD and the Adler D-function remains unbroken.
In the limit $\alpha\to 0$ general finite renormalizations can be presented in the form

$$\alpha'_s=\alpha'_s(\alpha_s);\qquad (\alpha')^{-1}=\alpha^{-1}+f(\alpha_s),$$

where $\alpha'_s(\alpha_s)$ and $f(\alpha_s)$ are finite functions of the strong coupling constant.
Requiring that the relation between the coupling constants in the limit $\alpha\to 0$ remains unbroken, we obtain the equation relating these functions,

$$\frac{1}{\alpha'_s(\alpha_s)}-\frac{1}{\alpha_s}+\frac{C_2}{2\pi}\ln\frac{\alpha'_s(\alpha_s)}{\alpha_s}=\frac{T(R)}{\dim R}\,f(\alpha_s).$$

Let us describe a group formed by the above finite renormalizations using the method proposed in

A. Kataev and K.S., arXiv:2404.15856 [hep-th].

For this purpose we present the infinitesimal changes of the coupling constants (in the limit $\alpha \to 0$) under the finite renormalizations in the form

$$\delta\alpha_s = -\sum_{n=1}^{\infty} a_{s,n}(\alpha_s)^{n+1} \equiv \sum_{n=1}^{\infty} a_{s,n}\hat{L}_{s,n}\alpha_s;$$

$$\delta\alpha = -\alpha^2\sum_{n=0}^{\infty} a_n(\alpha_s)^n \equiv \sum_{n=2}^{\infty} a_n\hat{L}_n\alpha,$$

where the operators

$$\hat{L}_{s,n} = -(\alpha_s)^{n+1}\frac{\partial}{\partial\alpha_s};\qquad \hat{L}_m = -\alpha^2(\alpha_s)^m\frac{\partial}{\partial\alpha}$$

with $n\geq 1$; $m\geq 0$ satisfy the commutation relations

$$[\hat{L}_{s,n},\hat{L}_{s,m}] = (n-m)\hat{L}_{s,n+m};\qquad [\hat{L}_{s,n},\hat{L}_m] = -m\hat{L}_{n+m};\qquad [\hat{L}_n,\hat{L}_m] = 0.$$

From the above constraint on the parameters of the finite renormalization we obtain the relations between the coefficients $a_{s,n}$ and a_n,

$$a_0 = \frac{\dim R}{T(R)} a_{s,1}; \qquad a_n = \frac{\dim R}{T(R)}\Big(a_{s,n+1} + \frac{C_2}{2\pi} a_{s,n}\Big), \quad n\ge 1.$$

Therefore, the renormalization prescriptions in which the considered exact equation is valid are related by finite renormalizations generated by the operators

$$\hat{\boldsymbol{L}}_{\boldsymbol{0}} \equiv \hat{L}_{s,1} + \frac{\dim R}{T(R)}\,\hat{L}_0; \qquad \hat{\boldsymbol{L}}_{\boldsymbol{n}} \equiv \hat{L}_{s,n} + \frac{\dim R}{T(R)}\,\Big(\hat{L}_{n-1} + \frac{C_2}{2\pi}\hat{L}_n\Big), \quad n\ge 1.$$

Then it is easy to verify that these generators satisfy the commutation relations of the Witt algebra

$$[\hat{\boldsymbol{L}}_n, \hat{\boldsymbol{L}}_m] = (n-m)\hat{\boldsymbol{L}}_{n+m}.$$

However, in the Witt algebra n is an arbitrary positive integer, while in the case under consideration $n\ge 0$. (This is the Borel subalgebra of the Witt algebra.)

The non-infinitesimal finite renormalizations can evidently be obtained with the help of the exponential map,

$$\alpha_s' = \exp\Big(\sum_{n=0}^{\infty} \boldsymbol{a}_n \hat{\boldsymbol{L}}_n\Big)\alpha_s; \qquad \alpha' = \exp\Big(\sum_{n=0}^{\infty} \boldsymbol{a}_n \hat{\boldsymbol{L}}_{\boldsymbol{n}}\Big)\alpha.$$

Conclusion

- We demonstrated that in $\mathcal{N}=1$ SQCD interacting with $\mathcal{N}=1$ SQED there are the exact all-loop equations relating the renormalization group running of the strong and electromagnetic coupling constants.
- If all matter superfields have the same absolute values of electromagnetic charges, then it is possible to relate two β-functions of the theory under consideration. In this case from the strong and electromagnetic coupling constants one can construct a renormalization group invariant.
- If the (absolute values of the) electromagnetic charges of matter superfields are different, then it is possible to construct an equation relating the β-function of $\mathcal{N}=1$ SQCD to the Adler D-function.
- All exact equations constructed here are valid in the HD+MSL schemes, when the theory is regularized by higher derivatives, and divergences are removed by minimal subtractions of logarithms.
- The whole class of the schemes in which the considered relations are valid is wider. Various schemes for that such relations are valid can be obtained from an HD+MSL scheme by finite renormalizations which satisfy a certain constraint. These finite renormalizations form an infinite dimensional Lie group, the corresponding Lie algebra being the Borel subalgebra of the Witt algebra.

Thank you for the attention!