One-point thermal conformal blocks from four-point conformal integrals

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RECENT DEVELOPMENTS IN CONFORMAL INVARIANT QUANTUM FIELD THEORY

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Abstract:

A review of the recent results concerning the kinematics of conformal fields, the analysis of dynamical equations and dynamical derivation of the operator product expansion is given.

The classification and transformational properties of fields which are transformed according to the representations of the universal covering group of the conformal group have been considered. A derivation of the partial wave expansion of Wightman functions is given. The analytical continuation to the Euclidean domain of coordinates is discussed. As shown, in the Euclidean space the partial wave expansion can be applied either to one-particle irreducible vertices or to the Green functions, depending on the dimensions of the fields.

The structure of Green functions, which contain a conserved current and the energy-momentum tensor, has been studied. Their partial wave expansion has been obtained. A solution of the Ward identity has been found. Special cases are discussed.

The program of the construction of exact solution of dynamical equations is discussed. It is shown, that integral dynamical equations for vertices (or Green's functions) can be diagonalized by means of the partial wave expansion. The general solution of these equations is obtained. The equations of motion for renormalized fields are considered. The way to define the product of renormalized fields at coinciding points (arising on the right-hand side) is discussed. A recipe for calculating this product is presented. It is shown, that this recipe necessarily follows from the renormalized equations.

The role of bare term and of canonical commutation relations (for unrenormalized fields) is discussed in connection with the problem of the field product determination at coinciding points. As a result an exact relation between fundamental field dimensions is found for various three-linear interactions (section 16 and Appendix 6). The problem of closing the infinite system of dynamical equations is discussed.

Al above said results are demonstrated using Thirring model as an example. A new approach to its solving is developed.

The program of closing the infinite system of dynamical equations is discussed. The Thirring model is considered as an example. A new approach to the solution of this model is discussed.

Methods are developed for the approximate calculation of dimensions and coupling constants in the 3-vertex and 5-vertex approximations. The dimensions are calculated in the $\lambda\varphi^3$ theory in 6-dimensional space.

The problem of calculating the critical indices in statistics (3-dimensional Euclidean space) is considered. The calculation of the dimension is carried out in the framework of the $\lambda\varphi^4$ model. The value of the dimension and the critical indices thus obtained coincide with the experimental ones.

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New developments in *D*-dimensional conformal quantum field theory

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Conformal correlators at finite temperature

Consider a correlation function at finite temperature $T = \beta^{-1}$ of a scalar primary $\phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau, \Omega)$ on a D-dimensional cylinder $\mathrm{d}s^2_{\mathbb{R}\times\mathbb{S}^{D-1}} = \mathrm{d}\tau^2 + \mathrm{d}\Omega^2_{D-1}$

$$\langle \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau, \Omega) \rangle_\beta = \mathrm{Tr}_{\mathcal{H}} \left[\phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau, \Omega) e^{-\beta D} \right] , \qquad \mathcal{H} = \oplus V_{\Delta, s} \,.$$

- We focus on scalar modules ($s = 0$) $V_{\Delta,s} \equiv V_\Delta$, constructed from a primary state $|\Delta\rangle$

$$D\,|\Delta\rangle = \Delta\,|\Delta\rangle \,, \qquad J_{\mu\nu}\,|\Delta\rangle = 0 \,, \qquad K_\mu\,|\Delta\rangle = 0 \,,$$

the dilatation operator D is a Hamiltonian within the usual radial quantization.

- Descendant states in V_Δ at level $n = 0, 1, 2, \ldots$ are

$$|\Delta + n\rangle_{\mu_1 \ldots \mu_n} = P_{\mu_1} \ldots P_{\mu_n}\,|\Delta\rangle \,, \quad D\,|\Delta + n\rangle_{\mu_1 \ldots \mu_n} = (\Delta + n)\,|\Delta + n\rangle_{\mu_1 \ldots \mu_n} \,.$$

- The thermal correlation function is periodic in time coordinate τ:

$$\langle \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau, \Omega) \rangle_\beta = \mathrm{Tr}_{\mathcal{H}} \left[e^{\tau D} \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(0, \Omega) e^{-\tau D} e^{-\beta D} \right]$$

$$= \mathrm{Tr}_{\mathcal{H}} \left[e^{-\beta D} e^{(\beta+\tau) D} \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(0, \Omega) e^{-(\beta+\tau) D} \right] = \langle \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau + \beta, \Omega) \rangle_\beta \,.$$

It means that this correlation function is actually defined on $S^1_\beta \times S^{D-1}_{L=1}$.

Thermal Ward identities

The cylinder $\mathbb{R} \times \mathbb{S}^{D-1}$ is related to $\mathbb{R}^D$ via the standard map $r = e^{\tau}$, where $r^2 = x_\mu x^\mu$. For a primary $\phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau,\Omega) = r^h \phi(x)$ of conformal dimension h one has

$$\langle \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau,\Omega)\rangle_\beta = r^h \,\mathrm{Tr}_{\mathcal{H}}\left[\phi(x) e^{-\beta D}\right] \equiv r^h \langle \phi(x)\rangle_\beta \,.$$

- One can work either in $\mathbb{R}^D$ or in $\mathbb{R} \times \mathbb{S}^{D-1}$ coordinates.
- We use Δ for internal and h for external dimensions.
- Introducing temperature partially breaks conformal invariance $O(D+1,1)$ down to a subgroup. Namely, consider the following manipulation

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left[D\phi(x) e^{-\beta D}\right] &= \mathrm{Tr}_{\mathcal{H}}\left[[D,\phi(x)] e^{-\beta D}\right] + \mathrm{Tr}_{\mathcal{H}}\left[\phi(x) e^{-\beta D} D\right] \\ &= \mathcal{D}\langle \phi(x)\rangle_\beta + \mathrm{Tr}_{\mathcal{H}}\left[D\phi(x) e^{-\beta D}\right] \Rightarrow\end{aligned}$$

- Thermal Ward idetities = residual symmetry $O(1,1) \oplus O(D)$ of the thermal correlator

$$\mathcal{D}\,\langle\phi(x)\rangle_\beta = 0\,, \qquad \mathcal{J}_{\mu\nu}\,\langle\phi(x)\rangle_\beta = 0\,.$$

  In cylindrical coordinates $\mathcal{D} = \partial_\tau \Rightarrow \langle \phi_{\mathbb{R}\times\mathbb{S}^{D-1}}(\tau,\Omega)\rangle_\beta$ is τ independent.
- The high-temperature ($\beta \to 0$) limit partially recovers conformal invariance(L.Iliesiu, et.al 2018)

$$\lim_{L\to\infty} \langle\phi\rangle_{S^1_\beta \times S^{D-1}_L} = \langle\phi\rangle_{S^1_\beta\times\mathbb{R}^{D-1}}\,.$$

  $\Rightarrow \langle\phi\rangle_{S^1_\beta\times\mathbb{R}^{D-1}}$ is fixed by symmetry up to a (model-dependent) constant.

Thermal conformal blocks

The thermal correlator can be expanded in conformal blocks

$$\langle\phi(x)\rangle_\beta = \sum_\Delta C_{\Delta,h,\Delta}\mathcal{F}^h_\Delta(q,x) + \text{ spinning contributions}\,.$$

The scalar thermal conformal block here is a power series in $q = \exp(-\beta)$

$$\mathcal{F}^h_\Delta(q,x) = (C_{\Delta,h,\Delta})^{-1}\sum_{n=0}^{\infty} q^{\Delta+n}\left(B^{-1}_\Delta\right)^{\mu_1\ldots\mu_n;\nu_1\ldots\nu_n}{}_{\nu_1\ldots\nu_n}\langle\Delta+n|\phi(x)|\Delta+n\rangle_{\mu_1\ldots\mu_n}\,,$$

where $B_{\nu_1\ldots\nu_n;\mu_1\ldots\mu_n} = {}_{\nu_1\ldots\nu_n}\langle\Delta+n|\Delta+n\rangle_{\mu_1\ldots\mu_n}$ is a Gram matrix in V_Δ at n-th level.

- The low-temperature ($\beta\to\infty$) limit: $\mathcal{F}^h_\Delta(q,x) = q^\Delta(1+\ldots)$ as $q\to 0$.
- Ward identities fix the x-dependence of the thermal conformal block.

How one can calculate thermal conformal blocks?

- Direct calculation of the matrix elemets $\Rightarrow$ quickly becomes complicated.
- Casimir equations (Y.Gobeil, et.al 2018)

$$\mathrm{Tr}_{\mathcal{H}}\left[C_2\phi(x)e^{-\beta D}\right] = \Delta(D-\Delta)\langle\phi(x)\rangle_\beta\,,$$

$\Rightarrow$ tractable only in $D=2$ (P. Kraus, et.al 2017, K. Alkalaev, SM, M. Pavlov 2022).

Shadow formalism

For a scalar primary operator $\mathcal{O}_\Delta(x) \equiv \mathcal{O}(x)$ one defines the shadow operator (S. Ferrara et.al '70)

$$\widetilde{\mathcal{O}}(x) = N_\Delta \int_{\mathbb{R}^D} \mathrm{d}^D x_0 \, (x_0 - x)^{-2\tilde{\Delta}} \, \mathcal{O}(x_0) \,, \qquad N_\Delta = \pi^{-D} \frac{\Gamma(\Delta)\Gamma(\tilde{\Delta})}{\Gamma(\frac{D}{2} - \Delta)\Gamma(\frac{D}{2} - \tilde{\Delta})} \,,$$

which is a primary operator of (dual/shadow) conformal dimension $\tilde{\Delta} = D - \Delta$.
This allows one to construct a projecting operator

$$\Pi_\Delta = \int_{\mathbb{R}^D} \mathrm{d}^D x \, \mathcal{O}(x) |0\rangle \langle 0| \widetilde{\mathcal{O}}(x) \,, \qquad \Pi_{\Delta_n} \Pi_{\Delta_m} = \delta_{\Delta_n \Delta_m} \Pi_{\Delta_m} \,.$$

Inserting the projector into the 4-point correlation function of primary scalar operators $\phi_{h_i}(x_i) \equiv \phi_i(x_i)$ one finds

$$\begin{aligned} \langle \phi_1(x_1) \phi_2(x_2) \Pi_\Delta \phi_3(x_3) \phi_4(x_4) \rangle &= \int_{\mathbb{R}^D} \mathrm{d}^D x_0 \langle \phi_1 \phi_2 \mathcal{O}(x_0) \rangle \langle \widetilde{\mathcal{O}}(x_0) \phi_3 \phi_4 \rangle \\ &= C_{h_1, h_2, \Delta} \, C_{\Delta, h_3, h_4} \, \Psi_\Delta^{h_1, h_2, h_3, h_4}(x_1, x_2, x_3, x_4) \,. \end{aligned}$$

- The 4-point conformal partial wave (CPW) $\Psi_\Delta^{h_1, h_2, h_3, h_4}$ is a linear combination of the conformal and shadow blocks (e.g. D. Simmons-Duffin 2012, V. Rosenhaus 2018)

$$\Psi_\Delta^{h_1, h_2, h_3, h_4}(x_1, x_2, x_3, x_4) = G_\Delta^{h_1, h_2, h_3, h_4}(x_1, x_2, x_3, x_4) + N_\Delta \, K_\Delta^{h_1, h_2} \, K_{\tilde{\Delta}}^{h_3, h_4} \, G_{\tilde{\Delta}}^{h_1, h_2, h_3, h_4}(x_1, x_2, x_3, x_4) \,.$$

Conformal integral

Substituting the 3-point function into CPW one finds ($\boldsymbol{x} = \{x_1, x_2, x_3, x_4\}$)

$$\Psi_{\Delta}^{h_1,h_2,h_3,h_4}(x_1,x_2,x_3,x_4) = N_\Delta K_\Delta^{h_3,h_4}\, X_{12}^{\frac{\Delta-h_1-h_2}{2}}\, X_{34}^{\frac{\tilde{\Delta}-h_3-h_4}{2}}\, I_4^{\boldsymbol{a}}(\boldsymbol{x})\,, \qquad \text{where}$$

- $I_4^{\boldsymbol{a}}(\boldsymbol{x})$ is a 4-point conformal integral (K. Symanzik 1972)

$$I_4^{\boldsymbol{a}}(\boldsymbol{x}) = \int_{\mathbb{R}^D} \mathrm{d}^D x_0 \prod_{i=1}^{4} X_{0i}^{-a_i}\,, \qquad \text{where } X_{ij} = (x_i - x_j)^2\,, \quad \boldsymbol{a} = \{a_1, a_2, a_3, a_4\}\,, \quad \sum_{j=1}^{4} a_j = D\,.$$

- It can be expressed in terms of special functions (e.g. F. Dolan, H. Osborn 2000)

$$I_4^{\boldsymbol{a}}(\boldsymbol{x}) = \left(1 + C_4 + (C_4)^2 + (C_4)^3\right) \frac{\pi^{\frac{D}{2}} L_4^{\boldsymbol{a}}(\boldsymbol{x})\, i_4^{\boldsymbol{a}}(u,v)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\,, \quad u = \frac{X_{12}X_{34}}{X_{13}X_{24}}\,, \quad v = \frac{X_{14}X_{23}}{X_{13}X_{24}}\,,$$

  where $C_4 = (1,2,3,4)$ is a cyclic permutation.
- $L_4^{\boldsymbol{a}}(\boldsymbol{x})$ is the leg-factor responsible for the conformal covariance of $I_4^{\boldsymbol{a}}(\boldsymbol{x})$, and

$$i_4^{\boldsymbol{a}}(u,v) \sim F_4\left[\begin{array}{c} \alpha_1, \alpha_2 \\ \gamma_1, \gamma_2 \end{array} \middle| u, v\right], \quad F_4\left[\begin{array}{c} \alpha_1, \alpha_2 \\ \gamma_1, \gamma_2 \end{array} \middle| u, v\right] = \sum_{m_1,m_2=0}^{\infty} \frac{(\alpha_1)_{m_1+m_2}(\alpha_2)_{m_2+m_1}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}} \frac{u^{m_1}}{m_1!} \frac{v^{m_2}}{m_2!}\,,$$

  where $(\alpha)_m = \Gamma(\alpha+m)/\Gamma(\alpha)$, and α_i, γ_j are expressed in terms of a_k.
- This allows one to express $G_\Delta^{\boldsymbol{h}}$ in terms of fourth Appell function F_4.

Thermal shadow formalism

Let us generalise the presented construction to thermal correlators. To this end, consider

$$\mathrm{Tr}_{\mathcal{H}}\left[\Pi_\Delta \phi(x) q^D\right] = \int_{\mathbb{R}^D} \mathrm{d}^D x_0 \sum_{\Delta_1} \sum_{n=0}^{\infty} (B^{-1}_{\Delta_1})^{\mu_1\dots\mu_n;\nu_1\dots\nu_n} \times {}_{\nu_1\dots\nu_n}\langle \Delta_1 + n|\mathcal{O}(x_0)\,|0\rangle\,\langle 0|\,\widetilde{\mathcal{O}}(x_0)\phi(x) q^D |\Delta_1 + n\rangle_{\mu_1\dots\mu_n}\,,$$

where we focused again on scalar modules. After some manipulations we find that

$$\mathrm{Tr}\left[\Pi_\Delta \phi(x) q^D\right] = q^\Delta \int_{\mathbb{R}^d} \mathrm{d}^D x_0 \langle \widetilde{\mathcal{O}}(x_0)\phi(x)\mathcal{O}(qx_0)\rangle = C_{\Delta,h,\Delta}\Upsilon^h_\Delta(q,x)\,.$$

- The 1-point thermal conformal partial wave is

$$\Upsilon^h_\Delta(q,x) = N_\Delta\, K^{h,\Delta}_{\tilde{\Delta}}\, \frac{q^{D-h-\Delta}}{(1-q)^{D-h}}\; T_2^{a_1,a_2;a_0}(x/q,x)\,,$$

where we have defined the thermal conformal integral

$$T_2^{a_1,a_2;a_0}(x_1,x_2) = \int_{\mathbb{R}^D} \mathrm{d}^D x_0 X_{01}^{-a_1} X_{02}^{-a_2} (x_0^2)^{-a_0}\,, \qquad a_1 + a_2 + 2a_0 = D\,.$$

Thermal conformal block

The expression for $T_2^{a_1,a_2;a_0}$ is known in terms of F_4 (E. Boos, A. Davydychev 1987), but it is also given by the limit of the conformal integral

$$T_2^{a_1,a_2;a_0}(x_1,x_2) = \lim_{\substack{x_3\to 0\\ x_4\to\infty}} \left(X_{14}^{-a_1} X_{24}^{\frac{D}{2}-a_2-a_4} X_{34}^{\frac{D}{2}-a_3-a_4} \right)^{-1} I_4^{a_1,a_2,a_0,a_0}(x_1,x_2,x_3,x_4)\,.$$

- Partial breaking conformal invariance by fixing two points $x_3 = 0$ and $x_4 = \infty \Longleftrightarrow$ the residual symmetry $(O(1,1) \oplus O(D))$ of the thermal correlation function.
- $\Rightarrow \Upsilon_\Delta^h(q,x)$ is a linear combination of thermal conformal and shadow blocks

$$\Upsilon_\Delta^h(q,x) = \mathcal{F}_\Delta^h(q,x) + K_\Delta^{h,\Delta} K_\Delta^{\tilde{h},\Delta} N_\Delta\, \mathcal{F}_{\tilde{\Delta}}^h(q,x)\,,$$

where the 1-point thermal block $\mathcal{F}_\Delta^h$ is expressed through F_4:

$$r^h\, \mathcal{F}_\Delta^h(q,x) = \frac{\Gamma(\Delta)\Gamma(h-\tilde{\Delta})}{\Gamma(\frac{h}{2})\Gamma(\Delta-\frac{\tilde{h}}{2})}\, q^\Delta (1-q)^{-h}\, F_4\left[\begin{array}{c} \Delta-\frac{h}{2}\,, \frac{D}{2}-\frac{h}{2} \\ 1+\frac{D}{2}-h\,, 1-\frac{D}{2}+\Delta \end{array} \middle| (1-q)^2\,, q^2 \right]$$

$$+(h \to \tilde{h} = D-h)\,.$$

Outlooks

The elaborated techniques can be extended in several directions

- One can consider operators with spin, e.g., for spin-1 exchange the shadow operator reads as

$$\widetilde{\mathcal{O}}_\mu(x) = N_{\Delta,s=1} \int_{\mathbb{R}^D} \mathrm{d}^D x_0 \, (x_0 - x)^{-2\tilde{\Delta}} \, \mathcal{I}_{\mu\nu}(x_0 - x) \, \mathcal{O}^\nu(x_0) \,, \qquad \mathcal{I}_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / r^2 \,.$$

  $\Rightarrow$ It complicates the integrals to be calculated.

- One can generalize the thermal correlator, by adding chemical potentials, e.g.

$$\langle \phi(x) \rangle_{\beta,\mu} \equiv \mathrm{Tr}_{\mathcal{H}} \left[\phi(x) e^{-\beta D} e^{-i\mu J_{12}} \right] \,.$$

  $\Rightarrow$ It complicates the x-dependence of the thermal conformal block, but the Casimir equations can be written (Y.Gobeil, et.al 2018, I. Buric, et.al 2024).

- One can consider the multipoint thermal correlators:

$$\langle \phi_1(x_1) ... \phi_n(x_n) \rangle_\beta = \mathrm{Tr}_{\mathcal{H}} \left[\phi_1(x_1) ... \phi_n(x_n) e^{-\beta D} \right] \,.$$

  $\Rightarrow$ It requires knowledge of the multipoint conformal integrals, but there is a partial result in $D = 2$ (K. Alkalaev, SM 2023)

$$\mathcal{F}^{h_1,\ldots,h_n}_{\Delta_1,\ldots,\Delta_n}(q, z_1, ..., z_n) \sim F_N \left[\begin{array}{c} a_1, ..., a_n \\ c_1, ..., c_n \end{array} \middle| \rho_1, ..., \rho_n \right] \,,$$ where F_N is a hypergeometric type function.

Thank you for your attention!