Wilson networks in AdS and global conformal blocks

Vladimir Khiteev

Moscow
03.09.2024

Based on: K.B. Alkalaev, A.O. Kanoda, V.S. Khiteev, 2307.08395
K.B. Alkalaev, V.S. Khiteev, to appear

- AdS_2 space and gravitational Wilson lines
- Wilson line networks
- HKLL formalism
- *n*-point AdS vertex function
- 2- and 3-point AdS vertex functions

AdS_2 space and gravitational Wilson lines

The AdS_2 gravity can be formulated in terms of $sl(2,\mathbb{R})$ gauge connections $A = A(\rho, z)$ with the zero curvature condition $dA + A \wedge A = 0$. The zero curvature condition can be realized dynaimcally via the BF action

$$S_{BF}[A, B] = \frac{1}{2} \int_{\mathcal{M}_2} Tr\, BF$$

where

- Tr is the $sl(2,\mathbb{R})$ Killing invariant form
- B is a scalar
- $F = dA + A \wedge A$

Introducing the $sl(2,\mathbb{R})$ commutation relations

$$[J_n, J_m] = (n-m)J_{n+m} \quad \text{with} \quad n, m = -1, 0, 1$$

the solution of the zero-curvature condition $F = 0$ can be cast into the form

$$A = e^{-\rho J_0} J_1 dz e^{\rho J_0} + J_0 d\rho$$ **(Banados 1995)**

The associated metric of the AdS_2 spacetime is given by

$$ds^2 = e^{2\rho} dz^2 + d\rho^2$$

The whole consideration can be extended to the AdS_3 case by introducing

- (anti)holomorphic coordinates $z, \bar{z}$
- gauge algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$
- Chern-Simons action
- anti-chiral gauge connections $\bar{A}$

Wilson lines and intertwiners

- Gravitational Wilson line:

$$W_j[L] = \mathbb{P}e^{-\int_L A}$$

  - L is a path from x_1 to x_2.
  - $\mathbb{P}$ is the path ordering operator.
  - The index $j \longrightarrow$ the connection A takes values in the $sl(2,\mathbb{R})$ module $\mathcal{R}_j$ of weight j.

  Properties:

  - Gauge transformation $W_j[L] \to g(x_2)W_j[L]g^{-1}(x_1)$.
  - Transitivity $W_j[L_1 + L_2] = W_j[L_2]W_j[L_1]$.
  - Wilson line associated with a flat connection depends only on the x_1, x_2.

  Direct calculation using A shows that

$$W_j[x_1, x_2] = e^{-\rho_2 J_0} e^{z_{12} J_1} e^{\rho_1 J_0} \qquad \text{where} \quad z_{ij} = z_i - z_j$$

- Intertwiners are defined as invariant tensors from $Inv(\mathcal{R}^*_{j_1} \otimes \mathcal{R}_{j_2} \otimes \mathcal{R}_{j_3})$, i.e

$$I_{j_1 j_2 j_3} : \quad \mathcal{R}_{j_2} \otimes \mathcal{R}_{j_3} \to \mathcal{R}_{j_1}$$

  with the invariance property

$$I_{j_1 j_2 j_3} U_{j_2} U_{j_3} = U_{j_1} I_{j_1 j_2 j_3}$$

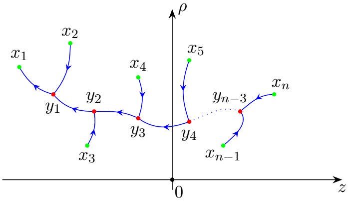
  where U_j are $SL(2,\mathbb{R})$ operators of the corresponding representations. Introducing the ladder basis $|j, m\rangle \in \mathcal{R}_j$ one obtains the matrix element of the intertwiner

$$\langle j_1, m_1 | I_{j_1 j_2 j_3} | j_2, m_2 \rangle \otimes | j_3, m_3 \rangle = \left[\frac{\Gamma(1-j_1+j_2+j_3)\Gamma(j_1+j_2-j_3+1)\Gamma(1+j_1-j_2+j_3)\Gamma(1+m_1+j_1)\Gamma(-m_1+j_1+1)}{\Gamma(j_1+j_2+j_3+2)} \right]^{\frac{1}{2}}$$

$$\times \delta_{m_1, m_2+m_3} \sum_{k=0}^{\infty} \frac{(-1)^{j_2+m_1-m_3+} \sqrt{\Gamma(-m_3+j_3+1)\Gamma(-m_2+j_2+1)\Gamma(1+m_2+j_2)\Gamma(1+m_3+j_3)}}{\Gamma(-m_2-j_1+j_3+k+1)\Gamma(-m_1+j_3-j_2+k+1)k!\Gamma(m_1+j_1-k+1)\Gamma(1+m_2+j_2-k)\Gamma(j_1+j_2-j_3-k+1)}$$

- AdS_2 space and gravitational Wilson lines
- Wilson line networks
- HKLL formalism
- *n*-point AdS vertex function
- 2- and 3-point AdS vertex functions

Wilson line networks



Wilson line network operator:

$$\widehat{W}^{j_1\dots j_n}_{\tilde{j}_1\dots\tilde{j}_{n-3}}(\mathbf{x},\mathbf{y}) := \Big(W_{j_1}[y_1,x_1]\, I_{j_1 j_2 \tilde{j}_1}\, W_{\tilde{j}_1}[y_2,y_1]\, I_{\tilde{j}_1 j_3 \tilde{j}_2} \dots W_{\tilde{j}_{n-3}}[y_{n-2},y_{n-3}]\, I_{\tilde{j}_{n-3} j_{n-1} j_n}\Big)$$

$$\times \Big(W_{j_2}[x_2,y_1] \dots W_{j_{n-1}}[x_{n-1},y_{n-2}]\, W_{j_n}[x_n,y_{n-2}]\Big)$$

AdS vertex function is matrix element of the Wilson line network operator with the cap states taken as Ishibashi states $|a_k\rangle$ **(Ishibashi 1989)**

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x}) := \langle a_1|\, \widehat{W}^{j_1\dots j_n}_{\tilde{j}_1\dots\tilde{j}_{n-3}}(\mathbf{x},\mathbf{y})\, |a_2\rangle \otimes |a_3\rangle \otimes \cdots \otimes |a_n\rangle$$

- Ishibashi states $\longrightarrow$ AdS_2 spacetime invariance of the AdS vertex function.
- The AdS vertex function is independent of positions of the vertices y_i **(Bhatta, et al. 2016; Besken, et al. 2016; Alkalaev, et al. 2020)**.

AdS vertex function parametrization, global conformal blocks and extrapolate dictionary relation

- The AdS vertex functions can be parameterized as

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x}) = \mathcal{V}_{j\tilde{j}}(c_{12}, ..., c_{n-1\, n}, c_{13}, ..., c_{n-2\, n})$$

where we introduced the AdS invariant variables:

$$c_{ij} = e^{\rho_i - \rho_j} + e^{\rho_i - \rho_j} + (z_i - z_j)^2 e^{\rho_i + \rho_j} - 2$$

- The conformal n-point correlators can be expanded into the conformal blocks as

$$\langle \hat{\mathcal{O}}_1(z_1) \cdots \hat{\mathcal{O}}_n(z_n) \rangle = \sum_{\tilde{h}} C_{12\tilde{h}_1} ... C_{\tilde{h}_{n-2}\, n-1\, n}\, \mathcal{F}_{h\tilde{h}}(\mathbf{z})$$

where C_{ijk} are model-dependent structure constant and $\mathcal{F}_{h\tilde{h}}(\mathbf{z})$ are conformal blocks. In general, exact expressions for the conformal blocks are not known. However, in the large-c regime $\mathcal{F}_{h\tilde{h}}(\mathbf{z})$ simplifies. The asymptotic of the block depends on the asymptotics of $h, \tilde{h}$:

$$h, \tilde{h} = \mathcal{O}(c^1) : \text{heavy operators}$$
$$h, \tilde{h} = \mathcal{O}(c^0) : \text{light operators}$$

Thus there are 3 types of conformal blocks
  - Global conformal block – all operators are light,
  - Classical conformal block – all operators are heavy,
  - Heavy-light conformal block – mix of heavy and light operators.

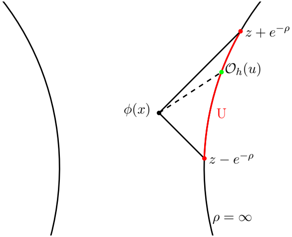
- The extrapolate dictionary relation (Alkalaev, Kanoda, Khiteev 2024)

$$\lim_{\rho \to \infty} e^{-\rho \sum_{i=1}^{n} j_i}\, \mathcal{V}_{j\tilde{j}}(\mathbf{x})|_{\rho_1 = \rho_2 = \ldots = \rho_n = \rho} = C_{j\tilde{j}}\, \mathcal{F}_{h\tilde{h}}(\mathbf{z})$$

where $\mathcal{F}_{h\tilde{h}}(\mathbf{z})$ is the global conformal block in the CFT_1 with $h = -j$.

- AdS_2 space and gravitational Wilson lines
- Wilson line networks
- HKLL formalism
- *n*-point AdS vertex function
- 2- and 3-point AdS vertex functions

HKLL formalism (Hamilton, et al. 2006)



HKLL representation of a scalar field in AdS_2:

$$\phi(x) = \int_U du\, K(x, u|h)\mathcal{O}_h(u)$$

$$K(x, u|h) = \frac{e^{-(1-h)\rho}}{(e^{-2\rho} + (z-u)^2)^{1-h}}$$

- $\mathcal{O}_h(u)$ is a primary operator of conformal weight h.
- Mass of the scalar field related with conformal weight as $m^2 = h(1-h)$.
- $K(x, u|h)$ is smearing function.
- Integration contour U lies on the conformal boundary of AdS_2.
- AdS_2 spacetime has two conformal boundaries but we consider only one.

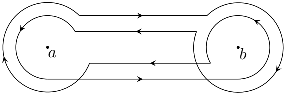
Matrix elements of the Wilson lines

Theorem 1: Matrix elements of the Wilson lines are building blocks of the AdS vertex function. Integral representation of the matrix elements is given by

$$\langle a|W_j[0,x]|j,m\rangle = \frac{(-)^{j+m+1}A_{jm}}{(1-e^{2\pi ij})}\oint_{P[w,\bar{w}]} du\,((u-z)^2e^{\rho}+e^{-\rho})^{-j-1}u^{j+m}$$

$$\langle j,m|W_j[x,0]|a\rangle = \frac{(-)^{j}A_{jm}}{(1-e^{2\pi ij})}\oint_{P[w,\bar{w}]} dv\,((v-z)^2e^{\rho}+e^{-\rho})^{-j-1}v^{j-m}$$

where $w_i = z_i + ie^{-\rho_i}$ and $P[w,\bar{w}]$ is the Pochhammer contour around points w и $\bar{w}$.



HKLL representation of the matrix elements

$$\langle a|W_j[0,x]|j,m\rangle = K_j\oint_{P[w,\bar{w}]} du\;\; K(x,u|-j)\,\langle a|W_j[0,(u,\rho)]|j,m\rangle_{\partial}$$

$$\langle j,m|W_j[x,0]|a\rangle = K_j\oint_{P[w,\bar{w}]} dv\;\; K(x,v|-j)\,\langle j,m|W_j[(v,\rho),0]|a\rangle_{\partial}$$

where $\langle a|W_j[0,y_1]|j,m\rangle_{\partial}$ are boundary asymptotics of the corresponding matrix elements.

- AdS_2 space and gravitational Wilson lines
- Wilson line networks
- HKLL formalism
- *n*-point AdS vertex function
- 2- and 3-point AdS vertex functions

n-point AdS vertex function

The direct calculation of the n-point AdS vertex function is technically difficult.

Theorem 2: The n-point AdS vertex function has the following HKLL representation:

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x}) = C_{j\tilde{j}} \prod_{k=1}^{n} K_{j_k} \oint_{P[w_k,\bar{w}_k]} du_k \; K(x_k, u_k| - j_k)\mathcal{F}_{h\tilde{h}}(\mathbf{u})$$

where $\mathcal{F}_{h\tilde{h}}(\mathbf{u})$ is n-point global conformal block. After integrating over u_k one obtains

$$\mathcal{V}_{j\tilde{j}}(\mathbf{x}) = C_{j\tilde{j}}(2i)^{\sum_{i=1}^{n} j_i}\mathcal{L}_{j\tilde{j}}(\mathbf{x}) \sum_{\substack{m_1,\ldots,m_{n-3}=0 \\ \{k_{i\,i-2},k_{i\,i-1},k_{i\,i+1},k_{i\,i+2}=0\}_{i=1,\ldots,n}}}^{\infty} D^{m_i\,k_{ij}}_{j\tilde{j}} \prod_{i=1}^{n-3} \chi_i^{m_i} \prod_{l=1}^{n-1} s_{l\,l+1}^{k_{l\,l+1}} s_{l+2\,l}^{k_{l+2\,l}} s_{l+1\,l}^{k_{l+1\,l}} s_{l\,l+2}^{k_{l\,l+2}}$$

where we introduced auxiliary variables

$$\chi_i := \frac{(\bar{w}_i - \bar{w}_{i+1})(\bar{w}_{i+2} - \bar{w}_{i+3})}{(\bar{w}_i - \bar{w}_{i+2})(\bar{w}_{i+1} - \bar{w}_{i+3})} \qquad s_{i\,j} := \frac{w_i - \bar{w}_i}{\bar{w}_i - \bar{w}_j} \qquad w_i := x_i + ie^{-\rho_i}$$

The leg factor:

$$\mathcal{L}_{j\tilde{j}}(\mathbf{x}) := \prod_{i=1}^{n-3} \chi_i^{-\tilde{j}_i} \prod_{l=1}^{n} (w_l - \bar{w}_l)^{-j_l}(\bar{w}_l - \bar{w}_{l+1})^{j_l + j_{l+1}}(\bar{w}_l - \bar{w}_{l+2})^{-j_{l+1}}$$

The coefficient:

$$D^{m_i\,k_{ij}}_{j\tilde{j}} := \prod_{s=1}^{n} \frac{(-j_s)_{k_{s\,s-2}+k_{s\,s-1}+k_{s\,s+1}+k_{s\,s+2}}}{k_{s\,s-2}!k_{s\,s-1}!k_{s\,s+1}!k_{s\,s+2}!m_s!(-2j_s)_{k_{s\,s-2}+k_{s\,s-1}+k_{s\,s+1}+k_{s\,s+2}}(-2\tilde{j}_s)_{m_s}}$$

$$\times \prod_{t=1}^{n}(-j_t - j_{t+1} + \tilde{j}_{t-2} + \tilde{j}_t - m_{t-2} - m_t)_{k_{t+1\,t}+k_{t\,t+1}}(j_{t+1} - \tilde{j}_{t-1} - \tilde{j}_t)_{m_{t-1}+m_t+k_{t+2\,t}+k_{t\,t+2}}$$

- AdS_2 space and gravitational Wilson lines
- Wilson line networks
- HKLL formalism
- *n*-point AdS vertex function
- 2- and 3-point AdS vertex functions

2-point AdS vertex function

The 2-point AdS vertex function:

$$\mathcal{V}_{j_1 j_2}(\mathbf{x}) = \frac{\delta_{j_1 j_2}}{(2j_1+1)^{\frac{1}{2}}} \left(\frac{\xi(x_1,x_2)}{2}\right)^{j_1} {}_2F_1\left(\frac{-j_1}{2}, \frac{-j_1}{2}+\frac{1}{2}; -j_1+\frac{1}{2} | \xi(x_1,x_2)^2\right)$$

where the AdS invariant distance is defined as

$$\xi(x,x') := \frac{2e^{-\rho-\rho'}}{e^{-2\rho}+e^{-2\rho'}+(z-z')^2}$$

The asymptotic expansion near the conformal boundary $\rho_1 = \rho_2 = \infty$ reads

$$e^{-2\rho j_1}\mathcal{V}_{j_1 j_2}(\rho, \mathbf{z}) = C_{j_1 j_2} z_{12}^{2j_1} + \mathcal{O}(e^{-\rho})$$

- The 2-point AdS vertex function coincide with the bulk-to-bulk propagator of free scalar fields in AdS_2.
- Similar results were obtained earlier **(Castro, et al. 2018)**, but the construction considered there is defined only in the case $n = 2$.
- Asymptotics of the AdS vertex function coincide with the 2-point conformal correlation function in agreement with the extrapolate dictionary relation.

3-point AdS vertex function

3-point AdS vertex function:

$$\mathcal{V}_{j_1j_2j_3}(\mathbf{x}) = C_{j_1j_2j_3}(2i)^{j_1+j_2+j_3}\mathcal{L}_{j_1j_2j_3}(\mathbf{x}) \sum_{k_{12},k_{21},k_{13},k_{31},k_{23},k_{32}=0}^{\infty} D^{k_{ij}}_{j_1j_2j_3}\, s_{12}^{k_{12}}\, s_{13}^{k_{13}}\, s_{21}^{k_{21}}\, s_{23}^{k_{23}}\, s_{31}^{k_{31}}\, s_{32}^{k_{32}}$$

The leg factor:

$$\mathcal{L}_{j_1j_2j_3}(\mathbf{x}) := (w_1-\bar{w}_1)^{-j_1}(w_2-\bar{w}_2)^{-j_2}(w_3-\bar{w}_3)^{-j_3}$$

$$\times(\bar{w}_1-\bar{w}_2)^{j_1+j_2-j_3}(\bar{w}_1-\bar{w}_3)^{j_1+j_3-j_2}(\bar{w}_2-\bar{w}_3)^{j_3+j_2-j_1}$$

The coefficient:

$$D^{k_{ij}}_{j_1j_2j_3} := \frac{(-j_1+j_2-j_3)_{k_{13}+k_{31}}(-j_2-j_3+j_1)_{k_{23}+k_{32}}(-j_1-j_2+j_3)_{k_{12}+k_{21}}}{k_{12}!k_{21}!k_{13}!k_{31}!k_{23}!k_{32}!}$$

$$\times\frac{(-j_1)_{k_{12}+k_{13}}(-j_2)_{k_{21}+k_{23}}(-j_3)_{k_{31}+k_{32}}}{(-2j_1)_{k_{12}+k_{13}}(-2j_2)_{k_{21}+k_{23}}(-2j_3)_{k_{31}+k_{32}}}$$

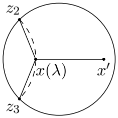
The asymptotic expansion near the conformal boundary $\rho_1=\rho_2=\rho_3=\infty$ reads

$$e^{(j_1+j_2+j_3)\rho}\mathcal{V}_{j_1j_2j_3}(\rho,\mathbf{z})|_{\rho\to\infty} = C_{j_1j_2j_3}\,(z_2-z_3)^{j_2+j_3-j_1}(z_3-z_1)^{j_3+j_1-j_2}(z_2-z_1)^{j_1+j_2-j_3}$$

$$+\mathcal{O}(e^{-\rho})$$

The expression on the right is the 3-point conformal correlation function.

- 2-point AdS vertex function = bulk-to-bulk propagator.
- 3-point AdS vertex function = ???



Proposition 1: In the case of two points on the boundary the AdS vertex function is proportional to the geodesic Witten diagram

$$\int_{\gamma_{23}} d\lambda G_{bb}(x(\lambda), x_1|h_1) G_{b\partial}(x(\lambda), z_2|h_2) G_{b\partial}(x(\lambda), z_3|h_3) \propto \mathcal{V}_{j_1 j_2 j_3}(x_1, z_2, z_3)$$

where $G_{bb}(x(\lambda), x_1|h_1)$ is bulk-to-bulk propagator and $G_{b\partial}(x(\lambda), z_2|h_2)$ is bulk-to-boundary propagator.

Proposition 2: In case of one point on the boundary the 3-point Witten diagramm can be expressed as a linear combination of the 3-point AdS vertex functions.

$$\int_{\mathrm{AdS}_2} d^2x G_{bb}(x, x_1|h_1) G_{bb}(x, x_2|h_2) G_{b\partial}(x, z_3|h_3) = \frac{\alpha(h_1, h_2, h_3)}{C_{j_1 j_2 j_3}} \mathcal{V}_{j_1 j_2 j_3}(x_1, x_2, z_3)\Big|_{j_i=-h_i}$$

$$+\beta(h_1; h_2, h_3) \sum_{n=0}^{\infty} \frac{a(h_1; h_2, h_3; n)}{C_{j_2+j_3-2n\ j_2 j_3}} \mathcal{V}_{j_2+j_3-2n\ j_2 j_3}(x_1, x_2, z_3)\Big|_{j_i=-h_i}$$

$$+\beta(h_2; h_1, h_3) \sum_{n=0}^{\infty} \frac{a(h_2; h_1, h_3; n)}{C_{j_1 j_1+j_3-2n\ j_3}} \mathcal{V}_{j_1 j_1+j_3-2n\ j_3}(x_1, x_2, z_3)\Big|_{j_i=-h_i}$$

Conclusion and outlooks

- Formulated the one-to-one holographic correspondence between the global conformal block and the AdS vertex function.
- Ishibashi states as cap states $\longrightarrow$ AdS_2 spacetime invariance of the AdS vertex function.
- In the case of two points on the boundary the AdS vertex function is proportional to the geodesic Witten diagram.
- In case of one point on the boundary the 3-point Witten diagramm expressed as a linear combination of the 3-point AdS vertex functions.

Further developments:

- Three points in the bulk.
- Linear combination of the n-point AdS vertex function = n-point Witten diagramm?
- Correspondence between scalar field theory in AdS_2 and theory of the $sl(2,\mathbb{R})$ flat connections.