Generalized fluid dynamics with non-relativistic conformal symmetries

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Plan

1. Non-relativistic perfect fluid and its symmetries
2. Schrödinger algebra and ℓ–conformal Galilei algebra
3. Generalized non-relativistic perfect fluid equations
4. Hamiltonian formulation TS, arXiv:2302.01565 (J. Geom. Phys.)
5. Lagrangian formulation TS, arXiv:2406.02952 (Phys. Rev. D)

1. Non-relativistic perfect fluid and its symmetries

Perfect fluid equations

In non-relativistic space-time $(t, x_i),\ i = 1, ..., d$ a compressible fluid is characterized by a density $\rho(t, x)$ and the velocity $\upsilon_i(t, x)$, which both enter the continuity equation

$$\partial_0 \rho + \partial_i(\rho \upsilon_i) = 0. \tag{1}$$

The perfect fluid dynamics is described by the Euler equation

$$\mathcal{D}\upsilon_i = -\frac{1}{\rho}\partial_i p, \quad \text{where} \quad \mathcal{D} = \partial_0 + \upsilon_i \partial_i \tag{2}$$

The pressure $p(t, x)$ is assumed to be related to $\rho(t, x)$ via an equation of state

$$p = p(\rho) \tag{3}$$

1. Non-relativistic perfect fluid and its symmetries

Hamiltonian formulation

The Hamiltonian=energy reads

$$H = \int dx \left(\frac{1}{2} \rho \upsilon_i \upsilon_i + V \right), \quad p = \rho V' - V \tag{4}$$

It generates the continuity equation and the Euler equation in the usual way

$$\partial_0 \rho = \{\rho, H\} = -\partial_i (\rho \upsilon_i), \qquad \partial_0 \upsilon_i = \{\upsilon_i, H\} = -\upsilon_j \partial_j \upsilon_i - \frac{1}{\rho} \partial_i p \tag{5}$$

provided the non-canonical Poisson brackets for ρ and υ_i are chosen
P. Morrison, J. Greene '80

$$\begin{aligned} \{\rho(x), \upsilon_i(y)\} &= -\partial_i \delta(x-y), \\ \{\upsilon_i(x), \upsilon_j(y)\} &= \frac{1}{\rho} \left(\partial_i \upsilon_j - \partial_j \upsilon_i \right) \delta(x-y). \end{aligned} \tag{6}$$

1. Non-relativistic perfect fluid and its symmetries

Conserved charges

Conserved energy, momentum, angular momentum, Galilei boost

$$H = \int dx \left(\frac{1}{2} \rho \upsilon_i \upsilon_i + V \right) \qquad P_i = \int dx \rho \upsilon_i$$

$$M_{ij} = \int dx \rho (x_i \upsilon_j - x_j \upsilon_i) \qquad C_i = P_i t - \int dx \rho x_i$$

For suitable equation of state L. O'Raifeartaigh, V. Sreedhar '01

$$\boxed{p = \nu \rho^{1+\frac{2}{d}}} \tag{7}$$

two more conserved charges exist corresponding to dilatation and special conformal transformation

$$D = tH - \frac{1}{2} \int \rho x_i \upsilon_i, \quad K = t^2 H - 2tD - \frac{1}{2} \int \rho x_i x_i \tag{8}$$

1. Non-relativistic perfect fluid and its symmetries

Algebra of conserved charges

Conserved charges under Poisson brackets form the Schrödinger algebra

$$\{H, D\} = H \qquad \{H, C_i\} = P_i$$
$$\{H, K\} = 2D \qquad \{D, C_i\} = \frac{1}{2}C_i \qquad \{D, P_i\} = -\frac{1}{2}P_i$$
$$\{D, K\} = K \qquad \{K, P_i\} = -C_i$$

with following rotation sector

$$\{M_{ij}, P_k\} = -\delta_{k[i}P_{j]}, \quad \{M_{ij}, C_k\} = -\delta_{k[i}C_{j]}, \quad \{M_{ij}, M_{kl}\} = -\delta_{ik}M_{jl} + ...$$

and centrally extended sector

$$\{P_i, C_j\} = \delta_{ij}M$$

Here H, D, K form the conformal $so(2,1)$ subalgebra. M is central charge corresponding to total mass $M = \int dx\rho$.

2. Schrödinger algebra and ℓ–conformal Galilei algebra

- The Schrödinger algebra is conformal extension of Galilei algebra which has been found to be relevant for a wide range of physical applications.
- However it does not reproduce the non-relativistic contraction of the relativistic conformal algebra.
- This stimulates interest in the study of other finite-dimensional conformal extensions of the Galilei algebra which are combined into a family known in the literature as the ℓ-conformal Galilei algebra J. Negro, M. del Olmo, A. Rodriguez-Marco '97
- Dynamical systems with the ℓ-conformal Galilei symmetries is of potential intrest in context of non-relativistic AdS/CFT.

2. Schrödinger algebra and ℓ–conformal Galilei algebra

The structure relations of ℓ-conformal Galilei algebra read

$$
\begin{aligned}
&[H, D] = H && [H, C_i^{(k)}] = kC_i^{(k-1)} \\
&[H, K] = 2D && [D, C_i^{(k)}] = (k-\ell)C_i^{(k)} \\
&[D, K] = K && [K, C_i^{(k)}] = (k-2\ell)C_i^{(k+1)}
\end{aligned}
$$

Realization in non-relativistic space-time (t, x_i)

$$
H = \partial_0, \quad D = t\partial_0 + \ell x_i\partial_i, \quad K = t^2\partial_0 + 2\ell t x_i\partial_i, \quad C_i^{(k)} = t^k\partial_i
$$

- To be finite-dimensional $k = 0, 1, .., 2\ell \;\Rightarrow\; \ell$ is (half)-integer. ℓ is sometimes called as conformal "spin".
- $C_i^{(k)}$ correspond to translation and Galilei boost for $k = 0, 1$ and accelerations for $k > 1$.
- $\ell = 1/2$ is the Schrödinger algebra, $\ell = 1$ is the non-relativistic limit of conformal algebra $so(2, d+1)$.

Example

Higher derivative Pais-Uhlenbeck oscillator Pais, Uhlenbeck 1950

$$\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)x_i(t)=0, \quad 0<\omega_1<...<\omega_n$$

enjoys the ℓ-conformal Galilei symmetry for a special choice of its frequencies Andrzejewski, Galajinsky, Gonera, Masterov '14

$$\omega_k=(2k-1)\omega_1, \quad k=1,...,n$$

with $\ell=n-\frac{1}{2}$

3. Generalized non-relativistic perfect fluid equations

Generalized perfect fluid equations which hold invariant under the action of the ℓ-conformal Galilei group were formulated by A. Galajinsky '22

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \upsilon_i)}{\partial x_i} = 0, \quad \mathcal{D}^{2\ell} \upsilon_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad p = \nu \rho^{1+\frac{1}{\ell d}}.$$

The energy density $\boxed{\ell = n + \frac{1}{2}}$

$$T^{00} = \frac{1}{2} \rho \sum_{k=0}^{2n} (-1)^k \mathcal{D}^k \upsilon_i \mathcal{D}^{2n-k} \upsilon_i + V, \quad V = \ell d p$$

Given a set of equations of motion, it is always desirable to have a Hamiltonian and Lagrangian formulation. The goal is to elaborate on this issue.

4. Hamiltonian formulation

To construct Hamiltonian formulation TS '23 we rewrite generalized equations in the equivalent first order form

$$\partial_0\rho + \partial_i(\rho \upsilon_i^{(0)}) = 0, \quad \mathcal{D}\upsilon_i^{(k)} = \upsilon_i^{(k+1)}, \quad \mathcal{D}\upsilon_i^{(2n)} = -\frac{1}{\rho}\partial_i p$$

with auxiliary fields $\upsilon_i^{(k)}$, $k = 0, 1, ..., 2n$, where $\upsilon_i^{(0)} = \upsilon_i$.

$$H = \int dx T^{00} = \int dx \left(\frac{1}{2}\rho \sum_{k=0}^{2n} (-1)^k \upsilon_i^{(k)} \upsilon_i^{(2n-k)} + V \right).$$

Equations in the Hamiltonian form

$$\begin{aligned}
\partial_0\rho &= \{\rho, H\} = -\partial_i(\rho \upsilon_i^{(0)}) \\
\partial_0\upsilon_i^{(k)} &= \{\upsilon_i^{(k)}, H\} = -\upsilon_j^{(0)}\partial_j\upsilon_i^{(k)} + \upsilon_i^{(k+1)} \\
\partial_0\upsilon_i^{(2n)} &= \{\upsilon_i^{(2n)}, H\} = -\upsilon_j^{(0)}\partial_j\upsilon_i^{(2n)} - \frac{1}{\rho}\partial_i p.
\end{aligned}$$

4. Hamiltonian formulation

Poisson brackets

$$
\begin{aligned}
\{\rho(x), \upsilon_i^{(k)}(y)\} &= -\delta_{(k)(2n)}\partial_i\delta(x-y) \\
\{\upsilon_i^{(k)}(x), \upsilon_j^{(m)}(y)\} &= \frac{1}{\rho}\left(\delta_{(k)(2n)}\partial_i\upsilon_j^{(m)} - \delta_{(m)(2n)}\partial_j\upsilon_i^{(k)}\right)\delta(x-y) \\
&\quad -(-1)^k\frac{1}{\rho}\delta_{(k+m)(2n-1)}\delta_{ij}\delta(x-y)
\end{aligned}
$$

where $\delta_{(k)(m)}$ and δ_{ij} are the Kronecker symbols.

4. Hamiltonian formulation

$\ell = 3/2$ example

Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \upsilon_i^{(0)})}{\partial x_i} = 0, \quad \mathcal{D}\upsilon_i^{(0)} = \upsilon_i^{(1)}, \quad \mathcal{D}\upsilon_i^{(1)} = \upsilon_i^{(2)}, \quad \mathcal{D}\upsilon_i^{(2)} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

Hamiltonian

$$H = \int dx \left(\rho \upsilon_i^{(0)} \upsilon_i^{(2)} - \frac{1}{2}\rho \upsilon_i^{(1)} \upsilon_i^{(1)} + V \right),$$

Poison brackets

$$\{\rho(x), \upsilon_i^{(2)}(y)\} = -\partial_i \delta(x-y), \quad \{\upsilon_i^{(0)}(x), \upsilon_j^{(2)}(y)\} = -\frac{1}{\rho}\partial_j \upsilon_i^{(0)} \delta(x-y)$$

$$\{\upsilon_i^{(0)}(x), \upsilon_j^{(1)}(y)\} = -\frac{1}{\rho}\delta_{ij}\delta(x-y), \quad \{\upsilon_i^{(1)}(x), \upsilon_j^{(2)}(y)\} = -\frac{1}{\rho}\partial_j \upsilon_i^{(1)} \delta(x-y)$$

$$\{\upsilon_i^{(2)}(x), \upsilon_j^{(2)}(y)\} = \frac{1}{\rho}\left(\partial_i \upsilon_j^{(2)} - \partial_j \upsilon_i^{(2)}\right)\delta(x-y)$$

4. Hamiltonian formulation

Conserved charges corresponding temporal translation, dilatation, special conformal transformations and vector generators read

$$
\begin{aligned}
H &= \int dx \left(\frac{1}{2}\rho \sum_{k=0}^{2n} (-1)^k \upsilon_i^{(k)} \upsilon_i^{(2n-k)} + V(p) \right) \\
D &= tH - \frac{1}{2} \int dx \rho \sum_{k=0}^{2n} (-1)^k (k+1) \upsilon_i^{(k)} \upsilon_i^{(2n-k-1)} \\
K &= t^2 H - 2tD \\
&\quad - \frac{1}{2} \int dx \rho \sum_{k=0}^{2n} (-1)^k \Big[(n+1)(2n+1) - k(k+1) \Big] \upsilon_i^{(k-1)} \upsilon_i^{(2n-k-1)} \\
C_i^{(k)} &= \sum_{s=0}^{k} (-1)^s \frac{k!}{(k-s)!} t^{(k-s)} \int dx \rho \upsilon_i^{(2n-s)}
\end{aligned}
$$

4. Hamiltonian formulation

They satisfy the structure relations of the ℓ-conformal Galilei algebra under Poisson brackets

$$\{H,D\} = H \qquad \{H, C_i^{(k)}\} = kC_i^{(k-1)}$$
$$\{H,K\} = 2D \qquad \{D, C_i^{(k)}\} = (k-\ell)C_i^{(k)}$$
$$\{D,K\} = K \qquad \{K, C_i^{(k)}\} = (k-2\ell)C_i^{(k+1)}$$

with central extensions A. Galajinsky, I. Masterov '11

$$\{C_i^{(k)}, C_j^{(m)}\} \;=\; (-1)^k k!m!\delta_{(k+m)(2n+1)}\delta_{ij}M, \quad M = \int dx\rho,$$

5. Lagrangian formulation

In order to demonstrate how the generalized perfect fluid equations can be obtained from the variational principle, let us first recall how the Lagrangian for a perfect fluid is built which correctly reproduces the continuity equation and the Euler equation (see e.g. review R. Jackiw, V.P. Nair, S.Y. Pi, A.P. Polychronakos '04)

$$\partial_0\rho + \partial_i(\rho\upsilon_i) = 0, \quad \mathcal{D}\upsilon_i = -\frac{1}{\rho}\partial_i p. \tag{9}$$

In three spatial dimensions this is achieved by making recourse to the Clebsch parametrization for the velocity vector field

$$\upsilon_i = \partial_i\theta + \alpha\partial_i\beta, \tag{10}$$

which involves three scalar functions θ, α and β. Then the Lagrangian reads

$$\begin{aligned} L &= -\int dx\rho\left(\partial_0\theta + \alpha\partial_0\beta\right) - H \\ &= -\int dx\rho\left(\partial_0\theta + \alpha\partial_0\beta\right) - \int dx\left(\frac{1}{2}\rho\upsilon_i\upsilon_i + V\right), \end{aligned} \tag{11}$$

5. Lagrangian formulation

The Euler-Lagrangian equations

$$\delta_\theta L = 0 \quad \to \quad \partial_0 \rho + \partial_i(\rho \upsilon_i) = 0 \tag{12}$$

$$\delta_{\alpha,\beta} L = 0 \quad \to \quad \mathcal{D}\alpha = 0, \quad \mathcal{D}\beta = 0 \tag{13}$$

$$\delta_\rho L = 0 \quad \to \quad \mathcal{D}\theta - \frac{1}{2}\upsilon_i \upsilon_i + V'_\rho = 0 \tag{14}$$

As a result, the Euler equation are satisfied

$$\mathcal{D}\upsilon_i = \mathcal{D}(\partial_i \theta + \alpha \partial_i \beta) = -\frac{1}{\rho}\partial_i p, \quad p = \rho V'_\rho - V. \tag{15}$$

5. Lagrangian formulation

In order to generalize the construction above to the ℓ-conformal perfect fluid, we go over to the equivalent first order system. In the case of half-integer $\ell = n + \frac{1}{2}$, the starting equations read

$$\partial_0 \rho + \partial_i (\rho \upsilon_i^{(0)}) = 0, \tag{16}$$

$$\mathcal{D} \upsilon_i^{(k)} = \upsilon_i^{(k+1)}, \quad k = 0, 1, ..., 2n - 1, \tag{17}$$

$$\mathcal{D} \upsilon_i^{(2n)} = -\frac{1}{\rho} \partial_i p, \quad p = \nu \rho^{1 + \frac{1}{\ell d}}. \tag{18}$$

Now one has a set of vector variables $\upsilon_i^{(k)}$. What suitable Clebsch-type parametrization should be used?

5. Lagrangian formulation

It turns out that in order to obtain the generalized equations from the variational principle only the highest component $\upsilon_i^{(2n)}$ should be Clebsch-decomposed, while the remaining vector variables $\upsilon_i^{(k)}$ with $k < 2n$ may remain intact. Up to a field redefinition, a suitable Clebsch-type decomposition can be chosen in the form TS '24

$$\upsilon_i^{(2n)} = \partial_i \theta + \alpha \partial_i \beta + \sum_{k=0}^{n-1} (-1)^{k+1} \upsilon_j^{(k)} \partial_i \upsilon_j^{(2n-k-1)}. \tag{19}$$

When $n = 0$, there is no sum on the right hand side and the decomposition for the Euler fluid is reproduced. The generalized Lagrangian reads

$$L = -\int dx \rho \left(\partial_0 \theta + \alpha \partial_0 \beta + \sum_{k=0}^{n-1} (-1)^{k+1} \upsilon_i^{(k)} \partial_0 \upsilon_i^{(2n-k-1)} \right) - H,$$

5. Lagrangian formulation

Thus, the basic variables for the Lagrangian are the scalar fields ρ, θ, α, β and a set of vector fields $\upsilon_i^{(k)}$ with $k < 2n$. The Euler-Lagrangian equations

$$\delta_\theta L = 0 \quad \rightarrow \quad \partial_0 \rho + \partial_i(\rho \upsilon_i) = 0 \tag{20}$$

$$\delta_{\alpha,\beta} L = 0 \quad \rightarrow \quad \mathcal{D}\alpha = 0, \quad \mathcal{D}\beta = 0 \tag{21}$$

$$\delta_{\upsilon_i^{(k)}} L = 0 \quad \rightarrow \quad \mathcal{D}\upsilon_i^{(k)} = \upsilon_i^{(k+1)} \tag{22}$$

$$\delta_\rho L = 0 \quad \rightarrow \quad \mathcal{D}\theta - \upsilon_i^{(0)}\upsilon_i^{(2n)} + \frac{(-1)^n}{2}\upsilon_i^{(n)}\upsilon_i^{(n)} + V'_\rho = 0 \tag{23}$$

As a result the last equation

$$\mathcal{D}\upsilon_i^{(2n)} = \mathcal{D}\left(\partial_i\theta + \alpha\partial_i\beta + \sum_{k=0}^{n-1}(-1)^{k+1}\upsilon_j^{(k)}\partial_i\upsilon_j^{(2n-k-1)}\right) = -\frac{1}{\rho}\partial_i p,$$

is satisfied as well, where $p = \rho V'_\rho - V$.

5. Lagrangian formulation

$\ell = 3/2$ **example**

Basic variables ρ, θ, α, β and $\upsilon_i^{(0)}$, $\upsilon_i^{(1)}$. Lagrangian is

$$L = -\int dx \rho \left(\partial_0 \theta + \alpha \partial_0 \beta - \upsilon_i^{(0)} \partial_0 \upsilon_i^{(1)} \right) - H$$

with Hamiltonian

$$H = \int dx \left(\rho \upsilon_i^{(0)} \upsilon_i^{(2)} - \frac{1}{2} \rho \upsilon_i^{(1)} \upsilon_i^{(1)} + V \right),$$

where $\upsilon_i^{(2)} = \partial_i \theta + \alpha \partial_i \beta - \upsilon_j^{(0)} \partial_i \upsilon_j^{(1)}$.
Transition to Hamiltonian formulation lead to the second-class constraints and following Dirac brackets

$$\{\rho(x), \theta(y)\}_D = \delta(x-y), \quad \{\theta(x), \alpha(y)\}_D = \frac{\alpha}{\rho} \delta(x-y)$$

$$\{\theta(x), \upsilon_i^{(0)}(y)\}_D = \frac{\upsilon_i^{(0)}}{\rho} \delta(x-y), \quad \{\alpha(x), \beta(y)\}_D = \frac{1}{\rho} \delta(x-y)$$

$$\{\upsilon_i^{(0)}(x), \upsilon_j^{(1)}(y)\}_D = -\frac{1}{\rho} \delta_{ij} \delta(x-y).$$

Conclusion

- The Hamiltonian and Lagrangian formulation for the generalized perfect fluid equations with the ℓ-conformal Galilei symmetry was constructed.
- The peculiarity of the Hamiltonian formulation is that the Poisson brackets among the physical fields related to the fluid density and the velocity vector with its descendants are non-canonical.
- In order to identify canonical variables and go over to a Lagrangian description, the Clebsch-type parametrization should be used.
- The non-relativistic perfect fluid dynamics is reproduced for $\ell = \frac{1}{2}$
- It would be interesting to construct supersymmetric extensions.

THANK YOU FOR ATTENTION!