# Unfolded formulation of *4d* Yang–Mills theory

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# outline

- unfolded dynamics approach of higher-spin gravity
- derivation of unfolded Yang–Mills equations
- interpretation of unfolded equations in terms of unfolding maps

# unfolded dynamics approach

- **higher-spin (HS) gravity** - a nonlinear gauge theory of interacting massless fields of all spins (including graviton) possessing $\infty$-dim HS gauge symmetry;
- to formulate the theory in a manifestly diffeomorphism- and gauge-invariant way [Vasiliev'89-94], a special first-order formalism was developed called **unfolded dynamics approach** [Vasiliev hep-th/0504090];
- it is interesting to apply it to different theories
- constructing unfolded formulations (especially nonlinear) is not easy
- many examples of unfolding linear theories are known, e.g. [Shaynkman, Vasiliev hep-th/0003123; Ponomarev, Vasiliev 1012.2903; Khabarov, Zinoviev 2001.07903; Buchbinder, Snegirev, Zinoviev 1606.02475; NM'19-'23].
- but not so many nonlinear theories beyond HS [Joung, Kim, Kim 2108.05535; NM 2208.04306, 2402.14164].
- method of quantization of unfolded field theories was proposed [NM 2208.04306].

# unfolded dynamics approach

- **Unfolded equations** are first-order exterior-form equations

$$\mathrm{d}W^A(x) + G^A(W) = 0, \tag{1}$$

- **Unfolded fields** are exterior forms; dynamical field theories require $\infty$ of unfolded fields encoding all d.o.f. (auxiliary generating variables are handy)
- **Consistency** condition

$$\mathrm{d}^2 \equiv 0 \quad \Rightarrow \quad G^B \frac{\delta G^A}{\delta W^B} \equiv 0. \tag{2}$$

- **Unfolded gauge symmetries**

$$\delta W^A = \mathrm{d}\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A}{\delta W^B}. \tag{3}$$

- Unfolded fields form modules for all symmetries of the theory, realized algebraically.
- Gauge symmetries are associated with $(n > 0)$-forms, 0-forms correspond to physical d.o.f. which transform only passively under 1-form symmetries.

# Yang–Mills equations in spinorial notations

- $4d$ YM equations + Bianchi identities in $sl(2,\mathbb{C})$-spinor notations

$$\mathrm{D}_{\beta\dot{\alpha}} F^{\beta}{}_{\alpha} = 0, \quad \mathrm{D}_{\alpha\dot{\beta}} \bar{F}^{\dot{\beta}}{}_{\dot{\alpha}} = 0, \tag{4}$$

- (anti-)selfdual YM strength tensor

$$F_{\alpha\alpha} := \frac{\partial}{\partial x^{\alpha\dot{\beta}}} A_{\alpha}{}^{\dot{\beta}} - i[A_{\alpha\dot{\beta}}, A_{\alpha}{}^{\dot{\beta}}], \quad \bar{F}_{\dot{\alpha}\dot{\alpha}} := \frac{\partial}{\partial x^{\beta\dot{\alpha}}} A^{\beta}{}_{\dot{\alpha}} - i[A_{\beta\dot{\alpha}}, A^{\beta}{}_{\dot{\alpha}}], \tag{5}$$

- covariant derivative

$$\mathrm{D}_{\alpha\dot{\alpha}} := \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} - i[A_{\alpha\dot{\alpha}}, \bullet], \tag{6}$$

$$[\mathrm{D}_{\alpha\dot{\alpha}}, \mathrm{D}_{\beta\dot{\beta}}] = -i\epsilon_{\alpha\beta}\bar{F}_{\dot{\alpha}\dot{\beta}} - i\epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}. \tag{7}$$

- $\epsilon_{\alpha\beta}$ is 2x2 antisymmetric spinor metric
- same-letter indices of multispinors are either contracted or symmetrized

$$T_{\alpha\alpha} := T_{(\alpha_1\alpha_2)}, \quad T_{\alpha}{}^{\alpha} := \epsilon^{\alpha\beta} T_{\alpha\beta}. \tag{8}$$

# unfolded Yang–Mills field and auxiliary spinors

- In order to unfold Yang–Mills theory, one has to introduce, on top of primaries $F_{\alpha\alpha}$ and $\bar{F}_{\dot{\alpha}\dot{\alpha}}$, the infinite towers of all their differential on-shell descendants.
- It's convenient to introduce auxiliary commuting spinors $Y = (y^{\alpha}, \bar{y}^{\dot{\alpha}})$
- Then the whole towers get packed into unfolded Yang–Mills master-fields, which we postulate to be of the form (here $\mathrm{D}y\bar{y} := y^{\alpha}\bar{y}^{\dot{\alpha}}\mathrm{D}_{\alpha\dot{\alpha}}$)

$$F(Y|x) = e^{\mathrm{D}y\bar{y}} F_{\alpha\alpha}(x) y^{\alpha} y^{\alpha} e^{-\mathrm{D}y\bar{y}}, \quad \bar{F}(Y|x) = e^{\mathrm{D}y\bar{y}} \bar{F}_{\dot{\alpha}\dot{\alpha}}(x) \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\alpha}} e^{-\mathrm{D}y\bar{y}}. \tag{9}$$

- Unfolded master-field $F$ contains the primary Yang-Mills tensor, together with an infinite sequence of its fully symmetrized traceless covariant derivatives of all orders. If $F_{\alpha\alpha}$ is on-shell, this constitutes a set of all its independent covariant descendants.
- By construction, $F$ takes values in the adjoint representation of the gauge algebra.
- To formulate the unfolded equations, one needs to express the derivatives of the unfolded fields in algebraic terms, which in this case means

$$\mathrm{D}_{\alpha\dot{\beta}} F = (Y, \partial/\partial Y)_{\alpha\dot{\beta}}(F, \bar{F}) \tag{10}$$

# unfolding map and operator relations

- Introduce the "unfolding map" for an arbitrary function $\mathrm{C}_{\alpha(n),\dot\beta(m)}(Y|x)$

$$\ll \mathrm{C}_{\alpha(n),\dot\beta(m)}(Y|x) \gg := e^{\mathrm{D}y\bar y}\mathrm{C}_{\alpha(n),\dot\beta(m)}(Y|x)e^{-\mathrm{D}y\bar y}, \tag{11}$$

- In particular,

$$F(Y|x) = \ll F_{\alpha\alpha}(x)y^\alpha y^\alpha \gg, \quad \bar F(Y|x) = \ll \bar F_{\dot\alpha\dot\alpha}(x)\bar y^{\dot\alpha}\bar y^{\dot\alpha} \gg . \tag{12}$$

- Making use of two relations,

$$[\hat{\mathrm{A}}, e^{\hat{\mathrm{D}}}] = \int_0^1 dt e^{t\hat{\mathrm{D}}}[\hat{\mathrm{A}}, \hat{\mathrm{D}}]e^{-t\hat{\mathrm{D}}}e^{\hat{\mathrm{D}}}, \quad \int_0^1 dt t^k F(tz) = \frac{1}{z\frac{\partial}{\partial z}+1+k}F(z), \tag{13}$$

one deduces

$$\ll \partial_\mu \mathrm{C} \gg = (\partial_\mu - \mathrm{D}_{\mu\dot\alpha}\bar y^{\dot\alpha}) \ll \mathrm{C} \gg + i y_\mu [\frac{1}{(N+1)(N+2)}\bar F, \ll \mathrm{C} \gg]. \tag{14}$$

where spinorial Euler operators are

$$N := y^\alpha \partial_\alpha, \quad \bar N := \bar y^{\dot\alpha}\bar\partial_{\dot\alpha}. \tag{15}$$

# separating the variables

- Direct application of $[\hat{\mathrm{A}}, e^{\hat{\mathrm{D}}}]$-formula gives

$$\mathrm{D}_{\alpha\dot{\beta}}F = [\frac{1}{N} \ll [\mathrm{D}_{\alpha\dot{\beta}}, \mathrm{D}y\bar{y}] \gg, F] + \ll \mathrm{D}_{\alpha\dot{\beta}}F_{\mu\mu}y^{\mu}y^{\mu} \gg . \quad (16)$$

- The task is to bring it to the form with separated variables

$$\mathrm{D}_{\alpha\dot{\beta}}F = (Y, \partial/\partial Y)_{\alpha\dot{\beta}}(F, \bar{F}) \quad (17)$$

- The tools available: YM e.o.m., $[\mathrm{D}_{\alpha\dot{\alpha}}, \mathrm{D}_{\beta\dot{\beta}}]$, the relation for $\ll \partial_{\mu}\mathrm{C} \gg$, Schouten indentities for spinors, Jacobi identity of the gauge Lie algebra.
- Applying some of them, one has

$$\begin{aligned}\mathrm{D}_{\mu\dot{\mu}}F \quad &= \quad \bar{y}_{\dot{\mu}}[\frac{i}{2N} \ll \partial_{\mu}F_{\alpha\alpha}y^{\alpha}y^{\alpha} \gg, F] + y_{\mu}[\frac{i}{2\bar{N}} \ll \bar{\partial}_{\dot{\mu}}\bar{F}_{\dot{\alpha}\dot{\alpha}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\alpha}} \gg, F] + \\ &+ \frac{1}{3} \ll \partial_{\mu}\bar{\partial}_{\dot{\mu}}\mathrm{D}y\bar{y}F_{\alpha\alpha}y^{\alpha}y^{\alpha} \gg . \quad (18)\end{aligned}$$

- Process the first term on the r.h.s. The problem is $\ll \partial_{\mu}F_{\alpha\alpha}y^{\alpha}y^{\alpha} \gg$.

# separating the variables

- Applying $\ll \partial_\mu \mathrm{C} \gg$-formula, one has

$$\ll \partial_\mu F_{\alpha\alpha} y^\alpha y^\alpha \gg = \partial_\mu F - \mathrm{D}_{\mu\dot{\alpha}} \bar{y}^{\dot{\alpha}} F + i y_\mu [\frac{1}{(N+1)(N+2)} \bar{F}, F]. \tag{19}$$

Now, problematic is $\mathrm{D}_{\mu\dot{\alpha}} \bar{y}^{\dot{\alpha}} F$. Contracting $\mathrm{D}_{\mu\dot{\mu}} F$ with $\bar{y}^{\dot{\mu}}$ yields

$$\mathrm{D}_{\mu\dot{\alpha}} \bar{y}^{\dot{\alpha}} F = i y_\mu [\frac{1}{N+1} \bar{F}, F] + \frac{1}{3} \ll \partial_\mu \mathrm{D} y \bar{y} F_{\alpha\alpha} y^\alpha y^\alpha \gg . \tag{20}$$

This way one finds

$$\ll \partial_\mu F_{\alpha\alpha} y^\alpha y^\alpha \gg = \frac{2}{N+1} (\partial_\mu F - i y_\mu [\frac{1}{N+2} \bar{F}, F]). \tag{21}$$

- Proceeding this way, the final result is

$$\begin{aligned} \mathrm{D}_{\mu\dot{\mu}} F = & \frac{1}{N+1} \partial_\mu \bar{\partial}_{\dot{\mu}} F + i N [\frac{1}{N(N+1)} \bar{y}_{\dot{\mu}} \partial_\mu F, \frac{1}{N} F] - i y_\mu \bar{\partial}_{\dot{\mu}} \frac{1}{N+2} [\frac{1}{N+2} \bar{F}, F] + \\ & + [\frac{i}{(N+1)(N+2)} y_\mu \bar{\partial}_{\dot{\mu}} \bar{F}, F] + \frac{1}{2} y_\mu \bar{y}_{\dot{\mu}} [\frac{N+3}{(N+1)(N+2)} [\frac{1}{N+2} \bar{F}, F], F] + \\ & + \frac{3}{2} y_\mu \bar{y}_{\dot{\mu}} [\frac{1}{(N+1)(N+2)} [\frac{1}{N} F, \bar{F}], F] + y_\mu \bar{y}_{\dot{\mu}} [\frac{1}{N+2} [\frac{1}{N+2} \bar{F}, F], \frac{1}{N} F]. \end{aligned} \tag{22}$$

# Poincaré symmetry and diffeomorphism-invariance

- Up to now, Cartesian coordinates was used ($\frac{\partial}{\partial x^{\mu\dot{\mu}}}$ in $D_{\mu\dot{\mu}}$). Manifest coordinate-independence requires exterior-form formalism.
- Switch to the fiber-space picture: $F(Y|x)$ and $\bar{F}(Y|x)$ are claimed to be 0-forms on the Minkowski space with base $x^{\underline{n}}$ and fiber $Y$.
- To implement global Poincaré symmetry, introduce a 1-form $\Omega(x) \in iso(1,3)$

$$\Omega = e^{\alpha\dot{\beta}} P_{\alpha\dot{\beta}} + \omega^{\alpha\alpha} M_{\alpha\alpha} + \bar{\omega}^{\dot{\alpha}\dot{\alpha}} \bar{M}_{\dot{\alpha}\dot{\alpha}}, \tag{23}$$

  where $P$, $M$ are $iso(1,3)$-generators, $e$ and $\omega$ are 1-forms of a vierbein and a Lorentz connection.
- $\Omega$ is subjected to the flatness condition

$$\mathrm{d}\Omega + \frac{1}{2}[\Omega, \Omega] = 0, \tag{24}$$

$$\delta\Omega = \mathrm{d}\varepsilon(x) + [\Omega, \varepsilon] \tag{25}$$

  with a gauge symmetry that describes $\infty$-dim freedom in switching between all possible local coordinates on $\mathbb{R}^{1,3}$.
- The gauge symmetry boils down to 10-dim global Poincaré symmetry after fixing some particular solution $\Omega_0$ and restricting to $\varepsilon(x)$ which leave it invariant

$$\mathrm{d}\varepsilon_0 + [\Omega_0, \varepsilon_0] = 0. \tag{26}$$

# Poincaré symmetry and diffeomorphism-invariance

- The simplest non-degenerate particular solution – Cartesian coordinates

$$e_{\underline{m}}{}^{\alpha\dot{\beta}} = (\bar{\sigma}_{\underline{m}})^{\dot{\beta}\alpha}, \quad \omega_{\underline{m}}{}^{\alpha\alpha} = 0, \quad \bar{\omega}_{\underline{m}}{}^{\dot{\alpha}\dot{\alpha}} = 0, \tag{27}$$

with global symmetries parameterized by $x$-independent $\xi^{\alpha\dot{\beta}}$, $\xi^{\alpha\alpha}$ and $\bar{\xi}^{\dot{\alpha}\dot{\alpha}}$ as

$$\varepsilon^{\alpha\alpha}_{glob} = \xi^{\alpha\alpha}, \quad \bar{\varepsilon}^{\dot{\alpha}\dot{\alpha}}_{glob} = \bar{\xi}^{\dot{\alpha}\dot{\alpha}}, \quad \varepsilon^{\alpha\dot{\beta}}_{glob} = \xi^{\alpha\dot{\beta}} + \xi^{\alpha}{}_{\gamma}(\bar{\sigma}_{\underline{m}})^{\dot{\beta}\gamma}x^{\underline{m}} + \bar{\xi}^{\dot{\beta}}{}_{\dot{\gamma}}(\bar{\sigma}_{\underline{m}})^{\dot{\gamma}\alpha}x^{\underline{m}}. \tag{28}$$

- To implement the Yang–Mills gauge symmetry, introduce a 1-form $A(x)$ as

$$A(x) = e^{\alpha\dot{\alpha}}A_{\alpha\dot{\alpha}}. \tag{29}$$

- An appropriate coordinate-independent generalization of $\mathrm{D}_{\mu\dot{\mu}}$ is a 1-form operator D

$$\mathrm{D} := \mathrm{d} + \omega^{\alpha\alpha}y_{\alpha}\partial_{\alpha} + \bar{\omega}^{\dot{\alpha}\dot{\alpha}}\bar{y}_{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} - i[A, \bullet]. \tag{30}$$

# unfolded Yang–Mills equations

- Unfolded consistent system for Yang–Mills theory

$$\mathrm{d}A + [A, A] = \frac{1}{4} e^{\alpha}{}_{\dot\beta} e^{\alpha\dot\beta} \partial_\alpha \partial_\alpha F|_{\bar y=0} + \frac{1}{4} e_{\beta}{}^{\dot\alpha} e^{\beta\dot\alpha} \bar\partial_{\dot\alpha} \bar\partial_{\dot\alpha} \bar F|_{y=0}, \quad (31)$$

$$\mathrm{D}F = \frac{1}{N+1} e\partial\bar\partial F + iN[\frac{1}{N(N+1)} e\partial\bar y F, \frac{1}{N} F] - iey\bar\partial \frac{1}{N+2}[\frac{1}{N+2}\bar F, F] +$$
$$+[\frac{i}{(N+1)(N+2)} ey\bar\partial\bar F, F] + \frac{1}{2} ey\bar y[\frac{N+3}{(N+1)(N+2)}[\frac{1}{N+2}\bar F, F], F] +$$
$$+\frac{3}{2} ey\bar y[\frac{1}{(N+1)(N+2)}[\frac{1}{N} F, \bar F], F] + ey\bar y[\frac{1}{N+2}[\frac{1}{N+2}\bar F, F], \frac{1}{N} F], \quad (32)$$

plus a conjugate equation for $\bar F$ and the $\Omega$-flatness equation.

- Spectrum of fields: 1-form $\Omega$, 1-form $A$, 0-form master-fields $F(Y|x)$ and $\bar F(Y|x)$

$$(N - \bar N)F = 2F, \quad (N - \bar N)\bar F = -2\bar F. \quad (33)$$

- Symmetries: diffeomorphism-invariance, global (after fixing $\Omega$) Poincaré associated to $\Omega$, local YM associated to $A$. The YM symmetry is

$$\delta A(x) = \mathrm{D}\varepsilon(x), \quad \delta F(Y|x) = i[\varepsilon(x), F(Y|x)], \quad \delta\bar F(Y|x) = i[\varepsilon(x), \bar F(Y|x)]. \quad (34)$$

- Two possible reductions:
anti-selfdual case $\bar F(Y|x) = 0$ (or $F = 0$); abelian case $[\bullet, \bullet] = 0$.

# unfolding maps

- One can think of the equation

$$\mathrm{D}F = \frac{1}{N+1} e\partial\bar{\partial}F + iN[\frac{1}{N(N+1)} e\partial\bar{y}F, \frac{1}{N}F] + \ldots \qquad (35)$$

as defining an unfolding map from $x$-space to $Y$-space

$$F_{\alpha\alpha}(x)|_{on-shell} \to F(Y|x) \to \mathcal{F}(Y) := F(Y|x=0). \qquad (36)$$

The first arrow is explicitly realized by

$$F(Y|x) = e^{\mathrm{D}y\bar{y}} F_{\alpha\alpha}(x) y^{\alpha} y^{\alpha} e^{-\mathrm{D}y\bar{y}}. \qquad (37)$$

The field $\mathcal{F}(Y)$ carries precisely the same information as on-shell $F_{\alpha\alpha}$ does.

- In a sense, spinors $Y$ effectively replace $x^{\alpha\dot{\alpha}}$ for on-shell configurations, hence being conjugate to helicity spinors resolving light-like momenta $p_{\alpha\dot{\alpha}} = \pi_{\alpha}\bar{\pi}_{\dot{\alpha}}$.
- This is how the unfolded system imposes e.o.m. on primary fields: via (36), it maps $4d$ space-time fields onto an effectively $3d$ hypersurface (in the sense that $y^{\alpha}\bar{y}^{\dot{\alpha}}$ is a light-like vector).

# unfolding maps

- Consider the abelian case in Cartesian coordinates

$$(\frac{\partial}{\partial x^{\mu\dot{\mu}}} - \frac{1}{N+1}\partial_{\mu}\bar{\partial}_{\dot{\mu}})F(Y|x) = 0, \tag{38}$$

$$F(Y|x) = e^{y\bar{y}\frac{\partial}{\partial x}}F_{\alpha\alpha}(x)y^{\alpha}y^{\alpha} = F_{\alpha\alpha}(x+y\bar{y})y^{\alpha}y^{\alpha}. \tag{39}$$

- A plane-wave solution is

$$A_{\alpha\dot{\alpha}}(x) = \pi_{\alpha}\bar{\mu}_{\dot{\alpha}}e^{i\pi_{\beta}\bar{\pi}_{\dot{\beta}}x^{\beta\dot{\beta}}} + c.c., \quad F(Y|x) = i\bar{\pi}_{\dot{\alpha}}\bar{\mu}^{\dot{\alpha}}(\pi_{\alpha}y^{\alpha})^2 e^{i\pi_{\beta}\bar{\pi}_{\dot{\beta}}(x^{\beta\dot{\beta}}+y^{\beta}\bar{y}^{\dot{\beta}})} \tag{40}$$

  with $\bar{\mu}_{\dot{\alpha}}$ being an arbitrary reference spinor, defined up to a gauge transformation $\bar{\mu}_{\dot{\alpha}} \to \bar{\mu}_{\dot{\alpha}} + const \cdot \bar{\pi}_{\dot{\alpha}}$.
- Putting $x=0$, one has

$$\mathcal{F}(Y) = i\overline{\pi\mu}(\pi y)^2 e^{i\pi y\overline{\pi y}} \tag{41}$$

  which is a plane-wave Maxwell tensor formulated purely in $Y$-terms.
- Although we have derived the unfolded equations starting from postulating the form of $F(Y|x)$, now this form per se is just one particular solution (namely, the solution to the $(Y=0)$-boundary problem).

# unfolding maps

- Alternatively, one can think of unfolded equations as defining a $Y$-to-$x$ map [Vasiliev 1404.1948]

$$\mathcal{F}(Y) \to F(Y|x) \to F_{\alpha\alpha}(x)|_{on-shell}. \tag{42}$$

  In this picture, $\mathcal{F}(Y)$ is completely unconstrained aside from the helicity condition

$$(N - \bar{N})\mathcal{F}(Y) = 2\mathcal{F}(Y) \quad \Rightarrow \quad \mathcal{F}(Y) = \mathcal{F}_{\alpha\alpha}(y\bar{y})y^{\alpha}y^{\alpha}. \tag{43}$$

- In the abelian case, the first arrow in (42) is implemented by

$$F(Y|x) = \exp(\frac{1}{N+1}x^{\beta\dot{\beta}}\partial_{\beta}\bar{\partial}_{\dot{\beta}})\mathcal{F}_{\alpha\alpha}(y\bar{y})y^{\alpha}y^{\alpha}. \tag{44}$$

  This generates a solution to the unfolded (and hence implicitly to Maxwell) equations for arbitrary $\mathcal{F}_{\alpha\alpha}(y\bar{y})$.

- In fact, relativistic dynamics can be realized without any reference to a space-time. An action principle for an arbitrary-mass bosonic field was constructed in [NM 2301.02207] on $(Y, \tau)$-space, where $\tau$ is an additional scalar coordinate.
- Twistor construction probably arises from treating an unfolded system as defining an unfolding map from some complex plane in $\mathbb{R}^{1,3} \times \mathbb{C}^2$ manifold, different from $Y = 0$ or $x = 0$.

# conclusions

- an unfolded **formulation** of 4*d* pure Yang–Mills theory was **constructed**
- certain **unfolding maps** solving the equations were **found**
- perspectives:
    - inclusion of **charged matter**: should be straightforward, like for scalar electrodynamics [NM 2402.14164].
    - introduce **supersymmetry** and manifest **conformal symmetry**: corresponding gauge 1-form of (super)conformal gravity should be introduced, which requires modifications of the unfolding technique
    - **quantization**: can be performed along the lines of [NM 2208.04306] but with necessary modifications in order to include ghosts.
    - **integrability**: construct an explicit map $\mathcal{F}(Y) \to F_{\alpha\alpha}(x)$ for non-abelian case
    - relation to **twistors**: reproduce twistor construction from unfolded formulation by projecting onto an appropriate complex plane