σ-model and singular tau function

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- σ-model and instanton sum (Fateev, Frolov, Schwartz 1979) which is singular
- instanton sum $=$ free massless fermionic tau function - singular tau function !
- regularization of singular tau function $=$ introduction of mass
- answers for correlation functions in terms of Bessel functions
- $\bar{\partial}$-problem and Dirac equation

σ model (Two-dimensional quantum ferromagnetic)

This model can be described by the action

$$S = \frac{1}{2f} \int \sum_{a=1}^{3} \left(\partial_\mu \sigma^a (x)\right)^2 \tag{1}$$

where σ^a, $a = 1, 2, 3$ are the components of the unit vector:
$\sum_{a=1}^{3} \sigma^a (x) \sigma^a (x) = 1$; $\mu = 0, 1.$
Let us note that the classical σ-model in Minkowski space is the well studied integrable model, see S.V. Manakov, S. P. Novikov, L. Pitaevski and V. E. Zakharov, "Theory of solitons" Nauka, 1979

$$w = \frac{\sigma_1 + i\sigma_2}{1 + \sigma_3} : \; s_1 = 2\frac{\mathrm{Re}w}{1 + |w|^2}, \; s_2 = 2\frac{\mathrm{Im}w}{1 + |w|^2}, \; s_3 = \frac{1 - |w|^2}{1 + |w|^2}$$

The model similar to a Yang-Mills theory and posesses exact multi-instanton solutions. The Euclidean Green functions can be represented in the form

$$\frac{\int \phi(\sigma) \exp(-S) \prod_x d\sigma(x)}{\int \exp(-S) \prod_x d\sigma(x)} \tag{2}$$

Here $\phi(\sigma)$ is an arbitrary functional of σ. If we parametrize $\sigma(x)$ with use of the complex function

$$\omega(z) = \frac{\sigma^1(z) + i\sigma^2(z)}{1 + \sigma^3(z)} \tag{3}$$

(the stereographic projection) obtained from the field $(\sigma^1, \sigma^2, \sigma^3)$ and the complex variable $z = \frac{1}{2}(x_0 + ix_1)$ instead of the time and space coordinate x_0 , x_1, then the instanton is the solution of the equation $\delta S = 0$ with the topological charge $q > 0$ is given [1]

$$\omega_q(a, b, z) = c\frac{(z - a_1) \dots (z - a_q)}{(z - b_1) \dots (z - b_q)} \tag{4}$$

where c, a_i, b_i are arbitrary complex parameters.

Instantons in 2D ferromagnets: history

A.A. Belavin and A.M. Polyakov, "Metastable states of two-dimensional isotropic ferromagnets", Pis'ma Zh. Eksp. Teor. Fiz. **22** N10 (1975) pp 503-506

V.A. Fateev, I.V. Frolov, A.S. Schwarz, "Quantum Fluctuations of Instantons in the Nonlinear σ model", Nuclear Physics B **154** N 1 (1979) pp 1-20

A.P. Bukhvostov and L.N. Litpatov, "Instanton-antiinstanton interaction in nonlinear σ-model and certain exactly solvable fermionic theory", Pis'ma v ZhETF vol 31 N 2 (1980) pp 138-142 (in Russian)

Fateev-Frolov-Schwarz (FFS) answer for the instanton contribution

The evaluation of the functional integral around the instanton vacuums yields [2] the answers written in form of multiple integrals over instanton parameters

$$\langle\phi\rangle_{\text{inst}} = \left[\frac{\sum_{q\geq 0}\frac{K^q}{(q!)^2}\int\phi(\omega_q)\prod_{i<j\leq q}\frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2|b_i-a_j|^2}\prod_{i=1}^{q}\frac{d^2a_id^2b_i}{|a_i-b_i|^2}}{\sum_{q\geq 0}\frac{K^q}{(q!)^2}\int\prod_{i<j\leq q}\frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2|b_i-a_j|^2}\prod_{i=1}^{q}\frac{d^2a_id^2b_i}{|a_i-b_i|^2}}\right]_{\text{reg}}, \tag{5}$$

where K is a real constant obtained as the result of the regularization procedure.
Each $\int\frac{d^2a_id^2b_i}{|a_i-b_i|^2}$ is UV and IR divergent as $L^2\epsilon$.

FFS sum as (singular) tau function of the two-component KP hierarchy

Kyoto group introduced n-component KP tau function with the help of two-component massless fermions

$$\psi^{(a)}(z) = \sum_{i\in Z} \psi_i^{(a)} z^i, \quad \psi^{\dagger(a)}(z) = \sum_{i\in Z} \psi_i^{\dagger(a)} z^{-i-1}, \ a = 1, 2 \quad (6)$$

$$[\psi^{(a)}(z_1), \psi^{(b)}(z_2)]_+ = \delta_{a,b}\delta(z_1/z_2) \quad (7)$$

where $\delta(z_1/z_2) = \frac{1}{z_1}\sum_{i\in Z} \frac{z_1^i}{z_2^i}$.
Fourier modes:

$$\psi_i^{(a)}|n^{(a)}, n^{(b)}\rangle = 0 = \psi_{-i-1}^{\dagger(a)}|n^{(a)}, n^{(b)}\rangle, \ i < 0, \ a, b = 1, 2, \ b \neq a \quad (8)$$

Tau function of 2-KP

Tau function has a form

$$\tau(n, n^{(1)}, n^{(2)}, t^{(1)}, t^{(2)}) =$$

$$\langle n^{(1)}, n^{(2)}| \, \Gamma\left(t^{(1)}\right) \Gamma\left(t^{(2)}\right) \, g \, |n^{(2)} - n^{(0)}, n^{(1)} + n^{(0)}\rangle, \tag{9}$$

where the function

$$\Gamma\left(t^{(\alpha)}\right) = e^{\sum_{i>0} t_i^{(\alpha)} J_i^{(\alpha)}} \tag{10}$$

yields the dependence of the tau function on the 2KP higher times. Here $J_i^{(a)}$ are Fourier modes of the fermionic currents:

$$: \psi^{(a)}(z)\psi^{(a)}(z) := \sum_{n \in Z} J_n^{(a)} z^{-n-1}$$

and g is the exponential of quadratic in fermions expression:

Fermi fields - known properties

we know

$$\psi(x)\psi(y) = (x-y)[***], \quad \psi(x)\psi^\dagger(y) = \frac{1}{x-y}[\text{regular}]$$

and

$$\psi^{(1)}(x)\psi^{\dagger(2)}(\bar{x})\psi^{(2)}(\bar{y})\psi^{\dagger(1)}(y) = \frac{1}{|x-y|^2}[\text{regular}]$$

so

$$\langle 0|\psi^{(1)}(x')\psi^{\dagger(2)}(\bar{x}')\psi^{(2)}(\bar{y}')\psi^{\dagger(1)}(y')\psi^{(1)}(x)\psi^{\dagger(2)}(\bar{x})\psi^{(2)}(\bar{y})\psi^{\dagger(1)}(y)|0\rangle =$$

$$= \frac{|x-x'|^2|y-y'|^2}{|x'-y'|^2|x-y|^2}(-1)^*$$

g defines the solution of the 2-KP hierarchy (initial data)

$$g = e^{K^{\frac{1}{2}} \int_{D^2} \psi^{(2)}(a)\psi^{\dagger(1)}(\bar{a})d^2a} e^{K^{\frac{1}{2}} \int_{D^2} \psi^{(1)}(\bar{b}-\epsilon)\psi^{\dagger(2)}(b+\epsilon)d^2b}$$

where I take finite integration domain D^2 and the (imaginary) shift ϵ in order to achieve a finite terms in the instanton sum. Now each integral of

$$\langle \psi^{(2)}(a)\psi^{\dagger(1)}(\bar{a})\psi^{(1)}(\bar{b}-\epsilon)\psi^{\dagger(2)}(b+\epsilon)d^2b \rangle = \frac{1}{|a_i - b_i - \epsilon|^2}$$

is finite. However, it is not correct way of regularization because we do not known what is the whole sum over q (q is the number of instantons).

The correct way is to re-write the tau function as the functional integral and notice that $K^{\frac{1}{2}}$ plays the role of mass in Dirac equation:

Lagrangian approach

$$\mathcal{L} = \int \left(\psi^{\dagger(1)}(\bar{z}) \partial_z \psi^{(1)}(\bar{z}) + \psi^{\dagger(2)}(z) \bar{\partial}_z \psi^{(2)}(z) \right) d^2 z$$

$$+ \left\{ \int \pi\mu \left(\psi^{(1)}(z) \psi^{\dagger(2)}(\bar{z}) + \psi^{(2)}(\bar{z}) \psi^{\dagger(1)}(z) \right) \right\}$$

$$\begin{pmatrix} 0 & \partial_{\bar{z}} \\ \partial_z & 0 \end{pmatrix} \begin{pmatrix} \psi^{(1)}(0,\bar{z}) \\ \psi^{(2)}(z,0) \end{pmatrix} = 0 \to \begin{pmatrix} \pi\mu & \partial_{\bar{z}} \\ \partial_z & \pi\mu \end{pmatrix} \begin{pmatrix} \psi_1(z,\bar{z}) \\ \psi_2(z,\bar{z}) \end{pmatrix} = 0$$

$$\begin{pmatrix} \psi^{(1)}(0,\bar{z}) \\ 0 \end{pmatrix} \to ?\,, \quad \begin{pmatrix} 0 \\ \psi^{(2)}(z,0) \end{pmatrix} \to ?$$

Dirac equation and adjoint Dirac equation

Now, we solve

$$\left(\psi_1^\dagger(z,\bar z;\mu),\psi_2^\dagger(z,\bar z;\mu)\right)\begin{pmatrix}\pi\mu & \leftarrow\partial_{\bar z}\\ \leftarrow\partial_z & \pi\mu\end{pmatrix}=0 \qquad (11)$$

where the derivatives act on the left (with the changing of signum). That is
Dirac adjoint equation:

$$\begin{pmatrix}\pi\mu & -\partial_{\bar z}\\ -\partial_z & \pi\mu\end{pmatrix}\begin{pmatrix}\psi_1^\dagger(z,\bar z;\mu)\\ \psi_2^\dagger(z,\bar z;\mu)\end{pmatrix}=0\,, \qquad (12)$$

Dirac equation:

$$\begin{pmatrix}\pi\mu & \partial_{\bar z}\\ \partial_z & \pi\mu\end{pmatrix}\begin{pmatrix}\psi_1(z,\bar z)\\ \psi_2(z,\bar z)\end{pmatrix}=0 \qquad (13)$$

Concerning (13) we introduce the variables ϕ, r: $z = e^{i\phi} r$. (In case (12) we have the same except $\phi \to \phi + \pi$)

$$n \geq 0: \; z^n \to z_n = \frac{\Gamma(n+1)}{(\pi\mu)^n} e^{i\phi n} I_n(2\pi\mu r) = z^n(1 + O(\mu r)) \quad (14)$$

$$n < 0: \; z^n \to z_n = 2\frac{(\pi\mu)^{|n|}}{\Gamma(|n|)} e^{i\phi n} K_{|n|}(2\pi\mu r) = z^n(1 + O(\mu r)) \quad (15)$$

where I_n, K_n are Bessel functions of the second kind (below $n \geq 0$ and ψ is digamma function):

$$I_n(z) = (\tfrac{z}{2})^n \sum_{k=0}^{\infty} \frac{(\frac{z^2}{4})^k}{k!\Gamma(n+k+1)} = \frac{\left(\frac{z}{2}\right)^n}{\Gamma(n+1)} + \cdots,$$

$$K_n(z) = \frac{1}{2} \left(\tfrac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{k!} \left(-\frac{z^2}{4}\right)^k + (-1)^{n+1} I_n(z) \log \tfrac{z}{2} +$$

$$\frac{(-1)^n}{2} \left(\tfrac{z}{2}\right)^n \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{(\frac{z^2}{4})^k}{k!(k+n)!} = \left(\tfrac{z}{2}\right)^{-n} \frac{\Gamma(n)}{2} + \cdots$$

Tails

Let us note that for $z = e^{i\phi}|z|$ we have

$$\frac{1}{\pi\mu}\partial_{\bar{z}} e^{i|n|\phi} I_{|n|}(2\pi\mu|z|) = e^{i(|n|+1)\phi} I_{|n|+1}(2\pi\mu|z|), \tag{16}$$

$$\frac{1}{\pi\mu}\partial_{\bar{z}} e^{-i|n|\phi} K_{|n|}(2\pi\mu|z|) = -e^{-i(|n|-1)\phi} K_{|n|-1}(2\pi\mu|z|), \tag{17}$$

which yields the upper and lower 'tails' (see the next page) which we need for the massivization of the solutions to Dirac equation (13):

$$\begin{pmatrix} 0, \\ \psi^{(2)}(z) \end{pmatrix} \to \begin{pmatrix} -\frac{1}{\pi\mu}\partial_{\bar{z}}\psi_2^{(2)}(\mathbf{z};\mu) \\ \psi_2^{(2)}(\mathbf{z};\mu) \end{pmatrix}, \tag{18}$$

$$\begin{pmatrix} \psi^{(1)}(\bar{z}) \\ 0 \end{pmatrix} \to \begin{pmatrix} \psi_1^{(1)}(\mathbf{z};\mu) \\ -\frac{1}{\pi\mu}\partial_{z}\psi_1^{(1)}(\mathbf{z};\mu) \end{pmatrix}, \tag{19}$$

Here $\mathbf{z} = z, \bar{z}$ and $\psi_2^{(2)}(\mathbf{z};\mu)$ is defined as the series (6) for $\psi^{(2)}(z)$ where

$$n \geq 0 : \; z^n \to z_n = \frac{\Gamma(n+1)}{(\pi\mu)^n} e^{i\phi n} I_n(2\pi\mu r) = z^n(1 + O(\mu r)) \tag{20}$$

$$n < 0 : \; z^n \to z_n = 2\frac{(\pi\mu)^{|n|}}{\Gamma(|n|)} e^{i\phi n} K_{|n|}(2\pi\mu r) = z^n(1 + O(\mu r)) \tag{21}$$

and the upper and lower 'tails' defined via (16)-(17). We obtain

$$\begin{pmatrix} \psi^{(1)}(0,\bar z) \\ 0 \end{pmatrix} \to \sum_{n\geq 0} \psi^{(1)}_n \frac{\Gamma(n+1)}{(\pi\mu)^n} \begin{pmatrix} e^{-i\phi n} I_n(2\pi\mu r) \\ -e^{-i\phi(n+1)} I_{n+1}(2\pi\mu r) \end{pmatrix} +$$

$$+2\sum_{n<0} \psi^{(1)}_n \frac{(\pi\mu)^{|n|}}{\Gamma(|n|)} \begin{pmatrix} e^{i\phi|n|} K_{|n|}(2\pi\mu r) \\ e^{i\phi(|n|-1)} K_{|n|-1}(2\pi\mu r) \end{pmatrix} =: \begin{pmatrix} \psi^{(1)}_1(z,\bar z;\mu) \\ \psi^{(1)}_2(z,\bar z;\mu) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \psi^{(2)}(z,0) \end{pmatrix} \to \sum_{n\geq 0} \psi^{(2)}_n \frac{\Gamma(n+1)}{(\pi\mu)^n} \begin{pmatrix} -e^{i\phi(n+1)} I_{n+1}(2\pi\mu r) \\ e^{i\phi n} I_n(2\pi\mu r) \end{pmatrix} +$$

$$2\sum_{n<0} \psi^{(2)}_n \frac{(\pi\mu)^{|n|}}{\Gamma(|n|)} \begin{pmatrix} e^{i\phi(1-|n|)} K_{|n|-1}(2\pi\mu r) \\ e^{-i\phi|n|} K_{|n|}(2\pi\mu r) \end{pmatrix} =: \begin{pmatrix} \psi^{(2)}_1(z,\bar z;\mu) \\ \psi^{(2)}_2(z,\bar z;\mu) \end{pmatrix}$$

The massivization of the solutions of the adjoint Dirac (12) is the same: we replace each $\psi^{\dagger(i)}$ by $\psi_i^{\dagger(i)}$, $i = 1, 2$ which differ by the replacements $z^n \to y_n$. The only difference is the sign of the derivatives, therefore, the signs in the right hand sides of (16) and (17). We obtain (see the next page):

$$\begin{pmatrix} \psi_1^{\dagger(1)}(z,\bar{z};\mu) \\ 0 \end{pmatrix}^{\mathrm{T}} \to 2\sum_{m\geq 0} \psi_m^{\dagger(1)} \frac{(\pi\mu)^{m+1}}{\Gamma(m+1)} \begin{pmatrix} e^{i\phi(m+1)} K_{m+1}(2\pi\mu r) \\ -e^{i\phi m} K_m(2\pi\mu r) \end{pmatrix}^{\mathrm{T}} +$$

$$+\sum_{m<0} \psi_m^{\dagger(1)} \frac{\Gamma(|m|)}{(\pi\mu)^{|m|-1}} \begin{pmatrix} e^{i\phi(1-|m|)} I_{|m|-1}(2\pi\mu r) \\ e^{-i\phi|m|} I_{|m|}(2\pi\mu r) \end{pmatrix}^{\mathrm{T}} =: \begin{pmatrix} \psi_1^{\dagger(1)}(z,\bar{z};\mu) \\ \psi_2^{\dagger(1)}(z,\bar{z};\mu) \end{pmatrix}^{\mathrm{T}}$$

$$\begin{pmatrix} 0 \\ \psi^{(2)}(z,0) \end{pmatrix}^{\mathrm{T}} \to 2\sum_{m\geq 0} \psi_m^{\dagger(2)} \frac{(\pi\mu)^{m+1}}{\Gamma(m+1)} \begin{pmatrix} -e^{-i\phi m} K_m(2\pi\mu r) \\ e^{-i\phi(m+1)} K_{m+1}(2\pi\mu r) \end{pmatrix}^{\mathrm{T}} +$$

$$+\sum_{m<0} \psi_m^{\dagger(2)} \frac{\Gamma(|m|)}{(\pi\mu)^{|m|-1}} \begin{pmatrix} e^{i\phi|m|} I_{|m|}(2\pi\mu r) \\ e^{i\phi(|m|-1)} I_{|m|-1}(2\pi\mu r) \end{pmatrix}^{\mathrm{T}} =: \begin{pmatrix} \psi_1^{(2)}(z,\bar{z};\mu) \\ \psi_2^{(2)}(z,\bar{z};\mu) \end{pmatrix}^{\mathrm{T}}$$

Massive fermions via massless ones

$$\begin{pmatrix} \psi_1(z,\bar{z};\mu) \\ \psi_2(z,\bar{z};\mu) \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)}(z,\bar{z};\mu) \\ \psi_2^{(1)}(z,\bar{z};\mu) \end{pmatrix} + \begin{pmatrix} \psi_1^{(2)}(z,\bar{z};\mu) \\ \psi_2^{(2)}(z,\bar{z};\mu) \end{pmatrix}$$

Also

$$\begin{pmatrix} \psi_1^{\dagger}(z,\bar{z};\mu) \\ \psi_2^{\dagger}(z,\bar{z};\mu) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \psi_1^{\dagger(1)}(z,\bar{z};\mu) \\ \psi_2^{\dagger(1)}(z,\bar{z};\mu) \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} \psi_1^{\dagger(2)}(z,\bar{z};\mu) \\ \psi_2^{\dagger(2)}(z,\bar{z};\mu) \end{pmatrix}^{\mathrm{T}}$$

Theorem.

$$[\psi_i(z_1), \psi_j^{\dagger}(z_2)]_+ = \delta_{i,j} \sum_{n\in Z} \frac{z_1^n}{z_2^{n+1}}, \quad i,j = 1,2$$

We use the summation formulas with Bessel functions: (see [4] N 5.9.2.9 page 699)

$$\sum_{k=-\infty}^{\infty} \left\{ \begin{matrix} \sin k\alpha \\ \cos k\alpha \end{matrix} \right\} I_k(w) K_{k+\nu}(z) = \left\{ \begin{matrix} \sin \nu\beta \\ \cos \nu\beta \end{matrix} \right\} K_\nu \left(\sqrt{w^2 + z^2 - 2wz \cos \alpha} \right) \tag{22}$$

where $\sin\beta = \frac{w \sin\alpha}{\sqrt{w^2+z^2-2wz\cos\alpha}}$ and where $|w \exp i\alpha|, |w \exp i\alpha| < |z|$.

$$\langle \psi_1^{(1)}(z,\bar z;\mu), \psi_2^{\dagger(1)}(z',\bar z';\mu)\rangle = 2\pi\mu \sum_{n<0} e^{i(\phi'-\phi)n} K_{|n|}(2\pi\mu r) I_{|n|}(2\pi\mu r') \tag{23}$$

$$\langle \psi_1^{(2)}(z,\bar z;\mu), \psi_2^{\dagger(2)}(z',\bar z';\mu)\rangle = 2\pi\mu \sum_{n\geq 0} e^{i(\phi'-\phi)n} K_{n}(2\pi\mu r) I_{n}(2\pi\mu r') \tag{24}$$

Therefore,

$$\langle \psi_1(z,\bar z;\mu), \psi_2^{\dagger}(z',\bar z';\mu)\rangle = \sum_{i=1,2} \langle \psi_1^{(i)}(z,\bar z;\mu), \psi_2^{\dagger(i)}(z',\bar z';\mu)\rangle = \tag{25}$$

$$= 2\pi\mu \sum_{n=-\infty}^{\infty} e^{i(\phi'-\phi)n} \frac{1}{\pi\mu} K_n(2\pi\mu r) I_n(2\pi\mu r') = 2\pi\mu K_0(2\pi\mu|z-z'|) = \tag{26}$$

$$= -\left(\log(\pi\mu|z-z'|) + \psi(1)\right)\left(1 + O(\mu^2|z-z'|^2)\right)$$

where $|z| > |z'|$ (in the last formula ψ is diGamma function).

$$\langle\psi_1^{\dagger(i)}(z',\bar z';\mu),\psi_2^{(i)}(z,\bar z;\mu)\rangle = -2\pi\mu\sum_{n<0} e^{i(\phi-\phi')n}K_n(2\pi\mu r)I_n(2\pi\mu r') \tag{27}$$

$$\langle\psi_1^{\dagger(2)}(z',\bar z';\mu),\psi_2^{(2)}(z,\bar z;\mu)\rangle = -2\pi\mu\sum_{n\geq 0} e^{i(\phi-\phi')n}K_n(2\pi\mu r')I_n(2\pi\mu r) \tag{28}$$

$$\langle\psi_1^{\dagger}(z',\bar z';\mu),\psi_2(z,\bar z;\mu)\rangle = \sum_{i=1,2}\langle\psi_1^{\dagger(i)}(z',\bar z';\mu),\psi_2^{(i)}(z,\bar z;\mu)\rangle = \tag{29}$$

$$= -2\pi\mu\sum_{n=-\infty}^{\infty} e^{i(\phi-\phi')n}K_n(2\pi\mu r)I_n(2\pi\mu r') = -2\pi\mu K_0(2\pi\mu|z-z'|) \tag{30}$$

Green function of massive fermions

$$G^{\mu}\left(z,\bar z,z',\bar z';\mu\right) = -\int \frac{d^2k}{(2\pi)^2}\left(\begin{array}{cc} 2\pi i\mu & k \\ \bar k & 2\pi i\mu \end{array}\right)\frac{\exp\left\{\frac{i}{2}\left(\bar k z + k\bar z\right)\right\}}{k\bar k + 4\pi^2\mu^2}$$

$$= \frac{1}{2\pi i}\left(\begin{array}{cc} \langle 0|\psi_1(z,\bar z)\psi_2^{\dagger}(z',\bar z';\mu)|0\rangle & \langle 0|\psi_1(z,\bar z;\mu)\psi_1^{\dagger}(z',\bar z';\mu)|0\rangle \\ \langle 0|\psi_2(z,\bar z;\mu)\psi_2^{\dagger}(z',\bar z';\mu)|0\rangle & \langle 0|\psi_2(z,\bar z;\mu)\psi_1^{\dagger}(z',\bar z')|0\rangle \end{array}\right)$$

$$= \left(\begin{array}{cc} -i\mu K_0(2\pi\mu r) & 2\pi\mu r K_1(2\pi\mu r)G^0_{1,2}(\bar z,\bar z') \\ 2\pi\mu r K_1(2\pi\mu r)G^0_{2,1}(z,z') & -i\mu K_0(2\pi\mu r), \end{array}\right) \tag{31}$$

where $r = |z-z'|$, $G^0_{2,1}(z,z') = \frac{1}{2\pi i}\frac{1}{z-z'}$, $G^0_{1,2}(\bar z,\bar z') = \frac{1}{2\pi i}\frac{1}{\bar z-\bar z'}$.

$$2i\left(\begin{array}{cc} \pi\mu & \bar\partial_z \\ \partial_z & \pi\mu \end{array}\right)G^{\mu}(z,z') = \delta^{(2)}(z-z')\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \tag{32}$$

$$w = \frac{\sigma_1 + i\sigma_2}{1 + \sigma_3}$$

$$\left\langle \overline{\omega}\left(x_1\right) ... \overline{\omega}\left(x_n\right) \overline{\omega}^{-1}\left(y_1\right) ... \overline{\omega}^{-1}\left(y_n\right) \right\rangle_{inst} = \quad (33)$$

$$= (2\pi\mu)^n \frac{\prod_{i,j=1}^{n} (\bar{x}_i - \bar{y}_j)}{\prod_{1 \le i < j \le n} (\bar{x}_i - \bar{x}_j)(\bar{y}_i - \bar{y}_j)} \det \left[K_1(2\pi\mu|x_i - y_j|) \right]_{i,j=1,\ldots,n} \quad (34)$$

$$\left\langle \left|\omega\left(x\right)\right|^2 \left|\omega\left(y\right)\right|^{-2} \right\rangle_{inst} \quad (35)$$

$$= \left(2\pi\mu|x-y|(K_1(2\pi\mu|x-y|)\right)^2 - \left(2\pi\mu|x-y|(K_0(2\pi\mu|x-y|)\right)^2 \quad (36)$$

Bilinear equations

Let $z = e^{i\phi} r$, $r = |z|$. Consider

$$S := \int_0^{2\pi} e^{-i\phi} d\phi \psi_1(\phi, r) \otimes \psi_1^\dagger(\phi, r) + \int_0^{2\pi} e^{i\phi} d\phi \psi_2(\phi, r) \otimes \psi_2^\dagger(\phi, r) =$$

$$\left(\sum_{n \in Z} A_n(2\pi\mu r) \sum_{i=1,2} \psi_n^{(i)} \otimes \psi_n^{\dagger(i)} \right)$$

where

$$A_n(2\pi\mu r) = I_{n+1}(2\pi\mu r) K_n(2\pi\mu r) + I_n(2\pi\mu r) K_{n+1}(2\pi\mu r), \quad n \in Z$$

then it follows:

$$S|0\rangle \otimes |0\rangle = 0$$

Discrete Hirota equation

$$
\begin{aligned}
&(z_2 - z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2,\mathtt{n}_3}(z_1,z_2,z_3)G_{\mathtt{n}_1,\mathtt{n}_2+1,\mathtt{n}_3+1}(z_1,z_2,z_3)\\
&+(z_3 - z_1)G_{\mathtt{n}_1,\mathtt{n}_2+1,\mathtt{n}_3}(z_1,z_2,z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2,\mathtt{n}_3+1}(z_1,z_2,z_3)\\
&+(z_1 - z_2)G_{\mathtt{n}_1,\mathtt{n}_2,\mathtt{n}_3+1}(z_1,z_2,z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2+1,\mathtt{n}_3}(z_1,z_2,z_3) = 0 \quad (37)
\end{aligned}
$$

where

$$G_{\mathtt{n}_1,\mathtt{n}_2,\mathtt{n}_3}(z_1,z_2,z_3) :=$$

$$\left\langle \left(\frac{\sigma^1(z_1)+i\sigma^2(z_1)}{1+\sigma^3(z_1)}\right)^{\mathtt{n}_1} \left(\frac{\sigma^1(z_2)+i\sigma^2(z_2)}{1+\sigma^3(z_2)}\right)^{\mathtt{n}_2} \left(\frac{\sigma^1(z_3)+i\sigma^2(z_3)}{1+\sigma^3(z_3)}\right)^{\mathtt{n}_3} \right\rangle^{A}_{\mathrm{in}}$$

- A.A. Belavin and A.M. Polyakov, “Metastable states of two-dimensional isotropic ferromagnets”, Pis’ma Zh. Eksp. Teor. Fiz. **22** N10 (1975) pp 503-506
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