Nambu-Goto string as a four-derivative Liouville theory

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Based on:

- Y. M. arXiv:2407.01136
- Y. M. Phys. Lett. B 845 (2023) 138170 [arXiv:2308.05030]
- Y. M. JHEP 09 (2023) 086 [arXiv:2307.06295]
- Y. M. JHEP 05 (2023) 085 [arXiv:2302.01954]
- Y. M. JHEP 01 (2023) 110 [arXiv:2212.02241]
- Y. M. IJMPA 38 (2023) 2350010 [arXiv:2204.10205]

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The beauty of 2D conformal theories

- Belavin-Polyakov-Zamolodchikov (1984):
  Virasoro algebra $\Longrightarrow$ Kac spectrum of the minimal models
- Knizhnik-Polyakov-Zamolodchikov (1988):
  $SL(2;R)$ Kac-Moody spectrum for the Liouville action
  (thanks to diffeomorphism invariance)

The number of surfaces of large area A embedded in d dimensions

$$\left\langle \delta\left(\int \sqrt{g} - A\right)\right\rangle \propto A^{\gamma_{\mathrm{str}}-3}\, \mathrm{e}^{CA}$$

with string susceptibility index of (closed) Polyakov's string

$$\gamma_{\mathrm{str}} = (h-1)\frac{25-d+\sqrt{(25-d)(1-d)}}{12} + 2 \qquad \boxed{\text{genus } h}$$

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Real in $d=4$ for the four-derivative Liouville theory and agrued to describe the Nambu-Goto string with $d_- = 15-4\sqrt{6} > 4$

$$\gamma_{\rm str} = (h-1)\frac{12+(d_+ + d_-)/2 - d + \sqrt{(d_+ - d)(d_- - d)}}{12} + 2$$

Content of the talk

- Nambu-Goto and Polyakov strings

- Generalized conformal anomaly ("massive" CFTs)
  - path-integrating over X^μ and ghosts
  - tracelessness of improved energy-momentum tensor
  - equivalence with four-derivative Liouville action
  - Salieri's check at one loop

- Exact solution and minimal models
  - singular products and universality of higher-derivative actions
  - BPZ null vectors and Kac's spectrum
  - Nambu-Goto string in d=4 as (4,3) minimal model

Nambu-Goto and Polyakov strings

Nambu-Goto string (imaginary Lagrange multiplier λ^{ab}) independent metric tensor g_{ab} Alvarez (1981)

$$K_0\int \mathrm{d}^2\omega\,\sqrt{\det\partial_a X\cdot\partial_b X} = K_0\int \mathrm{d}^2\omega\,\sqrt{g}+\frac{K_0}{2}\int \mathrm{d}^2\omega\,\lambda^{ab}\left(\partial_a X\cdot\partial_b X - g_{ab}\right)$$

Ground state $\lambda^{ab}=\bar\lambda\sqrt{g}g^{ab}$ classically $\bar\lambda=1$ $\Longrightarrow$ Polyakov string

$$\mathcal{S}=\frac{K_0}{2}\int \mathrm{d}^2\omega\,\sqrt{g}g^{ab}\partial_a X\cdot\partial_b X$$

Equivalence: classically Polyakov (1981), one loop Fradkin-Tseytlin (1982).

Closed bosonic string winding once around compactified dimension of circumference β, propagating (Euclidean) time L with topology of cylinder or torus (bagel). No tachyon if β is large enough.

Gaussian path integral over X^μ_{q} by splitting $X^\mu = X^\mu_{\text{cl}}+X^\mu_{\text{q}}$: $\Longrightarrow$ Emergent (or effective) action

$$S[\varphi,\lambda^{ab}] \;=\; K_0\int \mathrm{d}^2\omega\,\sqrt{g}+\frac{K_0}{2}\int \mathrm{d}^2\omega\,\lambda^{ab}\left(\partial_a X_{\text{cl}}\cdot\partial_b X_{\text{cl}} - g_{ab}\right)$$
$$+\frac{d}{2}\operatorname{tr}\log\left(-\frac{1}{\sqrt{g}}\partial_a\lambda^{ab}\partial_b\right)+\text{ ghosts}$$

Mean-field ground state with $\bar\lambda<\bar\lambda_{\text{cl}}=1$ is stable in $2<d<26$

2. “Massive” CFTs

Generalized conformal anomaly

Path-integrating over X^μ (and the usual ghosts)

$$S[g_{ab},\lambda^{ab}] = K_0 \int \left(\sqrt{g} - \frac{1}{2}\lambda^{ab} g_{ab}\right) + S_X[g_{ab},\lambda^{ab}],$$

$$\mathcal{S}_X = \frac{d}{96\pi}\int \left[-\frac{12\sqrt{g}}{a^2\sqrt{\det \lambda^{ab}}} + \sqrt{g}\, R \frac{1}{\Delta} R - (\beta \lambda^{ab} g_{ab} R + 2\lambda^{ab}\nabla_a \partial_b \frac{1}{\Delta} R)\right]$$

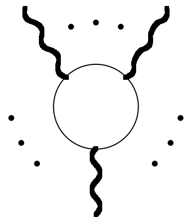
higher orders in Schwinger's proper-time ultraviolet cutoff τ dropped.
$\beta = 1$ for the Nambu-Goto string but kept arbitrary for generality.
The action is derived from the DeWitt-Seeley expansion of

$$\mathcal{O} = -(\sqrt{g})^{-1}\partial_a \lambda^{ab}\partial_b = -h^{ab}\partial_a\partial_b + A^a \partial_a$$

$$\left\langle \left| \mathrm{e}^{-\tau\mathcal{O}} \right| \right\rangle = \frac{1}{4\pi\tau} + \frac{1}{4\pi}\left(\frac{1}{6}R(h) + E(A,h)\right) + O(\tau)$$

Alternatively Coleman-Weinberg's effective action integrating out X^μ_q

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{1}{\sqrt{g}}\partial_a \lambda^{ab}\partial_b\right]_{\mathrm{reg}} = \sum_n \frac{1}{n}$$



wavy lines correspond to fluctuations $\delta\lambda^{ab}$ or δg_{ab} about ground state

Covariant Pauli-Villars regularization

Schwinger proper-time regularization of the trace

$$\operatorname{tr}\log\mathcal{O}\big|_{\text{reg}} = -\int_{a^2}^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}, \qquad \Lambda^2 = \frac{1}{4\pi a^2}$$

Pauli-Villars regularization of the trace

$$\det(\mathcal{O})\big|_{\text{reg}} \equiv \frac{\det(\mathcal{O})\det(\mathcal{O}+2M^2)}{\det(\mathcal{O}+M^2)^2}, \qquad \Lambda^2 = \frac{M^2}{2\pi}\log 2$$

$$\operatorname{tr}\log\mathcal{O}\big|_{\text{reg}} = -\int_0^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}\left(1-e^{-\tau M^2}\right)^2, \quad \left\langle\left|e^{-\tau\mathcal{O}}\right|\right\rangle = \frac{1}{4\pi\tau}+\ldots$$

Diaz, Troost, van Nieuwenhuizen, Van Proeyen (1989)

Covariant Pauli-Villars regulator Y (preserves conformal invariance)

$$\mathcal{S}^{(\text{reg})} = \int\left(\lambda^{ab}\,\partial_a\bar{Y}\cdot\partial_b Y + \boxed{M^2\sqrt{g}}\,\bar{Y}\cdot Y + \frac{1}{2}\lambda^{ab}\,\partial_a Z\cdot\partial_b Z + \boxed{M^2\sqrt{g}}\,Z^2\right)$$

Two anticommuting Grassmann Y and $\bar{Y}$ of mass squared M^2 and one Z of mass squared $2M^2$ with normal statistics:

Advantages over the proper-time regularization:
Feynman's diagrams apply for Pauli-Villars regularization
Gel'fand-Yaglom technique to compare with DeWitt-Seeley expansion

Conformal gauge and flat background

Emergent action becomes local in conformal gauge

$$g_{ab} = \hat{g}_{ab}\, \mathrm{e}^{\varphi}$$

where $\hat{g}_{ab}$ is background (or fiducial) metric tensor.
Usual ghosts and their usual contribution to effective action

Euclidean CFT: conformal coordinates z and $\bar{z}$ in flat background
$g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$ ($\beta = 1$ for the Nambu-Goto string)

$$\mathcal{S}[\varphi, \lambda^{ab}] \;=\; K_0 \int \mathrm{e}^{\varphi}(1 - \lambda^{z\bar{z}}) + \frac{1}{24\pi}\int [-\frac{3d\,\mathrm{e}^{\varphi}}{a^2\sqrt{\det \lambda^{ab}}} + (d-26)\varphi\partial\bar{\partial}\varphi$$
$$+ d\kappa(2(1+\beta)\lambda^{z\bar{z}}\partial\bar{\partial}\varphi + \lambda^{zz}\nabla\partial\varphi + \lambda^{\bar{z}\bar{z}}\bar{\nabla}\bar{\partial}\varphi)]$$

$\nabla = \partial - \partial\varphi$ is covariant derivative in conformal gauge so it describes a theory with interaction (no such interaction if only $\lambda^{z\bar{z}} = \lambda^{ab}\hat{g}_{ab}$)

Subtleties because of nonminimal interaction with background gravity

$$\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} - \hat{\Delta}\varphi\right)$$

It vanishes only if the background curvature $\hat{R}$ vanishes

Improved energy-momentum tensor

Callan-Coleman-Jackiw (1970)

Symmetric minimal energy-momentum tensor (by applying $\delta/\delta\hat{g}^{ab}$)

$$
\begin{aligned}
T^{(\mathrm{min})}_{zz} &= \frac{(d-26)}{24}(\partial\varphi)^2 + \frac{d\kappa}{24}[2(1+\beta)\partial\lambda^{z\bar{z}}\partial\varphi \\
&\quad + \bar{\partial}\lambda^{\bar{z}\bar{z}}\partial\varphi - \partial\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi - 2\lambda^{\bar{z}\bar{z}}\partial\bar{\partial}\varphi + 2\lambda^{\bar{z}\bar{z}}\partial\varphi\bar{\partial}\varphi] \\
T^{(\mathrm{min})}_{z\bar{z}} &= K_0\,\mathrm{e}^{\varphi}(1-\lambda^{z\bar{z}}) - \frac{d\,\mathrm{e}^{\varphi}}{2a^2\sqrt{\det\lambda^{**}}} + \frac{d\kappa}{24}[\bar{\partial}\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi \\
&\quad + \lambda^{\bar{z}\bar{z}}\bar{\partial}^2\varphi + \partial\lambda^{zz}\partial\varphi + \lambda^{zz}\partial^2\varphi]
\end{aligned}
$$

is conserved obeying $\bar{\partial}T^{(\mathrm{min})}_{zz} + \partial T^{(\mathrm{min})}_{\bar{z}z} = 0$ but not traceless.

Improved e-m tensor is given by the sum $T_{ab} = T^{(\mathrm{min})}_{ab} + T^{(\mathrm{add})}_{ab}$

$$
\begin{aligned}
T^{(\mathrm{add})}_{zz} &= -\frac{(d-26)}{12}\partial^2\varphi - \frac{d\kappa}{24}\left[2(1+\beta)\partial^2\lambda^{z\bar{z}} + \partial\bar{\partial}\lambda^{\bar{z}\bar{z}} + \partial(\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi)\right] \\
&\quad - \frac{d\kappa}{24}\left[\frac{1}{\bar{\partial}}\left(\partial^3\lambda^{zz} + \partial^2(\lambda^{zz}\partial\varphi)\right)\right]
\end{aligned}
$$

nonlocal term!

as a price for $\bar{\partial}T_{zz} = 0$ and $T_{z\bar{z}} = 0$.

Also from Nambu-Goto EMT: $T_{zz} = \left\langle \lambda^{z\bar{z}}\partial X\cdot\partial X + \lambda^{\bar{z}\bar{z}}\partial X\cdot\bar{\partial}X\right\rangle_X$

Non-local term gives classically an addition to Virasoro algebra

$$
\delta_\xi T_{zz} = \xi'''\frac{1}{2b^2} + 2\xi' T_{zz} + \xi\partial T_{zz} - \xi''\frac{1}{\bar{\partial}}\partial\nabla\lambda^{zz}
$$

Improved energy-momentum tensor (cont.)

Conservation and tracelessness of classical IEMT follows from

$$\frac{1}{\pi}\bar{\partial}T_{zz} = \partial\varphi\frac{\delta\mathcal{S}}{\delta\varphi} - \partial\frac{\delta\mathcal{S}}{\delta\varphi} - \lambda^{\bar{z}\bar{z}}\partial\frac{\delta\mathcal{S}}{\delta\lambda^{\bar{z}\bar{z}}} + \partial\lambda^{z\bar{z}}\frac{\delta\mathcal{S}}{\delta\lambda^{z\bar{z}}} + \partial(\lambda^{zz}\frac{\delta\mathcal{S}}{\delta\lambda^{zz}}) + \partial\lambda^{zz}\frac{\delta\mathcal{S}}{\delta\lambda^{zz}}$$

General property of improved energy-momentum tensor:

$$T_a^a \equiv \hat{g}^{ab}\frac{\delta\mathcal{S}}{\delta\hat{g}^{ab}} = -\frac{\delta\mathcal{S}}{\delta\varphi}$$

i.e. trace of IEMT = the classical equation of motion for φ.

In quantum theory variations of $\mathcal{S}$ replaced by variational derivatives.

IEMT does generate conformal transformation $\delta z = \xi(z)^*$

$$\hat{\delta}_\xi = \frac{1}{\pi}\int \xi\bar{\partial}T_{zz} = \int\Big[(\xi' + \xi\partial\varphi)\frac{\delta}{\delta\varphi} + (\xi'\lambda^{\bar{z}\bar{z}} + \xi\partial\lambda^{\bar{z}\bar{z}})\frac{\delta}{\delta\lambda^{\bar{z}\bar{z}}} + \xi\partial\lambda^{z\bar{z}}\frac{\delta}{\delta\lambda^{z\bar{z}}} + (-\xi'\lambda^{zz} + \xi\partial\lambda^{zz})\frac{\delta}{\delta\lambda^{zz}}\Big]$$

Classically it produces the right transformation laws of φ and λ^{ab} with components $\lambda^{\bar{z}\bar{z}}$, $\lambda^{z\bar{z}}$, λ^{zz} of conformal weights 1, 0, −1, respectively

*Note $\delta_\xi\lambda^{ab} = -(\partial_c\xi^a)\lambda^{bc} - (\partial_c\xi^b)\lambda^{ac} + (\partial_c\xi^c)\lambda^{ab} + \xi^c\partial_c\lambda^{ab}$ under diffeomorphisms

Improved energy-momentum tensor (cont.)

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Equivalence with four-derivative Liouville action

Path integral over $\delta\lambda^{ab}$ has a saddle point justified by small a^2 at

$$\delta\lambda^{ab} = \sqrt{g}a^2 \left(g^{ac}g^{bd}\nabla_c\partial_d\varphi + \frac{(\beta-1)}{4} g^{ab}\Delta\varphi \right) \frac{\kappa}{3} + \mathcal{O}(a^4)$$

Thus we arrive at four-derivative Liouville action (conformal gauge)

$$\mathcal{S}[\varphi] = \frac{1}{16\pi b_0^2} \int \sqrt{\hat{g}}[\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \varepsilon\, \mathrm{e}^{-\varphi}\hat{\Delta}\varphi \left(\hat{\Delta}\varphi - G\hat{g}^{ab}\,\partial_a\varphi\partial_b\varphi \right)]$$

with $G = -1/3$ for the Nambu-Goto string

$$b_0^2 = \frac{6}{26-d}, \qquad G = -\frac{1}{1+(1+\beta)^2/2}, \qquad \varepsilon = -\frac{2d\kappa^2\bar{\lambda}^3}{3G(26-d)} a^2$$

which was exactly solved previously Y.M. (2023)

Classically higher-derivative terms vanish for smooth $\varepsilon R \ll 1$.
Quantumly quartic derivative provides UV cutoff but also interaction with coupling ε $\Rightarrow$ uncertainties $\varepsilon \times \varepsilon^{-1}$ which revive $\Longrightarrow$ anomalies.
Yet higher terms which are primary scalars like R^n do not change – universality. $g^{ab}\,\partial_a\varphi\partial_b\varphi$ is not primary

Smallness of ε is compensated by change of the metric (shift of φ)

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Thus we arrive at four-derivative Liouville action (covariant)

$$\mathcal{S}[g] = \frac{1}{16\pi b_0^2} \int \sqrt{g} \left[-R\frac{1}{\Delta}R + \varepsilon R \left(R + G g^{ab}\, \partial_a \frac{1}{\Delta} R\, \partial_b \frac{1}{\Delta} R \right) \right]$$

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3. CFT á la KPZ-DDK

Review of KPZ-DDK

Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

Liouville action in fiducial (or background) metric $\hat{g}_{ab}$

$$S_L = \frac{1}{8\pi b^2}\int \sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + q\hat{R}\varphi\right) + \mu^2\int\sqrt{\hat{g}}\,\mathrm{e}^{\varphi}$$

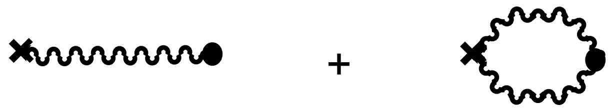
renormalized parameters b^2 and q called "intelligent" one (ione) loop

$$b^2 = b_0^2 + \mathcal{O}(b_0^4), \quad q = 1 + \mathcal{O}(b_0^2), \quad b_0^2 = \frac{6}{26-d}$$

Energy-momentum pseudotensor

$$T_{zz}^{(\varphi)} = -\frac{1}{4b^2}\left(\partial_z\varphi\partial_z\varphi - 2q\partial_z^2\varphi\right) \qquad \sqrt{g}R = \sqrt{\hat{g}}\,(q\hat{R} - \hat{\Delta}\varphi)$$

Background independence:



total central charge $\qquad c = d - 26 + 6\dfrac{q^2}{b^2} + 1 = 0$

conformal weight $\qquad \Delta(\mathrm{e}^{\alpha\varphi}) = \alpha q - \alpha^2 b^2 = 1$

$$\Longrightarrow \qquad b = \sqrt{\frac{25-d}{24}} - \sqrt{\frac{1-d}{24}}, \quad q = 1 + b^2 \quad \text{for } \alpha = 1$$

EMT for the four-derivative Liouville action

For minimal coupling to gravity

$$-4b_0^2 T_{ab}^{(\mathrm{min})} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\partial^c\varphi\partial_c\varphi - \mu_0^2 g_{ab} - \varepsilon\partial_a\varphi\partial_b\Delta\varphi - \varepsilon\partial_a\Delta\varphi\partial_b\varphi$$
$$+\varepsilon g_{ab}\partial^c\varphi\partial_c\Delta\varphi + \frac{\varepsilon}{2}g_{ab}(\Delta\varphi)^2 - G\varepsilon\partial_a\varphi\partial_b\varphi\Delta\varphi + G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c\varphi\partial_c\varphi)$$
$$+G\frac{\varepsilon}{2}\partial_a(\partial^c\varphi\partial_c\varphi)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d\varphi\partial_d\varphi)$$

"Improved" e-m tensor

$$-4b_0^2 T_{ab} = -4b_0^2 T_{ab}^{(\mathrm{min})} - 2(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\varphi - \varepsilon\Delta\varphi + G\frac{\varepsilon}{2}g^{ab}\partial_a\varphi\partial_b\varphi)$$
$$+2G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)\frac{1}{\Delta}\partial^d(\partial_d\varphi\Delta\varphi)$$

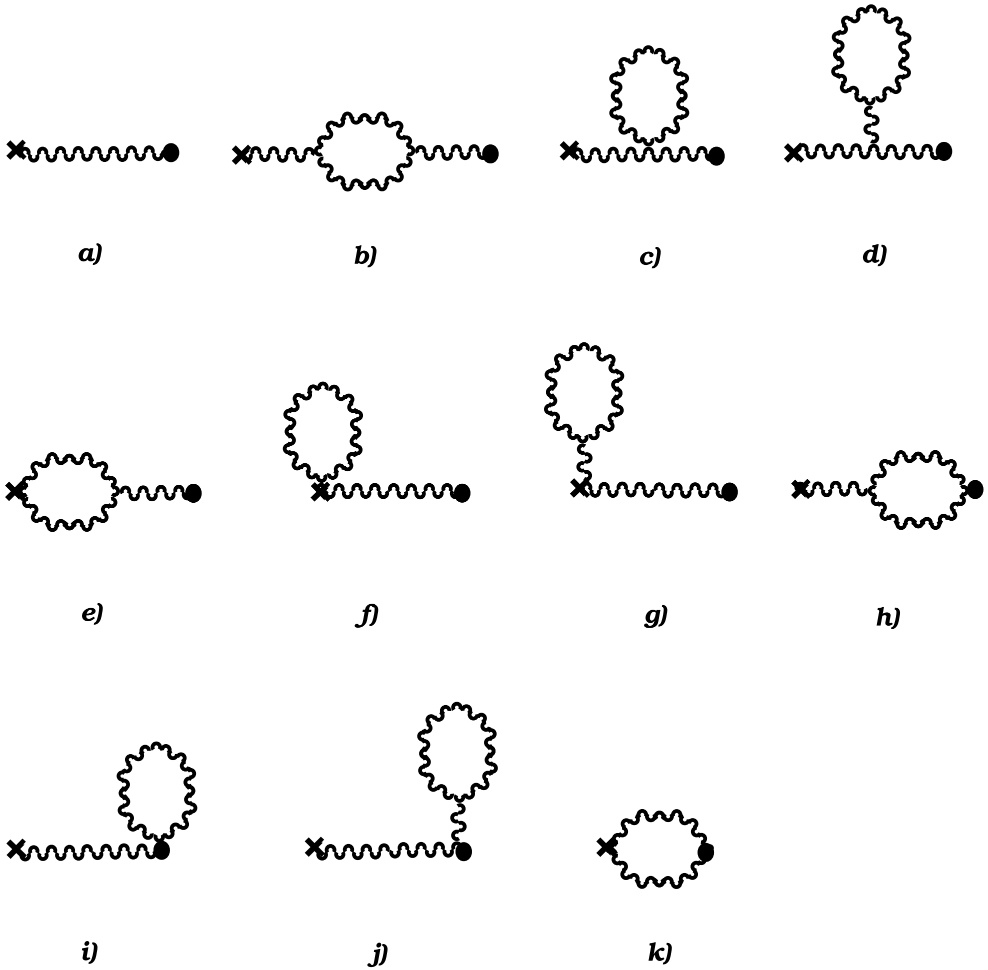
is conserved and traceless (!) thanks to diffeomorphism invariance

T_{zz} component (in two dimensions) Kawai, Nakayama (1993) at G=0

$$-4b_0^2 T_{zz} = (\partial\varphi)^2 - 2\varepsilon\partial\varphi\partial\Delta\varphi - 2\partial^2(\varphi - \varepsilon\Delta\varphi) - G\varepsilon(\partial\varphi)^2\Delta\varphi$$
$$+4G\varepsilon\partial\varphi\partial(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) - 4G\varepsilon\partial^2(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) + G\varepsilon\partial(\partial\varphi\Delta\varphi)$$
$$+G\varepsilon\frac{1}{\bar\partial}\partial^2(\bar\partial\varphi\Delta\varphi)$$

KPZ-DDK for the four-derivative Liouville action

One-loop operator products $T_{zz}(z)\,\mathrm{e}^{\varphi(0)}$ and $T_{zz}(z)T_{zz}(0)$



Conformal weight of $\mathrm{e}^{\varphi(0)}$: $\quad 1 = q - b^2$.

In central charge of φ nonlocal term revives: $\quad c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6Gq$

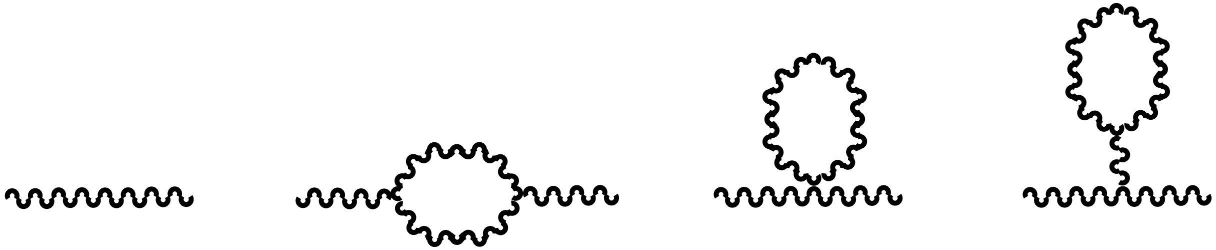
4. Algebraic check of DDK

Salieri:

"I checked the harmony with algebra.
Then finally proficient in the science,
I risked the rare delights of creativity."

A. Pushkin, *Mozart and Salieri*

One-loop propagator



a) b) c) d)

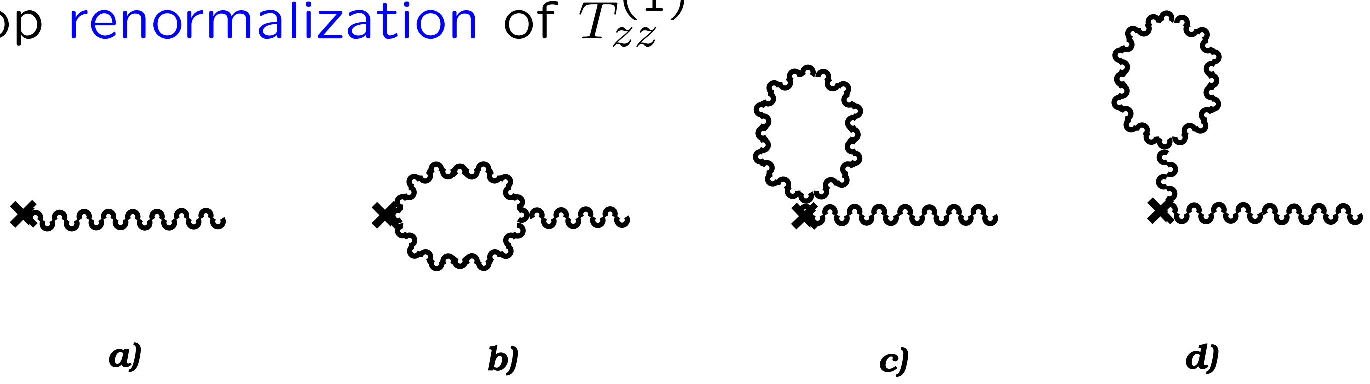
$$b) \;=\; -\frac{1}{4}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\frac{\varepsilon^2 k^2 (k-p)^2}{(1+\varepsilon k^2)[1+\varepsilon(k-p)^2]}\right.$$
$$-2\frac{(\varepsilon k^2(k-p)^2 - M^2)^2}{(k^2+M^2+\varepsilon k^4)[(k-p)^2+M^2+\varepsilon(k-p)^4]}$$
$$\left.+\frac{(\varepsilon k^2(k-p)^2 - 2M^2)^2}{(k^2+2M^2+\varepsilon k^4)[(k-p)^2+2M^2+\varepsilon(k-p)^4]}\right\}|\varphi(p)|^2$$
$$\to\; b)|_{\text{reg div}} - \frac{p^2}{96\pi}|\varphi(p)|^2$$

One-loop renormalization of b^2 where $A(\varepsilon M^2) \sim \varepsilon M^2 =$ tadpole $d)$

$$\frac{1}{b^2} = \frac{1}{b_0^2} - \left(\frac{1}{6} - 4 + A + 2G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} - \frac{1}{2}GA\right) + \mathcal{O}(b_0^2)$$

One-loop renormalization of T_{zz}

One-loop renormalization of $T_{zz}^{(1)}$



$$\frac{q}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 - \frac{1}{2}A - \frac{1}{2}G - G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} + \frac{1}{4}GA$$

or multiplying by b^2

$$\frac{q^2}{b^2} = \left(\frac{q}{b^2}\right)^2 \times b^2 = \frac{1}{b_0^2} - \frac{1}{6} - G + \mathcal{O}(b_0^2)$$

This precisely confirms the above shift of the central charge by $6G$ obtained by conformal field theory technique of DDK.

Tremendous cancellation due to diffeomorphism invariance proving "intelligent" one (ione) loop to be exact: (like Duistermaat-Heckman?)

$$-\frac{6}{b_0^2} + \frac{6q^2}{b^2} + 1 + 6Gq = 0, \qquad 1 = q - b^2$$

5. Method of singular products as pragmatic mixture of CFT and QFT

Conformal transformation revisited

Generator of conformal transformation for nonquadratic e-m tensor

$$\hat{\delta}_\xi \equiv \int_{D_1} \left(\xi' \frac{\delta}{\delta \varphi} + \xi \partial \varphi \frac{\delta}{\delta \varphi} \right) \overset{\text{w.s.}}{=} \int_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) T_{zz}(z)$$

where D_1 includes singularities of $\xi(z)$ and C_1 bounds D_1.
Equivalence of two forms is proved by integrating the total derivative

$$\bar{\partial} T_{zz} = -\pi \partial \frac{\delta S}{\delta \varphi} + \pi \partial \varphi \frac{\delta S}{\delta \varphi}$$

and using the (quantum) equation of motion

$$\frac{\delta S}{\delta \varphi} \overset{\text{w.s.}}{=} \frac{\delta}{\delta \varphi}$$

Actually, the form of $\hat{\delta}_\xi$ in the middle is primary.
It takes into account a tremendous cancellation of the diagrams, while there are subtleties associated with singular products

List of singular products

The simplest singular product

$$\frac{1}{b^2}\int \mathrm{d}^2z\,\xi(z)\,\langle\partial^n\varphi(z)\varphi(0)\rangle\,\delta^{(2)}(z) = (-1)^n\frac{2}{n(n+1)}\partial^n\xi(0)$$

arises already in a free CFT by the formulas

$$\delta^{(2)}(z) = \bar\partial\frac{1}{\pi z}, \qquad \frac{1}{z^n}\bar\partial\frac{1}{z} = (-1)^n\frac{1}{(n+1)!}\partial^n\bar\partial\frac{1}{z}$$

It can be alternatively derived introducing the regularization by ε

$$G_\varepsilon(k) = \frac{1}{k^2(1+\varepsilon k^2)}, \qquad \delta_\varepsilon^{(2)}(k) = \frac{1}{(1+\varepsilon k^2)}$$

We then have

$$8\pi\int \mathrm{d}^2z\,\xi(z)\partial^n G_\varepsilon(z)\delta_\varepsilon^{(2)}(z) = (-1)^n\frac{2}{n(n+1)}\partial^n\xi(0)$$

$$8\pi\int \mathrm{d}^2z\,\xi(z)[-4\varepsilon\partial^{n+1}\bar\partial G_\varepsilon(z)]\delta_\varepsilon^{(2)}(z) = (-1)^n\frac{2}{(n+1)}\partial^n\xi(0)$$

Computation of the central charge

Y.M. (2023)

Central charge $c^{(\varphi)}$ of φ can be computed for normal-ordered T_{zz} as

$$\left\langle \widehat{\delta}_\xi T_{zz}(\omega) \right\rangle = \frac{c^{(\varphi)}}{12} \xi'''(\omega)$$

For quadratic part of T_{zz}

$$\left\langle \widehat{\delta}_\xi T_{zz}^{(2)}(\omega) \right\rangle = \frac{1}{2b^2} \int \mathrm{d}^2 z \langle q^2 \xi'''(z) + \xi'(z) \partial^2 \varphi(z) \varphi(\omega) + \xi(z) \partial^3 \varphi(z) \varphi(\omega) \rangle$$
$$\times \delta^{(2)}(z-\omega) = \frac{\xi'''(\omega)}{2} \left(\frac{q^2}{b^2} + \frac{1}{3} - \frac{1}{6} \right) = \xi'''(\omega) \left(\frac{q^2}{2b^2} + \frac{1}{12} \right)$$

Here $1/12$ gives the usual quantum addition 1 to the central charge.
DDK formula for the central charge is reproduced for quadratic action.
Propagator is exact $\Longrightarrow$ this is why b^2 cancels

Computation of the central charge (cont.)

Computation for quartic part is lengthy but doable with Mathematica

$$\left\langle \hat{\delta}_\xi T^{(4)}_{zz}(\omega)\right\rangle_{G=0} = \frac{1}{b^2}\int d^2 z \left\langle [2q\alpha\varepsilon\xi'''(z)\partial\bar\partial\varphi(z) + (4q\alpha-2)\varepsilon\xi''(z)\partial^2\bar\partial\varphi(z) -6\varepsilon\xi'(z)\partial^3\bar\partial\varphi(z) - 4\varepsilon\xi(z)\partial^4\bar\partial\varphi(z)]\varphi(\omega)\right\rangle \delta^{(2)}_\varepsilon(z-\omega)$$

$$= \frac{\xi'''(\omega)}{4}\left(-2\cdot 2q\alpha + (4q\alpha-2)\cdot 1 + 6\frac{2}{3} - 4\frac{1}{2}\right) = 0$$

Central charge of φ equals 1 at $G=0$ as for quadratic action.
Computations is similar to one loop but higher loops are taken into account by b^2, q and α $\Longrightarrow$ why I call it "intelligent" one (ione) loop

Contribution from the G-term comes solely from the nonlocal part

$$\left\langle \hat{\delta}_\xi T^{(4)}_{zz}(\omega)\right\rangle_{G} = -\frac{2}{b^2}Gq\varepsilon\int d^2 z \left\langle [\xi'''(z)\partial\bar\partial\varphi(z) + \xi''(z)\partial^2\bar\partial\varphi(z)]\varphi(\omega)\right\rangle \times\delta^{(2)}_\varepsilon(z-\omega) = \frac{1}{2}Gq\xi'''(\omega)$$

The vanishing of total central charge results in the modified second DDK equation

$$\frac{6q^2}{b^2} + 1 + 6Gq = \frac{6}{b_0^2}$$

Universality of six and higher orders

To show the universality at order a^{2m} we use

$$G_\varepsilon(k) = \frac{1}{k^2(1+\varepsilon k^2)^m}, \qquad \delta_\varepsilon^{(2)} = \frac{1}{(1+\varepsilon k^2)^m}$$

$m=1$ for the four-derivative action. I have derived

$$8\pi \int \mathrm{d}^2 z\, \xi(z)[\partial^n(-4\partial\bar\partial)^k G_\varepsilon(z)]\delta_\varepsilon^{(2)}(z) = (-1)^n H_{n,m}^{(k)} \partial^n \xi(0)$$

$$H_{n,m}^{(k)} = 2\frac{\Gamma(n+k)\Gamma(2m-k)\Gamma(m+n)}{\Gamma(n+1)\Gamma(m)\Gamma(2m+n)}$$

It can be used for proving the universality of higher terms emerging for the Polyakov string

$$\begin{aligned}
\left\langle \hat\delta_\xi T_{zz}^{(\varphi,2)}(\omega)\right\rangle &= \frac{\xi'''(\omega)}{2}\left(\frac{q^2}{b^2} + H_{2,m}^{(0)} - H_{3,m}^{(0)}\right) = \xi'''(\omega)\left(\frac{q^2}{2b^2}+\frac{1}{12}\right)\\
\left\langle \hat\delta_\xi T_{zz}^{(\varphi,4)}(\omega)\right\rangle &= \frac{\xi'''(\omega)}{4}\left(-2H_{0,m}^{(1)} + 2H_{1,m}^{(1)} + 6H_{2,m}^{(1)} - 4H_{3,m}^{(1)}\right) = 0\\
\left\langle \hat\delta_\xi T_{zz}^{(\varphi,6)}(\omega)\right\rangle &= \xi'''(\omega)\left(-\frac{1}{2}H_{0,m}^{(2)} - H_{1,m}^{(2)} + 3H_{2,m}^{(2)} - \frac{3}{2}H_{3,m}^{(2)}\right) = 0
\end{aligned}$$

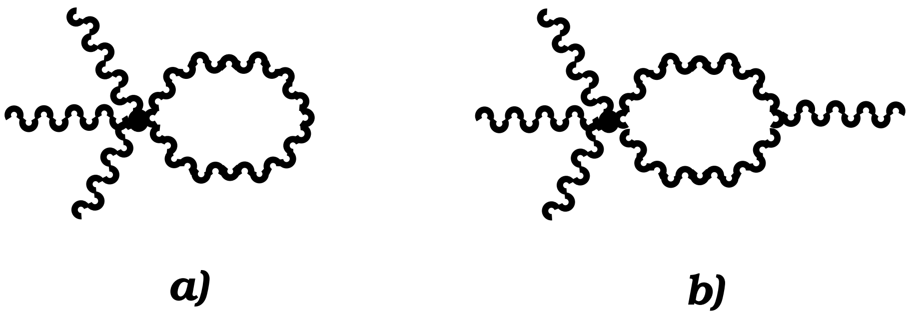
Heuristic understanding of universality

For general action $(F(x) = (1+x)/8\pi b_0^2$ for four-derivative)

$$S^{\text{gen}}[\varphi] = -\frac{1}{2}\int\sqrt{\hat{g}}\varphi\hat{\Delta}F(-\varepsilon\,\mathrm{e}^{-\varphi}\hat{\Delta})\varphi, \quad F(0) = \frac{1}{8\pi b_0^2}$$

propagator: $k^2F(\varepsilon k^2)$ and triple vertex: $\varepsilon k^4F'(\varepsilon k^2)$.

One-loop renormalization $\mathrm{e}^{\varphi} \Rightarrow \mathrm{e}^{\alpha\varphi}$ (wavy lines represent φ)

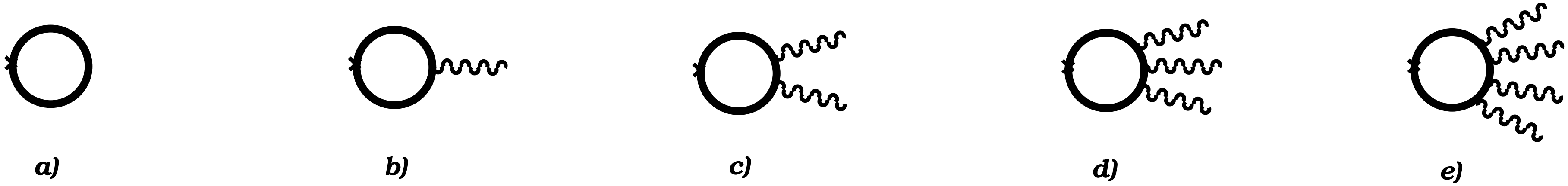


$$b) = \frac{\mathrm{e}^{\varphi}}{2}\times\varphi\int\frac{d^2k}{(2\pi)^2}\frac{\varepsilon k^4F'(\varepsilon k^2)}{[k^2F(\varepsilon k^2)]^2} = \frac{1}{8\pi F(0)}\mathrm{e}^{\varphi}\varphi = \mathrm{e}^{\varphi}b_0^2\varphi$$

independently on the choice of F like anomalies in QFT

Universality of nonlocal terms?

Nonlocal term comes from averaging EMT of the Nambu-Goto string over X^μ due to interaction $\lambda^{zz}\partial X \cdot \partial X$. Emerging T_{zz}: X^μ = solid line



$$b) \;=\; d\int \mathrm{d}^2\omega[\partial^2 G_0(\omega-\omega_0)]^2\lambda^{zz}(\omega) = d\int \mathrm{d}^2\omega\frac{1}{(\omega-\omega_0)^4}\lambda^{zz}(\omega)$$

$$= d\,\partial^3\int \mathrm{d}^2\omega\frac{1}{(\omega-\omega_0)}\lambda^{zz}(\omega) \equiv d\,\partial^3\frac{1}{\bar\partial}\lambda^{zz}(\omega_0)$$

$$c) \;=\; d\int \mathrm{d}^2\omega_1\mathrm{d}^2\omega_2\,\partial^2 G_0(\omega_1-\omega_0)\partial^2 G_0(\omega_2-\omega_0)\partial\bar\partial G_0(\omega_1-\omega_2)$$

$$\times\lambda^{zz}(\omega_1)\delta\lambda^{z\bar z}(\omega_2) = d\,\partial^3\frac{1}{\bar\partial}\lambda^{zz}(\omega_0)\delta\lambda^{z\bar z}(\omega_0)$$

d), e) etc. do not contribute at one-loop order

$$\left\langle \hat\delta_\xi T^{(\mathrm{NL})}_{zz}\right\rangle \;=\; \frac{1}{2b^2}\partial^2\frac{1}{\bar\partial}\int d^2z\,\xi'(z)[\langle\partial\lambda^{zz}(z)\varphi(0)\rangle + 2\left\langle\partial\lambda^{zz}(z)\lambda^{z\bar z}(0)\right\rangle]\delta^{(2)}(z)$$

$$= \frac{1}{2}\xi'''(0)[H^{(1)}_{1,2}+2H^{(2)}_{1,2}] = \frac{1}{2}\xi'''(0)[\frac{1}{3}+2\times\frac{1}{3}] = \frac{1}{2}\xi'''(0)$$

6. Relation to minimal models

Exact solution for four-derivative action

Solution to two modified DDK equations

$$
\begin{aligned}
b^{-2} &= \frac{13-d-6G+\sqrt{(d-d_+)(d-d_-)}}{12} \\
q &= 1+b^2 \\
d_\pm &= 13-6G\pm 12\sqrt{1+G}
\end{aligned}
$$

where $d=26-6/b_0^2$ to comply with the Liouville action.
KPZ barriers are shifted to $d_\pm$ which depend on $G\in[-1,0]$.
For $G=-1/3$ (the Nambu-Goto string) $\Longrightarrow$ $d_-=15-4\sqrt{6}\approx 5.2>4$

The string susceptibility equals

$$
\gamma_{\text{str}}=(h-1)\frac{q}{b^2}+2=(h-1)\frac{25-d-6G+\sqrt{(d-d_+)(d-d_-)}}{12}+2
$$

It is real for $d<d_-$ with $d_->1$ increasing from 1 at $G=0$ to 19 at $G=-1$ for $0\geq G\geq -1$ required for stability as it follows from the identity (modulo boundary terms)

$$
\int \mathrm{e}^{-\varphi}\left[(\partial\bar\partial\varphi)^2-G\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right]=\int \mathrm{e}^{-\varphi}\left[(1+G)(\partial\bar\partial\varphi)^2-G\nabla\partial\varphi\bar\nabla\bar\partial\varphi\right]
$$

BPZ null-vectors and Kac's spectrum

Like in usual Liouville theory the operators

$$V_\alpha = \mathrm{e}^{\alpha\varphi}, \qquad \alpha = \frac{1-n}{2} + \frac{1-m}{2b^2}$$

are the BPZ null-vectors for integer n and m obeying

$$\left(L_{-1}^2 + b^2 L_{-2}\right) \mathrm{e}^{-\varphi/2} = 0, \quad \left(L_{-1}^2 + b^{-2} L_{-2}\right) \mathrm{e}^{-b^{-2}\varphi/2} = 0, \quad \ldots$$

Their conformal weights

$$\Delta_\alpha = \alpha + (\alpha - \alpha^2) b^2$$

coincide with Kac's spectrum

$$\Delta_{m,n}(c) = \frac{c-1}{24} + \frac{1}{4}\left((m+n)\sqrt{\frac{1-c}{24}} + (m-n)\sqrt{\frac{25-c}{24}}\right)^2$$

for

$$c = 26 - d + G\frac{\left[25 - d - 6G + \sqrt{(d-d_+)(d-d_-)}\right]}{2(1+G)} = 1 + 6(b + b^{-1})^2$$

Minimal models from four-derivative action

To describe minimal models we choose like in usual Liouville theory

$$c = 25 + 6\frac{(p-q)^2}{pq} \quad \Longrightarrow \quad G = \frac{(1 - d - 6\frac{(p-q)^2}{pq})q}{6(q+p)}$$

with coprime $q > p$

If $G = 0$ this would imply

$$d = 1 - 6\frac{(p-q)^2}{pq}$$

for central charge of matter but now d is a free parameter obeying

$$1 - 6\frac{(p-q)^2}{pq} \leq d \leq 19 - 6\frac{p}{q} \quad \Longleftarrow \quad 0 \geq G \geq -1$$

Contrary to the Liouville theory now Kac's $c \neq c^{(\varphi)} = 26 - d$

Remarkably, $G = -1/3$ is associated in $d = 4$ with $p = 3$, $q = p+1 = 4$
unitary minimal model like critical Ising model on a random lattice

Minimal models from four-derivative action (cont.)

From the above formula for b^2

$$b^{-2} = \begin{cases} \dfrac{q}{p} & \text{perturbative branch} \\ -1 + \dfrac{(25-d)p}{6(q+p)} & \text{the other branch} \end{cases} \qquad \text{for } d > 25 - 6\frac{(p+q)^2}{p^2}$$

Perturbative branch is as in the usual Liouville theory, but the second branch is no longer $p \leftrightarrow q$ with it. It is $b^{-2} = p/q$ for $d = 1 - 6\frac{(p-q)^2}{pq}$

There are no obstacles against $d = 4$ for $q = p + 1$ (unitary case)!

$$d_+ = d_- = 19 \quad \text{for} \quad d = d_c = 13 - \frac{6}{p}$$

For $1 \leq d < d_c$ (d_c is always $>$10) we have $d \leq d_-$ and γ_{str} is REAL.

The perturbative branch is as in the usual Liouville theory but the domain of applicability is now broader which may have applications of the four-derivative Liouville action in Statistical Mechanics á la Kogan-Mudry-Tsvelik (1996)

7. Why ione loop?

Operatorial central charge otherwise

Y.M. (2022)

Generator of conformal transformation

$$\widehat{\delta}_\xi \equiv \int_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) T_{zz}(z) = \frac{1}{\pi} \int_{D_1} \xi \bar{\partial} T_{zz} \overset{\text{w.s.}}{=} \int_{D_1} \left(q \xi' \frac{\delta}{\delta\varphi} + \xi \partial\varphi \frac{\delta}{\delta\varphi} \right)$$

with the commutator (where $\zeta = \xi\eta' - \xi'\eta$ as it should)

$$\left\langle (\widehat{\delta}_\eta \widehat{\delta}_\xi - \widehat{\delta}_\xi \widehat{\delta}_\eta) X \right\rangle = \left\langle \widehat{\delta}_\zeta X \right\rangle + \int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2 \omega$$

$$\times \left\langle [q\xi'(z) + \xi(z)\partial\varphi(z)][q\eta'(\omega) + \eta(\omega)\partial\varphi(\omega)] \frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} X \right\rangle$$

$$= \left\langle \widehat{\delta}_\zeta X \right\rangle + \frac{1}{24} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} [\xi'''(z)\eta(z) - \xi(z)\eta'''(z)] \langle cX \rangle$$

DDK is reproduced for quadratic action S

Still usual central charge c for higher-derivative action with $G = 0$ but field-dependent for $G \neq 0$. Usual Virasoro algebra at one loop with

$$c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6G + \mathcal{O}(b_0^2)$$

Where is $SL(2,R)$ Kac-Moody algebra at higher loops?

Conclusion

- "Massive" CFTs exist and solved by (almost) usual CFT technique except for nonlocality in improved EMT

- Nambu-Goto and Polyakov strings are told apart by higher-derivative terms which revive quantumly like anomalies in QFT

- Emergence of the four-derivative Liouville action alludes to (4,3) minimal model like critical Ising model on a random lattice

- Any suggestions for gravity like Riegert-Fradkin-Tseytlin action?

Conclusion

- "Massive" CFTs exist and solved by (almost) usual CFT technique except for nonlocality in improved EMT $\approx 100\%$
- Nambu-Goto and Polyakov strings are told apart by higher-derivative terms which revive quantumly like anomalies in QFT $\approx 100\%$
- Emergence of the four-derivative Liouville action alludes to (4,3) minimal model like critical Ising model on a random lattice $\approx 75\%$
- Any suggestions for gravity like Riegert-Fradkin-Tseytlin action?