Meson mass spectrum in 't Hooft's QCD_2 model

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- Main goal: understand basic mechanism of confiniment

- We study toy model: 't Hooft's model or $N_c = \infty$ QCD$_2$

$$\mathcal{L} = \frac{N_c}{4g^2}\text{tr } F_{\mu\nu}F^{\mu\nu} + \sum_{k=1}^{N_f} \overline{\Psi}_k(i\gamma^\mu D_\mu - m_k)\Psi_k.$$

- At $N_c = \infty$ the Bethe Salpeter equation is exact

$$\left[\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right]\phi(x) - \fint_0^1 dy \frac{\phi(y)}{(x-y)^2} = 2\pi^2\lambda\ \phi(x),$$

$$\alpha_i = \frac{\pi m_i^2}{g^2} - 1, \qquad M_n^2 = 2\pi g^2 \lambda_n.$$

- Local goal: analytic properties of $\lambda_n(\alpha_1, \alpha_2)$ at complex α_k?

- Chiral limit $m_k \to 0$ corresponds to critical point (Gepner 1988, Affleck 1989):

$$N_c \text{WZW}[U(N_f)]$$

- It is interesting to study the vicinity of this fixed point: Hamiltonian truncation method is very efficient (Fitzpatrick et al)

- Other fixed points? Requires analytic properties of $\lambda_n(\alpha_1, \alpha_2)$.

- Fateev, Lukyanov, Zamolodchikov 2009: $\alpha_1 = \alpha_2 = 0$

- This talk: $\alpha_1 = \alpha_2 = \alpha$.

- Fourier transform

$$\phi(x) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \left(\frac{x}{1-x}\right)^{\frac{i\nu}{2}} \Psi(\nu), \qquad \Psi(\nu) = \int_{0}^{1} \frac{dx}{2x(1-x)} \left(\frac{x}{1-x}\right)^{-\frac{i\nu}{2}} \phi(x).$$

The Fourier form of 't Hooft's equation ($\alpha_1 = \alpha_2 = \alpha$)

$$\left(\frac{2\alpha}{\pi} + \nu \coth\frac{\pi\nu}{2}\right) \Psi(\nu) - \lambda \int_{-\infty}^{\infty} d\nu' \frac{\pi(\nu' - \nu)}{2\sinh\frac{\pi(\nu'-\nu)}{2}} \Psi(\nu') = 0.$$

In this form it is amenable for numeric solution.

- $\Psi(\nu)$ is a meromorphic function of ν with simple poles at

$$i\nu_k^* + 2iN, \qquad -i\nu_k^* - 2iN, \qquad N \geq 0,$$

where $\pm i\nu_k^*$ are roots of the equation ($\mathrm{Re}\,\nu_k^* > 0$ for $\mathrm{Re}\,\alpha > -1$)

$$\frac{2\alpha}{\pi} + \nu \coth\frac{\pi\nu}{2} = 0.$$

- If we define the new function $Q(\nu)$ by the following expression

$$Q(\nu) \stackrel{\text{def}}{=} \cosh\frac{\pi\nu}{2}\left(\nu + \frac{2\alpha}{\pi}\tanh\frac{\pi\nu}{2}\right)\Psi(\nu),$$

we conclude that the Q-function has to satisfy the following properties

1. be analytic in the strip $\text{Im}\,\nu \in [-2, 2]$;

2. grow slower than any exponential at $|\text{Re}\,\nu| \to \infty$

$$Q(\nu) = \mathcal{O}(e^{\epsilon|\nu|}), \quad \forall\epsilon > 0, \quad |\text{Re}\,\nu| \to \infty;$$

3. obey the quantization conditions

$$Q(0) = Q(\pm 2i) = 0.$$

- $Q(\nu)$ satisfies finite difference equation (Baxter's TQ equation)

$$Q(\nu + 2i) + Q(\nu - 2i) - 2Q(\nu) = -\frac{2z}{\nu + \alpha x} Q(\nu),$$

$$z = 2\pi\lambda \tanh\left(\frac{\pi\nu}{2}\right) \quad \text{and} \quad x = \frac{2}{\pi} \tanh\left(\frac{\pi\nu}{2}\right).$$

- Integrability of large N_c QCD$_2$?

- We will derive the spectral sums

$$G_+^{(1)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left[\frac{1}{\lambda_{2n}} - \frac{1}{n+1}\right] \quad \text{and} \quad G_-^{(1)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left[\frac{1}{\lambda_{2n+1}} - \frac{1}{n+1}\right],$$

$$G_+^{(s)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n}^s} \quad \text{and} \quad G_-^{(s)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{\lambda_{2n+1}^s}, \quad s > 1,$$

and WKB expansion.

- If we drop the quantization condition $Q(0)=Q(\pm 2i)=0$ then $\Psi(\nu|\lambda)$ will satisfy the inhomogeneous integral equation

$$\left(\frac{2\alpha}{\pi}+\nu\coth\frac{\pi\nu}{2}\right)\Psi(\nu)-\lambda\fint_{-\infty}^{\infty}d\nu'\frac{\pi(\nu'-\nu)}{2\sinh\frac{\pi(\nu'-\nu)}{2}}\Psi(\nu')=F(\nu|\lambda),$$

$$F(\nu|\lambda)=\frac{q_+(\lambda)\nu+q_-(\lambda)}{\sinh\frac{\pi\nu}{2}},$$

where $q_\pm(\lambda)$ are linear combinations of $Q(0)$ and $Q(\pm 2i)$.

- It can be shown that for given $q_\pm(\lambda)$ the solution of is unique.

- We can choose a basis of symmetric and antisymmetric functions

$$\Psi_\pm(-\nu|\lambda)=\pm\Psi_\pm(\nu|\lambda),$$

corresponding to

$$F_+(\nu|\lambda)=\frac{\nu}{\sinh\frac{\pi\nu}{2}}\quad\text{and}\quad F_-(\nu|\lambda)=\frac{1}{\sinh\frac{\pi\nu}{2}}.$$

- At the spectral points one has to recover the homogeneous equation.

- Thus $\Psi_{\pm}(\nu|\lambda)$ are meromorphic functions of λ

$$\Psi_{+}(\nu|\lambda) = \sum_{n=0}^{\infty} \frac{c_{2n}\Psi_{2n}(\nu)}{\lambda - \lambda_{2n}}, \quad \Psi_{-}(\nu|\lambda) = \sum_{n=0}^{\infty} \frac{c_{2n+1}\Psi_{2n+1}(\nu)}{\lambda - \lambda_{2n+1}},$$

- Quantum Wronskian

$$W(\nu|\lambda) \stackrel{\text{def}}{=} Q_{+}(\nu+i)Q_{-}(\nu-i) - Q_{+}(\nu-i)Q_{-}(\nu+i) = 2i.$$

- We have

$$Q_{+}(i) \sim \prod_{n=0}^{\infty} \frac{(\lambda - \lambda_{2n+1})}{(\lambda - \lambda_{2n})} \quad \text{and} \quad Q_{-}(i) \sim \prod_{n=0}^{\infty} \frac{(\lambda - \lambda_{2n})}{(\lambda - \lambda_{2n+1})}.$$

- Spectral determinants

$$D_+(\lambda) \stackrel{\text{def}}{=} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n}}\right) e^{\frac{\lambda}{n+1}},$$

$$D_-(\lambda) \stackrel{\text{def}}{=} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n+1}}\right) e^{\frac{\lambda}{n+1}}.$$

can be written in terms of the spectral sums as

$$D_\pm(\lambda) = \exp\left[-\sum_{s=1}^{\infty} s^{-1} G_\pm^{(s)} \lambda^s\right].$$

- We propose the following relations

$$\partial_\lambda \log D_-(\lambda) + q(\alpha) = 2i\, \partial_\nu \log Q_+(\nu)\Big|_{\nu=i},$$

$$\partial_\lambda \log D_+(\lambda) + q(\alpha) = 2i \left(1 - \frac{2\alpha}{\pi^2}\lambda^{-1}\right) \partial_\nu \log Q_-(\nu)\Big|_{\nu=i},$$

where $q(\alpha) = \ldots$

We have to find a solution of

$$Q(\nu+2i)+Q(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q(\nu),$$

$$z=2\pi\lambda\,\mathrm{tanh}\left(\frac{\pi\nu}{2}\right)\quad\text{and}\quad x=\frac{2}{\pi}\,\mathrm{tanh}\left(\frac{\pi\nu}{2}\right).$$

analytic in the strip $\mathrm{Im}\nu\in[-2,2]$.

For small $\lambda\to 0$ there are two solutions (FLZ)

$$\Xi(\nu|\lambda)=(\nu+\alpha x)\sum_{k=0}^{\infty}\frac{\left(1+\frac{i(\nu+\alpha x)}{2}\right)_k}{k!(k+1)!}(-iz)^k,$$

$$\Sigma(\nu|\lambda)=1+\sum_{k=1}^{\infty}\frac{\left(\frac{i(\nu+\alpha x)}{2}\right)_k}{k!(k-1)!}\left(\psi\Big(k+\frac{i(\nu+\alpha x)}{2}\Big)-\psi(k)-\psi(k+1)\right)(-iz)^k.$$

Both of them fail to satisfy the analytic requirements.

We notice that $\Sigma(\nu|\lambda)$ solves TQ equation even if we replace

$$\psi\Big(k+\frac{i(\nu+\alpha x)}{2}\Big)\to \psi_\alpha\Big(\nu-2i(k-1)\Big),\quad k=1,2,\ldots$$

where $\psi_\alpha\Big(\nu\Big)$ is a function analytic in the strip $\mathrm{Im}\,\nu\in[0,2)$ which obeys the functional relation

$$\psi_\alpha\Big(\nu+2i\Big)=\psi_\alpha\Big(\nu\Big)+\frac{2i}{\nu+\alpha x}.$$

Such a function is unique up to a constant shift:

$$\psi_\alpha\Big(\nu+i\Big)=-\gamma_E-\log 4+\frac{1}{2}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{1}{t+\frac{2\alpha}{\pi}\tanh\frac{\pi t}{2}}\left(\tanh\frac{\pi t}{2}-\tanh\frac{\pi(t-\nu)}{2}\right)dt.$$

Then we define

$$M_+(\nu|\lambda) = e^{\frac{iz}{2}}\Xi(\nu|\lambda),$$
$$M_-(\nu|\lambda) = \frac{1}{2}\left(e^{\frac{iz}{2}}\Sigma(\nu|\lambda) + e^{-\frac{iz}{2}}\Sigma(-\nu|\lambda)\right),$$

so that

$$M_\pm(-\nu|\lambda) = \mp M_\pm(\nu|\lambda).$$

But still we have poles at $\nu = \pm i$ of growing order.

We look for solutions of TQ equation in the form

$$Q_\pm(\nu|\lambda) = A_\pm(\tau|\lambda)M_\pm(\nu|\lambda) + B_\pm(\tau|\lambda)zM_\mp(\nu|\lambda),$$
$$\tau = \frac{\pi^2}{4}\tanh^2\left(\frac{\pi\nu}{2}\right)$$

where $A_\pm(\tau|\lambda)$ and $B_\pm(\tau|\lambda)$ admit the expansion

$$A_\pm(\tau|\lambda) = 1 + \sum_{s=1}^{\infty} a_\pm^{(s)}(\tau)\lambda^s, \quad B_\pm(\tau|\lambda) = -(1\pm1)\frac{\alpha}{2\pi^2}\lambda^{-1} + \sum_{s=0}^{\infty} b_\pm^{(s)}(\tau)\lambda^s,$$

The functions $a_\pm^{(s)}(\tau)$ and $b_\pm^{(s)}(\tau)$ are polynomials in τ of degree s and $s+1$ respectively. They are uniquely determined by the requirement of absence of poles at $\nu = \pm i$. For example

$$a_+^{(1)}(\tau) = -\frac{8\alpha}{\pi^2}\,\tau,\; a_+^{(2)}(\tau) = \left[1 - \frac{24\alpha\left(\pi^2 - 7\alpha\zeta(3)\right)}{\pi^4} - \frac{12\alpha^3 \mathtt{u}_3(\alpha)}{\pi^2}\right]\tau - \frac{80\alpha^2}{\pi^4}\tau^2$$

$$b_+^{(0)}(\tau) = \left[\frac{1}{4} - \frac{2\alpha\left(\pi^2 - 7\alpha\zeta(3)\right)}{\pi^4} - \frac{\alpha^3 \mathtt{u}_3(\alpha)}{\pi^2}\right] - \frac{4\alpha^2}{\pi^4}\tau$$

where

$$\mathtt{u}_{2k-1}(\alpha) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\sinh^2 t}{t\cosh^{2k-1} t\left(\alpha \sinh t + t\cosh t\right)} dt.$$

Using the relations

$$\partial_\lambda \log D_-(\lambda) + q(\alpha) = 2i\,\partial_\nu \log Q_+(\nu)\Big|_{\nu=i},$$

$$\partial_\lambda \log D_+(\lambda) + q(\alpha) = 2i\left(1-\frac{2\alpha}{\pi^2}\lambda^{-1}\right)\partial_\nu \log Q_-(\nu)\Big|_{\nu=i},$$

we obtain

$$G_+^{(1)} = \log(8\pi) - 1 - \frac{7\alpha\zeta(3)}{\pi^2} - \frac{\alpha}{2}\left(\mathrm{u}_1(\alpha) - \alpha \mathrm{u}_3(\alpha)\right),$$

$$G_-^{(1)} = \log(8\pi) - 3 + \frac{7\alpha\zeta(3)}{\pi^2} - \frac{\alpha}{2}\left(\mathrm{u}_1(\alpha) + \alpha \mathrm{u}_3(\alpha)\right),$$

$$G_+^{(2)} = 7\zeta(3) + 8\alpha\left[\frac{1}{3} - \frac{7\zeta(3)}{\pi^2}\right] + \frac{4\alpha^2}{\pi^2}\left[-\frac{28\zeta(3)}{3} + \frac{49\zeta^2(3)}{\pi^2} + \frac{62\zeta(5)}{\pi^2}\right] +$$
$$+\left[-\frac{\pi^2}{2} + 4\alpha + 4\alpha^2 - \frac{28\alpha^2\zeta(3)}{\pi^2}\right]\alpha \mathrm{u}_3(\alpha) + \alpha^4 \mathrm{u}_3^2(\alpha) - 4\alpha^3 \mathrm{u}_5(\alpha),$$

$$G_-^{(2)} = 2 - \frac{4\alpha}{3} + \frac{4\alpha^2}{\pi^2}\left[\frac{14}{3}\zeta(3) - \frac{31}{\pi^2}\zeta(5)\right] - 2\alpha^3 \mathrm{u}_3(\alpha) + 2\alpha^3 \mathrm{u}_5(\alpha),$$

etc. We have computed and verified numerically $G_\pm^{(s)}$ for $s \le 7$.

Ar large **negative** λ another hypergeometric series solve our TQ equation

$$S(\nu) = (-\lambda)^{-\frac{i\nu}{2}} S_0(\nu) \sum_{k=0}^{\infty} \frac{\left(1+\frac{i(\nu+\alpha x)}{2}\right)_k \left(\frac{i(\nu+\alpha x)}{2}\right)_k}{k!} (iz)^{-k},$$

provided that $S_0(\nu)$ satisfies the functional relation

$$S_0(\nu+2i) = \frac{4\pi \tanh\left(\frac{\pi\nu}{2}\right)}{\nu + \frac{2\alpha}{\pi}\tanh\left(\frac{\pi\nu}{2}\right)} S_0(\nu).$$

We take

$$S_0(\nu+i) = \exp\left[\frac{i}{4}\int_{-\infty}^{\infty} \log\left(\frac{4\pi \tanh\left(\frac{\pi t}{2}\right)}{t + \frac{2\alpha}{\pi}\tanh\left(\frac{\pi t}{2}\right)}\right)\left(\tanh\frac{\pi(t-\nu)}{2} - \tanh\frac{\pi t}{2}\right) dt\right],$$

which is analytic in the strip $\mathrm{Im}\,\nu \in [-2,2]$ *except* one point $\nu = -2i$ where it has a simple pole.

It is impossible to meet the analyticity requirements because of poles at $\nu = 0$, $\nu = \pm i$ and $\nu = \pm 2i$ of growing order.

We resolve this problem similar to the small λ case

$$Q_{\pm}(\nu|\lambda) = T(c^{-1}|\lambda)R_{\pm}(c|\lambda)S(\nu) \mp T(-c^{-1}|\lambda)R_{\pm}(-c|\lambda)S(-\nu),$$

where

$$c(\nu) = i\pi \coth\left(\frac{\pi\nu}{2}\right).$$

We look for the functions $T(c^{-1}|\lambda)$ and $R_{\pm}(c|\lambda)$ in the form of asymptotic expansion at large λ

$$T(c^{-1}|\lambda) = 1 + \sum_{k=1}^{\infty} T^{(k)}(c^{-1})\lambda^{-k} \quad R_{\pm}(c|\lambda) = 1 + \sum_{k=1}^{\infty} R_{\pm}^{(k)}\Big(c|\log(-\lambda)\Big)\lambda^{-k},$$

where $T^{(k)}(c^{-1})$ and $R_{\pm}^{(k)}\Big(c|\log(-\lambda)\Big)$ are polynomials in their variables.

We have found exact formula for $T(y|\lambda)$

$$T(y|\lambda) = \exp\left[\alpha f\left(\frac{\alpha}{\pi^2\lambda}\right) y\right],$$

where

$$f(t) = \frac{\sqrt{1-2t} - 1 + t(1+\log 4) - 2t\log(1+\sqrt{1-2t})}{t}.$$

While

$$R_{\pm}(c|\lambda) = 1 \pm \frac{(1+\alpha)c}{4\pi^4}\lambda^{-2} \pm \frac{(1+\alpha)c\,(6c+6q_1 \mp (1+\alpha))}{24\pi^6}\lambda^{-3} + \ldots,$$
$$q_1 = 3(1+\alpha) - 2\log(-\lambda) - 4is_1(\alpha).$$

Here

$$s_1(\alpha) = -\frac{i}{2}(1+\log 2\pi + \gamma_E) + \frac{i\alpha}{8}\mathtt{i}_1(\alpha),$$

with

$$\mathtt{i}_{2k-1}(\alpha) \overset{\text{def}}{=} \fint_{-\infty}^{\infty} \frac{\sinh 2t - 2t}{t\sinh^{2k-1} t\Big(\alpha\sinh t + t\cosh t\Big)}dt.$$

We have computed the coefficients $R_{\pm}^{(k)}\Big(c|\log(-\lambda)\Big)$ for $k \le 8$.

We obtain the asymptotic expansion of the spectral determinants $D_\pm(\lambda)$

$$D_\pm(\lambda) = d_\pm \left(8\pi e^{-2+\gamma_E}\right)^\lambda (-\lambda)^{\lambda-\frac{1}{8}\pm\frac{1}{4}} \exp\Big(F_\pm^{(0)}(L) + F_\pm^{(1)}(L)\lambda^{-1} + \dots\Big),$$

where $F_\pm^{(k)}(L)$ are polynomials in $L = \log(-2\pi\lambda) + \gamma_E$

$$F_\pm^{(0)}(L) = -\frac{\alpha L^2}{2\pi^2} - \frac{(3\pi^2 + 16\alpha\log 2 - 2\alpha^2 \mathtt{i}_2(\alpha))L}{8\pi^2},$$
$$F_\pm^{(1)}(L) = \frac{\alpha^2 L}{2\pi^4} + \frac{\alpha\pi^2 + 4\alpha^2(\log 16 - 1) - 2\alpha^3 \mathtt{i}_2(\alpha) \mp \pi^2(4+8\alpha)}{16\pi^4},$$

where

$$\mathtt{i}_{2k}(\alpha) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\sinh t(\sinh 2t - 2t)}{t\cosh^{2k} t\Big(\alpha\sinh t + t\cosh t\Big)} dt,$$

and

$$\frac{d_-}{d_+} = \frac{\sqrt{2(1+\alpha)}}{\pi}.$$

We have constructed the expansion of $D_\pm(\lambda)$ for $\lambda \to -\infty$. However, the physical values of the meson masses belong to the sector of positive values of λ. We conjecture that the correct analytic continuation is

$$\mathcal{D}_\pm(\lambda) = \frac{1}{2}\left(D_\pm(-e^{-i\pi}\lambda) + D_\pm(-e^{+i\pi}\lambda)\right).$$

We have

$$\mathcal{D}_\pm(\lambda) = 2d_\pm \left(8\pi e^{-2+\gamma_E}\right)^\lambda \lambda^{\lambda-\frac{1}{8}\pm\frac{1}{4}} \exp\left(\sum_{k=0}^{\infty} \Xi_\pm^{(k)}(l)\lambda^{-k}\right)\times$$
$$\times \cos\left(\frac{\pi}{2}\left[2\lambda - \frac{1}{4} \pm \frac{1}{2} + \sum_{k=0}^{\infty} \Phi_\pm^{(k)}(l)\lambda^{-k}\right]\right),$$

where the coefficients $\Xi_\pm^{(k)}(l)$ and $\Phi_\pm^{(k)}(l)$ are polynomials in

$$l = \log(2\pi\lambda) + \gamma_E.$$

They are related to the polynomials $F_{\pm}^{(k)}(L)$ in a simple way

$$\Xi_{\pm}^{(k)}(l) \stackrel{\text{def}}{=} \frac{1}{2}\left(F_{\pm}^{(k)}(l+i\pi) + F_{\pm}^{(k)}(l-i\pi)\right),$$
$$\Phi_{\pm}^{(k)}(l) \stackrel{\text{def}}{=} \frac{i}{\pi}\left(F_{\pm}^{(k)}(l-i\pi) - F_{\pm}^{(k)}(l+i\pi)\right).$$

The polynomials $\Phi_{\pm}^{(k)}(l)$ are responsible for quantization conditions

$$2\lambda - \frac{1}{4} \pm \frac{1}{2} + \Phi_{\pm}^{(0)}(l) + \Phi_{\pm}^{(1)}(l)\lambda^{-1} + \cdots = 2m+1, \quad m = 0,1,2,\ldots$$

We have computed $\Phi_{\pm}^{(s)}(l)$ for $s \leq 7$. For example

$$\Phi_{\pm}^{(0)}(l) = -\frac{3}{4} - \frac{\alpha(4\log 4 - \alpha \mathrm{i}_2(\alpha))}{2\pi^2} - \frac{2\alpha}{\pi^2}l, \quad \Phi_{\pm}^{(1)}(l) = \frac{\alpha^2}{\pi^4}, \quad \Phi_{\pm}^{(2)}(l) = \frac{\alpha^3 \pm \pi^2(1+\alpha)}{2\pi^6},$$
$$\Phi_{\pm}^{(3)}(l) = \frac{1}{12\pi^8}\left[5\alpha^4 + \pi^2(1+\alpha)^2 \mp 12\pi^2(1+\alpha)\left(l - \frac{1}{2} - \frac{3\alpha}{2} - \frac{\alpha}{4}\mathrm{i}_1(\alpha)\right)\right],$$
$$\Phi_{\pm}^{(4)}(l) = -\frac{(1+\alpha)^2}{4\pi^8}l + \frac{7\alpha^5 + \pi^2(1+\alpha)^2(1+5\alpha) + \pi^2(1+\alpha)^2\alpha\mathrm{i}_1(\alpha)}{16\pi^{10}} \mp \left[-\frac{3(\alpha+1)}{2\pi^8}l^2 + \right.$$
$$\left. + \frac{(1+\alpha)(22\alpha + 3\alpha\mathrm{i}_1(\alpha) + 10)}{4\pi^8}l + \frac{8\pi^2(4+3\alpha) - (1+\alpha)(8 + 24\alpha(4+5\alpha) - (20 + 44\alpha + 3\alpha\mathrm{i}_1(\alpha))\alpha\mathrm{i}_1(\alpha))}{32\pi^8}\right],$$

Thus we derive the expansion

$$\lambda_n = \frac{1}{2}\mathfrak{n} + \frac{\alpha}{\pi^2}\log\rho + \frac{\alpha^2}{\pi^4}\frac{2\log\rho - 1}{\mathfrak{n}} -$$

$$- \frac{1}{\pi^4}\frac{1}{\mathfrak{n}^2}\left[\frac{2\alpha^3}{\pi^2}\log^2\rho - \frac{6\alpha^3}{\pi^2}\log\rho + \frac{3\alpha^3}{\pi^2} + (-1)^n(1+\alpha)\right] +$$

$$+ \frac{1}{\pi^6}\frac{1}{\mathfrak{n}^3}\left[\frac{8\alpha^4}{3\pi^2}\log^3\rho - \frac{16\alpha^4}{\pi^2}\log^2\rho + \frac{24\alpha^4}{\pi^2}\log\rho - \frac{29\alpha^4 + \pi^2(1+\alpha)^2}{3\pi^2} +\right.$$

$$\left. +(-1)^n(1+\alpha)\left(4(1+\alpha)\log\rho - (2 + 8\alpha + 8\log 2 + \alpha\mathrm{i}_1(\alpha))\right)\right] + \mathcal{O}\left(\frac{\log^4\mathfrak{n}}{\mathfrak{n}^4}\right),$$

where

$$\mathfrak{n} = n + \frac{3}{4} - \frac{\alpha^2}{2\pi^2}\mathrm{i}_2(\alpha), \qquad \rho = 4\pi e^{\gamma_E}\left(n + \frac{3}{4} - \frac{\alpha^2}{2\pi^2}\mathrm{i}_2(\alpha)\right).$$

We derived WKB formula up to $\frac{1}{\mathfrak{n}^6}$. Numerics shows that already at $n = 1$ the accuracy is quite high!

What about analytic properties of $\lambda_n(\alpha)$?

We remind that all the spectral sums are expressed in terms of

$$\mathrm{u}_{2k-1}(\alpha) = \int\limits_{-\infty}^{\infty} \frac{\sinh^2 t}{t \cosh^{2k-1} t \Big(\alpha \sinh t + t \cosh t\Big)} dt.$$

The integrand in $\mathrm{u}_{2k-1}(\alpha)$ has poles at t_k:

$$\alpha \sinh t + t \cosh t = 0.$$

For real $\alpha > -1$ they are imaginary: $2k - 2 < \operatorname{Im} t_k < 2k$ for $k = 1, 2, \ldots$

However, for complex α, poles from the lower half-plane can penetrate into the upper half-plane and hit some pole there. This gives rise to singularities of $\mathrm{u}_{2k-1}(\alpha)$ at

$$\alpha_k = -\cosh^2 \tau_k, \quad 2\tau_k = \sinh[2\tau_k].$$

At any of α_k (for example $\alpha_1 = -1$) some mass has to vanish!

Future plans:

- Proof of

$$\partial_\lambda \log D_-(\lambda) + q(\alpha) = 2i\,\partial_\nu \log Q_+(\nu)\Big|_{\nu=i},$$
$$\partial_\lambda \log D_+(\lambda) + q(\alpha) = 2i\left(1 - \frac{2\alpha}{\pi^2}\lambda^{-1}\right)\partial_\nu \log Q_-(\nu)\Big|_{\nu=i}.$$

- $\alpha_1 \neq \alpha_2$: work in progress

- $1/N_c$ corrections: $M_k^2(\alpha) \sim (\alpha - \alpha_k)^{\beta_k}$?

**Thank you
for your attention!**