Strings, non-compact CY and N=2 Liouville theory

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Main points:

joint with P. Gavrylenko, A. Yung, E. Ievlev, I. Monastyrskii; arXiv:2307.02929

- Strings: fundamental and vortex from SQCD;
- World-sheet sigma-model, case of 2d SCFT;
- Higgs and Coulomb branches;
- Effective theory on Coulomb branch, N=2 Liouville;
- Relation with geometry: complex and Kähler deformations;
- Liouville "explanations".

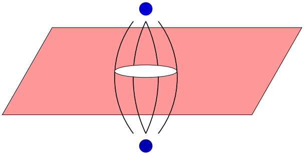
Fundamental strings and Liouville

- Liouville theory from Polyakov's path integral:
  - 2d quantum gravity (non-critical string);
  - 2d CFT (matter) "in general position".
- $\mathcal{N} = 2$ Liouville has to be discovered by Fradkin & Tseytlin in 1981 as non-critical $U(1)$-string (Kutasov & Seiberg 1990);
- Found earlier (Ivanov-Krivonos 1983, Girardello-Pasquinucci-Porrati, ???)
- Below: $\mathcal{N} = 2$ Liouville as "matter" (in critical $\mathcal{N} = 1$ superstring).

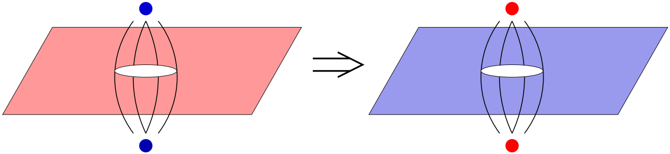
SUSY vortex strings

Conformal vortex strings in $\mathcal{N} = 2$ 4d SQCD

Vortex solutions (ANO, SUSY $\to$ BPS) and strings (slow dependence on (t, z)):

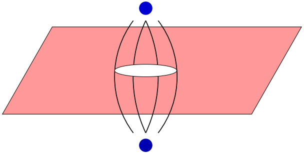


confinement as a (dual) Meissner effect in $\mathcal{N} = 2$ 4d SQCD
(ideas of Polyakov, 't Hooft and Mandelstam in SW theory)

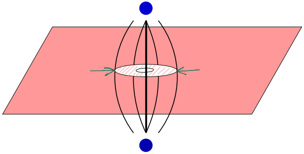


Conformal case: vortex versus fundamental

Conformal: thick versus thin



versus



- No mass parameter: already in 4d theory $m = 0$ and $b_g = 0$;
- 2d conformal (effective?) theory on world-sheet;
- Field theory realizations? Strong-coupling QFT versus *critical* string ...

Non-Abelian strings

Effective world-sheet theory contains “non-Abelian” modes, fluctuations in flavor space $i, j = 1, \ldots, N$ (non-compact CY in our case).

Action (bosonic part of $\mathcal{N} = (2,2)$): GLSM

$$S = \int d^2 z \left\{ \left| \nabla_\alpha n^i \right|^2 + \left| \widetilde{\nabla}_\alpha \rho^j \right|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} \left| \partial_\alpha \sigma \right|^2 + \frac{1}{2e_0^2} D^2 \right.$$
$$\left. - 2 \left| \sigma \right|^2 \left(\left| n^i \right|^2 + \left| \rho^j \right|^2 \right) + D \left(\left| n^i \right|^2 - \left| \rho^j \right|^2 - \mathrm{Re}\, \beta \right) - \frac{\vartheta}{2\pi} F_{01} \right\}$$

with $(\alpha = 1, 2)$

$$\nabla_\alpha = \partial_\alpha - i A_\alpha \,, \qquad \widetilde{\nabla}_\alpha = \partial_\alpha + i A_\alpha \,, \tag{1}$$

for the charges $\mathtt{Q} = +1$ and $\mathtt{Q} = -1$, bare $e_0 \to \infty$.

Conformal theory

- $\#\{n^i\} = \#\{\rho^j\} = N$, ($4N$ real fields with $\mathtt{Q} = \pm 1$), $b_\beta = \sum \mathtt{Q} = 0$;
- Higgs branch:
  - D-term condition $\left|n^i\right|^2 - \left|\rho^j\right|^2 = \mathrm{Re}\,\beta$, with complex FI $\beta = \mathrm{Re}\,\beta + i\,\frac{\vartheta}{2\pi}$;
  - For $\beta \neq 0$ necessarily either $\langle n \rangle \neq 0$ or $\langle \rho \rangle \neq 0$ (dependently on $\beta \gtrless 0$), then $\langle \sigma \rangle = 0$, since it is massive;
  - The dimension (with D-term constraint and "eaten" $U(1)$-phase)

$$\dim_{\mathbb{R}} \mathcal{H} = 4N - 1 - 1 = 2\,(2N-1) = 2\dim_{\mathbb{C}} \mathcal{H}.$$

- For vortex strings – from conformal invariance $b_g = 2N - N_f = 0$ of $N_f = 2N$ SQCD in 4d;
- The central charge

$$\hat{c} \equiv \frac{c}{3} = \dim_{\mathbb{C}} \mathcal{H} = 2N - 1,$$

  of $\mathcal{N} = (2,2)$ 2d SCFT is given by dimension of the Higgs branch;
- Critical string: $\hat{c} = 3$ i.e. $N = 2$, the case of (resolved if $\beta \neq 0$) *conifold*.

Conifold, deformed and resolved

- Conifold (in *many* problems of String Theory): $xv - uy = 0$ in $\mathbb{C}^4$.
- Resolution: from

$$\det \begin{pmatrix} x & y \\ u & v \end{pmatrix} = 0$$

to

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \vec{\lambda} = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0, \quad \vec{\lambda} \in \mathbb{P}^1$$

Non-compact CY 3-fold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$ with holomorphic $\Omega = dy \wedge dv \wedge d\xi = du \wedge dx \wedge d\chi$, where $\xi = \lambda_2/\lambda_1 = 1/\chi$.

- Deformation: $xv - uy = \epsilon$
  Obviously has $S^3 \simeq SU(2) \subset SL(2, \mathbb{C})$, may be thought as T^*S^3.

Coulomb branch

When $\beta \to 0$, at $\langle n \rangle = \langle \rho \rangle = 0$ a "Coulomb branch" with $\langle \sigma \rangle \neq 0$ can develop.

- A delicate point in 2d, no "strict border" between Higgs and Coulomb branches, large IR fluctuations;
- Any geometric definition? (a common belief for $N = 2$ is that it is *deformed* conifold). Arbitrary $N \geq 3$?

Obvious idea:

- To integrate out massive "matter" $\{n, \rho\}$-fields and to study effective theory ($\mathcal{N} = 2$ Liouville?!) on the Coulomb branch;
- To study complex deformations and understand their field-theory origin (different at $N = 2$ and $N \geq 3$?!).

Effective action

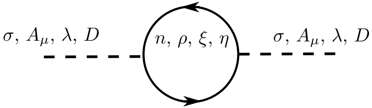
Quantum renormalization, $e_0^2 \to e^2(\sigma)$

$$S_{\rm eff}^{\rm kin} = \frac{1}{e^2}\int d^2x \left\{ -\frac{1}{4}F_{\alpha\beta}^2 + |\partial_\alpha \sigma|^2 + \frac{1}{2}D^2 \right\},$$

by 1-loop contribution (large-N)

$$\frac{1}{e^2} = \left(\frac{1}{e_0^2} + \frac{2N}{4\pi}\,\frac{1}{2|\sigma|^2} \right)\Bigg|_{e_0^2\to\infty} = \frac{2N}{4\pi}\,\frac{1}{2|\sigma|^2}\,.$$

following from (proportional to $\sum \mathtt{Q}^2 = 2N$)



Effective action

Results in

- Kinetic term $\int \bar{\Sigma}\Sigma$ runs to $2N \int \overline{\log \Sigma} \log \Sigma$ or after $\sigma = e^{-\frac{\phi + iY}{Q}}$ with compact $Y + 2\pi Q \sim Y$ into

$$S_{\text{eff}}^{\sigma} = \frac{1}{4\pi} \int d^2x \sqrt{h} \left(\frac{1}{2} h^{\alpha\beta} (\partial_\alpha \phi \partial_\beta \phi + \partial_\alpha Y \partial_\beta Y) - \frac{Q}{2} \phi R^{(2)} \right)$$

  with $Q \underset{N \to \infty}{\approx} \sqrt{2N}$, in nontrivial world-sheet metric linear dilaton comes from $\sigma \to \sigma(h)^{1/4}$ and metric-dependent determinants;
- Inclusion of twisted superpotential $Q^2 \int \overline{\log \Sigma} \log \Sigma + \left(\mu \int \Sigma + cc\right)$ completes it to $\mathcal{N} = 2$ Liouville theory, consistent with

$$\Delta(\sigma) = \left(\frac{1}{2}, \frac{1}{2} \right).$$

Exact equivalence?

Main conjecture:

- The effective IR action on the Coulomb branch is *exact* beyond large-N, if

$$Q^2 = 2(N-1) \underset{N\to\infty}{\approx} 2N$$

- Then

$$\hat{c}_{\mathcal{H}} = 2N - 1 = 1 + Q^2 = \hat{c}_L,$$

  and there is indeed "no border" between the Higgs and Coulomb "branches";
- The issues of complex deformations can be understood via the effective Liouville description.

Complex deformations

In gauge-invariant "mesonic" variables $w^{ij} = n^i \rho^j$, $i, j = 1, \ldots, N$

- The conifold $N = 2$ with $w = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$ can be described by a single equation $\det_{2\times 2} w = 0$, which can be deformed to further (to the Coulomb branch?) as $\det_{2\times 2} w = \epsilon$.
- For $N \geq 3$ there are $N_e = \left(\frac{N(N-1)}{2}\right)^2$ equations

$$F^{[ij][kl]} = w^{ik} w^{jl} - w^{jk} w^{il} = 0.$$

for $N_v = N^2$ variables, not a complete intersection, i.e.
$N_v - N_e \leq d = 2N - 1$.

Can it still be deformed?

Complex deformations

- Small deformations locally $\delta F^{[ij][kl]}(\vec{n},\vec{\rho}) \in \mathrm{Im} D_0(\vec{n},\vec{\rho})$ for $D_0(\vec{n},\vec{\rho})^{[ij][kl]}_{ab} = \frac{\partial F^{[ij][kl]}(\vec{w})}{\partial w^{ab}}\Big|_{w^{ab}=n^a\rho^b}$ to satisfy

$$D_0(\vec{n},\vec{\rho})^{[ij][kl]}_{ab}\delta w^{ab} = \delta F^{[ij][kl]}(\vec{w})\Big|_{w^{ab}=n^a\rho^b}$$

- To describe $\delta F^{[ij][kl]}(\vec{w})$ explicitly, construct $D_1(\vec{n},\vec{\rho})$ such that $\mathrm{Im} D_0(\vec{n},\vec{\rho}) = \ker D_1(\vec{n},\vec{\rho})$ for non-zero $\vec{n},\vec{\rho}$.
- Look for $\ker D_1$:
  - Although locally $\mathrm{Im} D_0(\vec{n},\vec{\rho}) = \ker D_1(\vec{n},\vec{\rho})$ (except for zero), it is possible that $\mathrm{Im} D_0 \neq \ker D_1$ in the space of polynomial functions $\mathbb{C}[\vec{n},\vec{\rho}]^{\deg_n=\deg_\rho}$;
  - Non-trivial deformations are described by $\ker D_1/\mathrm{Im} D_0$.

Complex deformations

- Formulas for (linear choice of) $D_1(\vec{n}, \vec{\rho})$:

$$D_1(\vec{n}, \vec{\rho})^{[i_1,i_2,i_3]\rho}_{[ij][kl]} = \sum_{\sigma \in S_3} (-1)^\sigma \rho^{\sigma(i_1)} \delta_{i\sigma(i_2)} \delta_{j\sigma(i_3)}$$

$$D_1(\vec{n}, \vec{\rho})^{[i_1,i_2,i_3]n}_{[ij][kl]} = \sum_{\sigma \in S_3} (-1)^\sigma n^{\sigma(i_1)} \delta_{k\sigma(i_2)} \delta_{l\sigma(i_3)}$$

- Example $N = 3$

$$D_1 = \begin{pmatrix} n^1 & -n^2 & n^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n^1 & -n^2 & n^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n^1 & -n^2 & n^3 \\ \rho^1 & 0 & 0 & -\rho^2 & 0 & 0 & \rho^3 & 0 & 0 \\ 0 & \rho^1 & 0 & 0 & -\rho^2 & 0 & 0 & \rho^3 & 0 \\ 0 & 0 & \rho^1 & 0 & 0 & -\rho^2 & 0 & 0 & \rho^3 \end{pmatrix}. \tag{2}$$

Non-trivial complex deformations

Two cases:

For $N \geq 3$ no deformations ker $D_1 = \mathrm{Im} D_0$ due to
Lemma: A general solution to $\sum_{k=1}^{M} x_k f_k(\vec{x}) = 0$ in polynomial functions is given by $f_k(\vec{x}) = \sum_{m=1}^{M} \Omega_{km}(\vec{x}) x_m$, with $\Omega_{km} = -\Omega_{mk}$.

Exceptional $N = 2$ case:

- $D_1 = \mathcal{V}$ (Voronov's operator); $D_0(\vec{n}, \vec{\rho}) = \begin{pmatrix} n^2\rho^2 & -n^2\rho^1 & -n^1\rho^2 & n^1\rho^1 \end{pmatrix}$
- Hence $\mathrm{Im} D_0 = \mathbb{C}[\vec{n}, \vec{\rho}]^{\deg_n = \deg_\rho \geq 1}$, but due to B.L.Voronov's **Lemma**: ker $D_1 =$ ker $\mathcal{V} = \mathbb{C}[\vec{n}, \vec{\rho}]^{\deg_n = \deg_\rho}$;
- Hence ker $D_1 / \mathrm{Im} D_0 = \mathbb{C} \cdot 1$ is non-empty.

The complex deformation exists only for $N = 2$.

Liouville side

$\mathcal{N} = 2$ Liouville action

$$S_L = \int d^2x \left\{ \frac{1}{4\pi} \left[\frac{1}{2} (\partial_\alpha \phi)^2 + \frac{1}{2} (\partial_\alpha Y)^2 - \frac{Q}{2} \phi R^{(2)} + \bar{\psi}_L i \partial_R \psi_L + \bar{\psi}_R i \partial_L \psi_R \right] \right.$$
$$\left. + \frac{2\tilde{\mu}}{Q^2} \psi_R \bar{\psi}_L e^{-\frac{\phi + iY}{Q}} + \frac{2\bar{\tilde{\mu}}}{Q^2} \psi_L \bar{\psi}_R e^{-\frac{\phi - iY}{Q}} - 4\pi \frac{|\tilde{\mu}|^2}{Q^2} : e^{-\frac{\phi - iY}{Q}} :: e^{-\frac{\phi + iY}{Q}} : \right\}$$

is reproduced by 1-loop computation with fermions.

- The central charge

$$c_L = 3 + 3Q^2, \qquad \hat{c}_L \equiv \frac{c_L}{3} = 1 + Q^2 = 2N - 1$$

- The mirror description is by (a SUSY version of) 2d black hole or the $SL(2, \mathbb{R})_k / U(1)$ coset with

$$k = \frac{2}{Q^2} = \frac{1}{N - 1}$$

N=1,2

Two important particular cases:

- $N = 1$, $Q^2 = 2(N-1) = 0$, $k \to \infty$, free field theory: e.g. the dual black-hole metric

$$ds^2 = dr^2 + \tanh^2 r d\vartheta^2 \to dr^2 + r^2 d\vartheta^2$$

- Conifold: $N = 2$, $Q = \sqrt{2}$, $k = 1$ i.e. "self-dual" case.

$N \geq 3$ is "generic" case, $k < 1$.

Spectrum

At $\phi \to \infty$

$$V_{j;m_L,m_R} \simeq e^{Q[j\phi + i(m_L Y_L + m_R Y_R)]},$$

corresponding to the target-space wave function $V(\phi, Y) = g_s(\phi)\Psi(\phi, Y)$

$$\Psi_{j;m_L,m_R}(\phi, Y) \underset{\phi\to\infty}{\sim} e^{Q(j+\frac{1}{2})\phi + iQ(m_L Y_L + m_R Y_R)}$$

- $j = -\frac{1}{2}$ distinguished case of (logarithmically) normalized state;
- Generally

$$V_{j,m_L,m_R} \simeq e^{iQ(m_L \tilde{Y}_L + m_R \tilde{Y}_R)} \left[e^{Qj\phi} + R(j, m_{L,R}; k) e^{-Q(j+1)\phi} \right] \tag{3}$$

with extra (reflected) exponent, and

$$R(j, m_{L,R}; k) \sim \prod_{m=m_L,m_R} \frac{\Gamma(m+j+1)}{\Gamma(m-j)}$$

$\mathcal{N} = 2$ Liouville operator

Liouville interaction exponent $\sigma = e^{-\frac{\phi+iY}{Q}} = e^{Q\left(-\frac{\phi}{Q^2}-\frac{iY}{Q^2}\right)}$ corresponds (formally?) to

$$j = m = -\frac{1}{Q^2} = -\frac{k}{2} = -\frac{1}{2(N-1)}$$

i.e. it is in the spectrum (with exactly $j = -\frac{1}{2}$) only for $N = 2$ i.e. in the case of conifold.

The Coulomb branch

$$\langle \sigma \overline{\sigma} \rangle \neq 0$$

only for $N = 2$ (reflection coefficient!).

The dual black-hole picture is “sick” for $k = \frac{1}{N-1} < 1$.

Conclusions

- GLSM describes the Higgs branch of effective 2d conformal string, for $N = 2$ this is resolved conifold for generic FI parameter β;
- At $\beta = 0$ the "Coulomb branch" can open up, effective $\mathcal{N} = 2$ Liouville in IR with $Q^2 = 2(N-1)$;
- It correspond to deformed conifold at $N = 2$, and there are no deformations at $N \geq 3$;
- It has a simple explanation, since only for $N = 2$ the Liouville operator $\sigma = e^{-\frac{\phi + iY}{Q}}$ is in the spectrum.