Differential equations for classical Virasoro blocks with heavy and light operators

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Introduction and motivation

Based on the following work
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Motivation: 4-pt Virasoro conformal blocks are functions which are unknown in closed form. But some (limiting or particular cases) the blocks are subjected by non-linear differential equations (NDE).

1. Global blocks ($sl(2,\mathbb{C})$) are hypergeometric functions (Dolan, Osborn'03)
2. Painlevé VI tau function can be interpreted as four-point correlator with $c = 1$ (Gamayun, Iorgov, Lisovyy'12). $c = -2$ case (Bershtein, Shchechkin'18) is related to the particular Painlevé III equation.
3. Non-linear equations determine blocks: a classical 4-pt block can be computed as a classical action of system, governed by the Painlevé VI equation (Litvinov, Lukyanov, Nekrasov, Zamolodchikov'13)

Our frame:

1. Classical blocks within the HL approximation
2. NDE for them

Plan

1. Definitions
2. NDE for the HHLL identity block and diagrams
3. Generalizations from the monodromy method
4. AdS/CFT correspondence

Classical Virasoro blocks and the monodromy method - I

A 4-pt conformal block ($\mathcal{F}_4(z|\Delta_i, \tilde{\Delta}, c), i = 1,...4$) is a contribution of one conformal family to the 4-pt correlation function of primaries. The *classical limit* of the block implies $c \to \infty$ and $\Delta_i/c, \tilde{\Delta}/c$ to be finite, and it was claimed that the block $\mathcal{F}_4(z|\Delta_i, \tilde{\Delta}, c)$ has the exponential form (Zamolodchikov'86; Besken, Datta, Kraus'19)

$$\mathcal{F}_4(z|\Delta_i, \tilde{\Delta}, c) = \exp\left[\frac{c}{6} f_4(z|\epsilon_i, \tilde{\epsilon})\right] + \mathcal{O}\left(\frac{1}{c}\right) \quad \text{at} \quad c \to \infty \,, \qquad (2.1)$$

where $\epsilon_i \equiv 6\Delta_i/c$, $\tilde{\epsilon} \equiv 6\tilde{\Delta}/c$ are internal/intermediate classical dimensions, and $f_n(z|\epsilon_i, \tilde{\epsilon})$ stands for the 4-pt classical conformal block. Within the monodromy method, we consider an auxiliary 5-pt block with the degenerate operator $V_{(2,1)}(y)$. The large-c contribution of $V_{(2,1)}(y)$ denoted by $\psi(y|z)$ satisfies the (classical) BPZ equation

$$\left[\frac{d^2}{dy^2} + T(y|z)\right]\psi(y|z) = 0 \,, \quad T(y|z) = \sum_{i=1}^{4}\left(\frac{\epsilon_i}{(y-z_i)^2} + \frac{c_i}{y-z_i}\right), \qquad (2.2)$$

where $c_i = \partial_{z_i} f_4(z|\epsilon_i, \tilde{\epsilon})$ are accessory parameters.

Classical Virasoro blocks and the monodromy method - II

On the other hand, one can consider a contour Γ, encircling points $\{0, z\}$. By traversing the argument y of the degenerate operator $V_{(2,1)}(y)$ along the contour, we should have the following monodromy for $\psi(y|z)$

$$\widetilde{M} = -\begin{pmatrix} e^{i\pi\gamma} & 0 \\ 0 & e^{-i\pi\gamma} \end{pmatrix}, \qquad \gamma = \sqrt{1 - 4\tilde{\epsilon}}. \tag{2.3}$$

In the light of these, the essence of the monodromy method is to find solutions of the BPZ equation with the prescribed monodromy along contour Γ. Essentially, it imposes algebraic relations on the accessory parameters c_i, which are called *monodromy equations*. Once these relations are resolved, we end up with a system of the first order PDEs, which can be integrated for finding the classical block.

HL approximation

It is possible to obtain classical blocks functions in closed form within the HL approximation (Fitzpatrick'14). More precisely, let $(n-k)$ heavy operators with classical dimensions ϵ_j have larger classical dimensions than remaining k

$$\epsilon_i \ll \epsilon_j \ , \qquad i = 1,..,k \ , \qquad j = k+1,..,n \ . \tag{2.4}$$

We mainly deal with $k = 1$ and $k = 2$, i.e. HHLL and HHHL blocks. Due to the $sl(2,\mathbb{C})$ invariance these blocks are functions of one cross ratio z.

We focus on the first order in the HL approximation only. In what follows, we will call blocks with $\tilde{\epsilon} = 0$ *identity blocks*.

Now, we turn to HHLL blocks $f_4(z|\epsilon_H)$ which are parameterized by two external classical dimensions ϵ_1, ϵ_2 and an intermediate dimension $\tilde{\epsilon}$. Heavy operators are located at points $(1, \infty)$.

the HHLL identity block and diagrams

The first example is a HHLL identity blocks $\tilde{\epsilon} = 0, \epsilon_1 = \epsilon_2 = \epsilon_L$ (Fitzpatrick, Kaplan, Walters and Wang'15). They study matrix elements between heavy and light operators, which are contributions to conformal blocks. These contributions can be written in terms of new on-shell tree diagrams. The Virasoro identity conformal block, which is the sum of all the tree diagrams, obeys a differential recursion relation generalizing that of the Catalan numbers

$$\frac{1}{2\epsilon_L}\left(\frac{d}{dz}\right)^2 \log F_4(z) = \frac{\epsilon_H}{(1-z)^2} + \frac{1}{4\epsilon_L^2}\left(\frac{d}{dz}\log F_4(z)\right)^2 . \tag{2.5}$$

The differential recursion relation above is a Riccati equation for the "dimensionless"accessory parameter

$$C'(z) - C^2(z) = T_2(z), \quad C(z) = \frac{\partial_z f(z|\epsilon, \tilde{\epsilon})}{2\epsilon_L}, \quad T_2(z) = \frac{\epsilon_H}{(1-z)^2}. \tag{2.6}$$

HHLL non-identity blocks from the monodromy method

Our idea is to step aside from the diagrammatic technique and try to derive ODEs for the HHLL non-identity block. We exploit the monodromy method itself, claiming that the ODE considered above is a differential consequence of the monodromy equations.
The monodromy equation for the HHLL block is (Fitzpatrick,14)

$$I_{+-}^{(2)} I_{-+}^{(2)} = -4\pi^2 \tilde{\epsilon}^2. \tag{2.7}$$

where

$$\frac{\alpha I_{+-}^{(2)}}{2\pi i} = \alpha\epsilon_1 + c_2(z)(1-z) - \epsilon_2 + (1-z)^{\alpha}\left(c_2(z)(1-z) - \epsilon_2(1+\alpha)\right),$$

$$I_{-+}^{(2)}[\alpha] = -I_{+-}^{(2)}[-\alpha], \qquad \alpha = \sqrt{1-4\epsilon_H}. \tag{2.8}$$

an ODE for HHLL non-identity blocks

One finds that

$$
\begin{gathered}
(C'(z) - C^2(z) - T_2(z))^2 = \kappa^2\Big(\frac{\alpha^2((1-z)C(z)-1/2)^2}{4} \\
-\left(\frac{\alpha^2-4(1-z)C(z)}{4(1-z)^2}+C'(z)\right)^2\Big), \qquad \kappa = \frac{\tilde{\epsilon}}{2\epsilon_L}.
\end{gathered} \tag{2.9}
$$

The change

$$
C(z) = \frac{i}{1-z}\left(\frac{i}{2} - k(\alpha w)\right), \quad w = i\log(1-z), \tag{2.10}
$$

yields (a dot stands for d/dw)

$$
\left(k(w)^2 - \dot{k}(w) + \frac{1}{4}\right)^2 = \kappa^2\left(\left(\frac{1}{4} - \dot{k}(w)\right)^2 + \frac{k(w)^2}{4}\right). \tag{2.11}
$$

HHHL blocks from the monodromy method

The same can be done for the HHHL blocks (Alkalaev, Pavlov'19). Consider a HHHL block with three heavy operators $\epsilon_2 \ll \epsilon_1, \epsilon_3 = \epsilon_4 = \epsilon_H$, heavy operators located at $(0{,}1, \infty)$, respectively. Also, $\tilde{\epsilon} = \epsilon_H$. The resulting ODE has the form

$$C''(z) = -C^3(z) + 3C(z)C'(z) - 4T_3(z)C(z) + 2T_3'(z), \qquad C(z) = \frac{c_2(z)}{\epsilon_2}, \tag{2.12}$$

where

$$T_3(z) = \frac{\epsilon_1}{z^2} + \frac{\epsilon_H}{(1-z)^2} + \frac{\epsilon_1}{z(1-z)}. \tag{2.13}$$

To consider a limit from HHHL blocks to HHLL blocks, we set $\epsilon_1 \to 0$ but for the latter blocks we assume that $\epsilon_1 \ll \epsilon_2 \ll \epsilon_H$. We checked that the ODE for such a HHLL block is a limit of the ODE for the HHHL block.

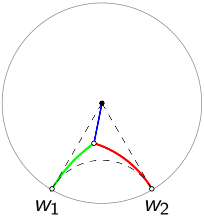
AdS/CFT for HL blocks

AdS/CFT for H^2L^{n-2} classical blocks (Alkalaev'15, Alkalaev'18)

$$f_n(z(w)|\epsilon_h,\epsilon,\tilde{\epsilon}) = -L_{n-1}(\alpha w|\epsilon,\tilde{\epsilon}) + i\sum_{k=1}^{n-2}\epsilon_k w_k\,, \quad z(w) = 1-\exp[-iw],$$

$$\alpha = \sqrt{1-4\epsilon_H}.$$

Here $L_{n-1}(\alpha w|\epsilon,\tilde{\epsilon})$ denotes the length of the Steiner tree, which is stretched on the Poincare disk with the conical defect ($\mathbb{D}_\alpha = \{t,\phi : t\in[0,1], \phi\in[0,2\pi\alpha)\}$); $(n-2)$ endpoints of the tree belong to the boundary and one endpoint is located at $t=0$.



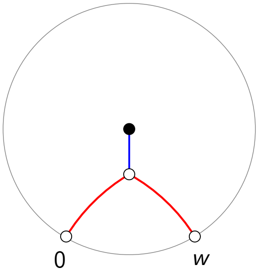
Derivation of ODEs from AdS/CFT - the HHLL blocks

Consider the "normalized"length of the tree below (the dual tree for HHLL blocks with $\epsilon_1 = \epsilon_2$)

$$l(w) = Y(w) + \kappa X(w), \tag{2.14}$$

where Y denotes the length of a red segment and X - a blue one. We find that $l(w)$ satisfies

$$\left(\dot{l}(w)^2 - \ddot{l}(w) + \frac{1}{4}\right)^2 = \kappa^2 \left(\left(\frac{1}{4} - \ddot{l}(w)\right)^2 + \frac{\dot{l}(w)^2}{4}\right). \tag{2.15}$$



Conclusion and future directions

Conclusion:

1. ODEs for the HHLL blocks and the HHHL blocks
2. AdS-like derivation

Future directions:

1. PDEs for many-point blocks
2. The next order corrections in heavy and light classical dimensions
3. Diagrammatic technique for non-identity HHLL blocks

Thank you for your attention!