Conformal Triangles and Zig-Zag Diagrams.

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6. Proof of zig-zag conjecture
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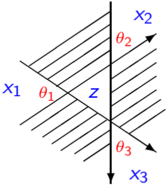
A.B. Zamolodchikov, "Fishing-net" Diagrams as a Completely Integrable System, Phys.Lett. B 97 (1980) 63-66. INSPIRE 112 citations

The star-triangle relation can be visualized as Yang-Baxter Eq.

$$\int \frac{d^D z}{f(\theta_1,\theta_2,\theta_3)} \cdot$$



$$=$$

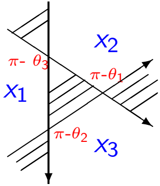
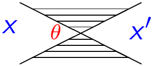


Fig. **1**

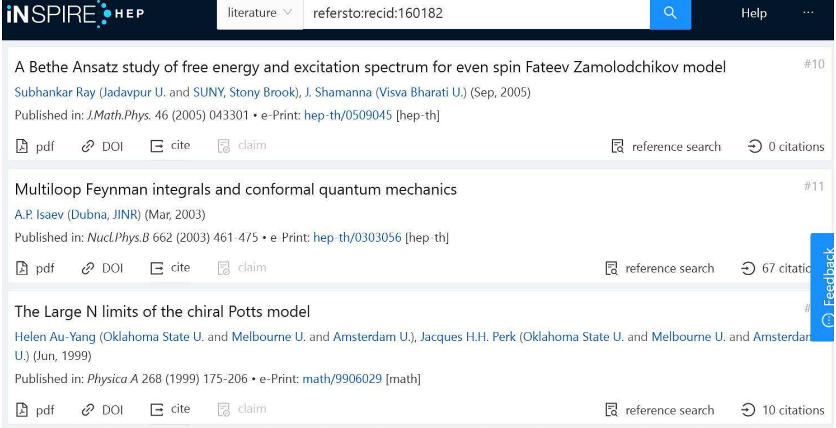
where $f(\theta_1, \theta_2, \theta_3) = (2\pi)^{2D}\, a(\alpha_1)\, a(\alpha_2)\, a(\alpha_3)$, $a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\pi^{D/2}\, 2^{2\alpha}\, \Gamma(\alpha)}$ and

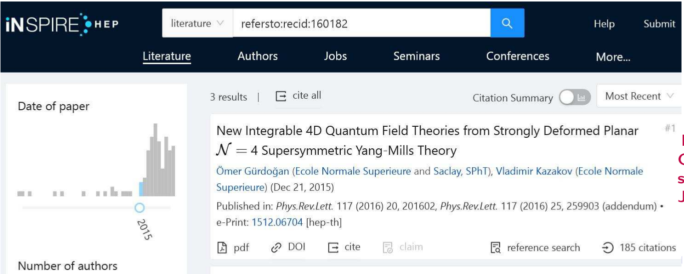


$$= \qquad \frac{1}{(x-x')^{2\alpha(\theta)}}, \qquad \alpha(\theta) := \frac{D}{2\pi}(\pi - \theta).$$

Fig. **2**

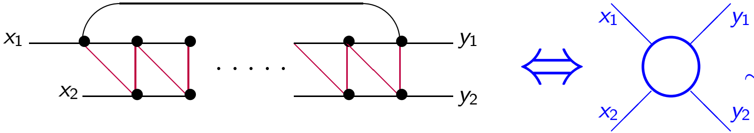
E.S. Fradkin and M.Y. Palchik, Recent Developments in Conformal Invariant Quantum Field Theory, Phys. Rept. 44 (1978) 249.





D.Chicherin, S.Derkachov, A.P.I. Conformal algebra: R-matrix and star-triangle relation JHEP 04 (2013) 020

The 4-dimensional ϕ^4 field theory (and its multicomponent generalizations) serves the Brout-Englert-Higgs mechanism and thus is an essential part of the Standard Model of particle physics. It was shown by explicit evaluation (in MS scheme) of the Gell-Mann-Low β-function in $\phi^4_{D=4}$ theory that special Feynman diagrams – so-called zig-zag diagrams (in fact the residue $\text{Res}_\epsilon = Z(M+1)$ of the corresponding 4-point perturbative integral)



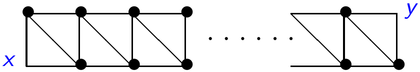
$\sim \frac{1}{\epsilon} Z(M+1)$

where

$$x_1 \overset{\beta}{\text{——}} x_2 = \frac{1}{(x_1 - x_2)^{2\beta}} , \quad x_i, y_i \in \mathbb{R}^D, \qquad \bullet = \int d^D x, \quad D = 4 - 2\epsilon ,$$

give 44%, 46% and 47% of numerical contributions, respectively, to the $3, 4$ and 5 loop orders of β [D.J. Broadhurst and D. Kreimer (1995)].

One can show that $Z(M+1)$ (($M+1$)-loop contribution to the β-function) is also given by the integral for M-loop 2-point zig-zag diagrams (ZZD):

$G_2(x,y) =$ 

and it has the general form for $D=4$:

$$G_2(x,y) = \frac{\pi^{2M}}{(x-y)^2}\, Z(M+1)\,, \tag{1}$$

where π^{2M} is the normalization factor, $x, y \in \mathbb{R}^4$ and $Z(M+1)$ is **the same constant** that gives $(M+1)$-loop order contribution to the β-function in the $\phi^4_{D=4}$ theory.

History. The first $Z(3) = 6\zeta_3 \sim$  and $Z(4) = 20\zeta_5 \sim$  in (1) were analytically evaluated by [K.G.Chetyrkin, A.L.Kataev, F.V.Tkachov, **1980**][1] and [K.G.Chetyrkin, F.V.Tkachov, **1981**], respectively. The constant $Z(5) = \frac{441}{8}\zeta_7$ of the ZZD with 4 loops  was calculated by D.Kazakov in **1983**. The 5 loop ZZD  contribution $Z(6) = 168\zeta_9$ to the β-function (in 6-loop order) was found by D.Broadhurst in **1985**. Here $\zeta_k := \sum_{n\geq 1} 1/n^k$.

[1] The "two-loop fish diagram" was firstly evaluated in [E.De Rafael, J.L.Rosner, 1974].

Then D.Broadhurst and D.Kreimer in **1995** evaluated $Z(M+1)$ numerically up to $(M+1)=10$ loops, and based on these data they formulated a remarkable conjecture that the constant $Z(M+1)$ is given by the sign alternating sum

$$Z(M+1) = 4C_M \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)(M+1)}}{n^{2M-1}} = \\ = \begin{cases} 4\, C_M\, \zeta_{2M-1}\,, & \text{for } M = 2N+1\,; \\ 4\, C_M\, (1-2^{2(1-M)})\, \zeta_{2M-1}\,, & \text{for } M = 2N\,; \end{cases} \qquad \zeta_k = \sum_{n>1} \frac{1}{n^k}\,, \tag{2}$$

where M is the number of loops in ZZDs and $C_M = \frac{(2M)!}{(M+1)!\,M!}$ is the Catalan number. Finally, the very nontrivial proof of the Broadhurst-Kreimer conjecture was found by [F.Brown and O.Schnetz in **2013,2015**; based on J.M.Drummond (2012)].

In this report, by using methods of D-dimensional CFT, the concise integral presentations for 4-point and 2-point zig-zag Feynman graphs are deduced. It gives a possibility to compute exactly a special class of 2- and 4-point Feynman diagrams (ZZDs for any M) in ϕ^4_D theory. In particular we find new rather simple proof of the Broadhurst-Kreimer conjecture.

Operator formalism for massless diagrams.

Let $\{\hat{q}_a^{\mu},\ \hat{p}_b^{\nu}\}$ $(a, b = 1, ..., n)$ be generators of the D-dimensional Heisenberg algebras $\mathcal{H}_a$ (a=1,...,n)

$$[\hat{q}_a^{\mu},\ \hat{q}_b^{\nu}] = 0 = [\hat{p}_a^{\mu},\ \hat{p}_b^{\nu}]\ , \qquad [\hat{q}_a^{\mu},\ \hat{p}_b^{\nu}] = i\,\delta^{\mu\nu}\,\delta_{ab} \qquad (\mu, \nu = 1, ..., D)\ .$$

We introduce states $|x_a\rangle$ which diagonalize coordinates $\hat{q}_a^{\mu}$:

$$\hat{q}_a^{\mu}|x_a\rangle = x_a^{\mu}|x_a\rangle\ .$$

These states form the basis in the representation space V_a of subalgebra $\mathcal{H}_a$. We also introduce the dual states $\langle x_a|$ such that the orthogonality and completeness conditions are valid

$$\langle x_a|x'_a\rangle = \delta^D(x_a - x'_a), \qquad \int d^D x_a\ |x_a\rangle\langle x_a| = I_a\ ,$$

where I_a is the unit operator in V_a and there are no summations over indices a. So, we have the algebra $\mathcal{H}^{(n)} = \oplus_{a=1}^{n}\mathcal{H}_a$ which acts in the space $V_1 \otimes \cdots \otimes V_n$ with basis vectors $|x_1\rangle \otimes \cdots |x_n\rangle$.

We use operators $(\hat{q}_a)^{2\alpha} = (\sum_\mu \hat{q}_a^\mu \hat{q}_a^\mu)^\alpha$ and $(\hat{p}_a)^{2\beta} = (\sum_\mu \hat{p}_a^\mu \hat{p}_a^\mu)^\beta$ with non-integer α and β. These operators are understood as integral operators defined via their integral kernels: $\langle x| \, (\hat{q})^{-2\alpha} \, |y\rangle = (x)^{-2\alpha} \, \delta^D(x-y)$ and

$$\langle x| \frac{1}{(\hat{p})^{2\beta}} |y\rangle = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik(x-y)}}{(k)^{2\beta}} = \frac{a(\beta)}{(x-y)^{2\beta'}} \, ,$$

$$a(\beta) := \frac{2^{-2\beta}}{\pi^{D/2}} \frac{\Gamma(\beta')}{\Gamma(\beta)} \, , \qquad \beta' := D/2 - \beta \, .$$

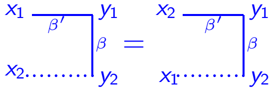
Consider the algebra $\mathcal{H}^{(2)} = \mathcal{H}_1 + \mathcal{H}_2$, which acts in $V_1 \otimes V_2$ with basis $|x_1, x_2\rangle := |x_1\rangle \otimes |x_2\rangle$. To evaluate ZZDs in the operator approach we introduce the main object – graph building operator:

$$\hat{Q}_{12}^{(\beta)} := \frac{1}{a(\beta)} \, \mathcal{P}_{12} \, (\hat{p}_1)^{-2\beta} \, (\hat{q}_{12})^{-2\beta} \, ,$$

where $(\hat{q}_{12})^2 = (\hat{q}_1^\mu - \hat{q}_2^\mu)(\hat{q}_1^\mu - \hat{q}_2^\mu)$ and $\mathcal{P}_{12}$ is the permutation operator

$$\mathcal{P}_{12} \, \hat{q}_1 = \hat{q}_2 \, \mathcal{P}_{12} \, , \quad \mathcal{P}_{12} \, \hat{p}_1 = \hat{p}_2 \, \mathcal{P}_{12} \, , \quad \mathcal{P}_{12} |x_1, x_2\rangle = |x_2, x_1\rangle \, , \quad (\mathcal{P}_{12})^2 = I \, .$$

We depict the kernel $\langle x_1, x_2|\hat{Q}^{(\beta)}_{12}|y_1, y_2\rangle$ of the graph building operator (GBO) $\hat{Q}^{(\beta)}_{12}$ as

$\mathcal{P}_{12}\cdot$  $= \frac{1}{a(\beta)} \langle x_1, x_2|\, \mathcal{P}_{12}\, (\hat{p}_1)^{-2\beta}\, (\hat{q}_{12})^{-2\beta}\, |y_1, y_2\rangle =$

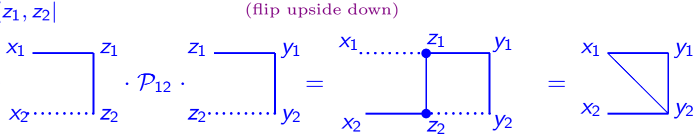
$$= \frac{1}{(x_2-y_1)^{2\beta'}\,(y_1-y_2)^{2\beta}}\; \delta^D(x_1 - y_2)\;,$$

where

$$x_1 \cdots\cdots\cdots\; x_2 \;=\; \delta^D(x_1 - x_2)\;, \qquad x_1 \overset{\beta}{\text{———}} x_2 \;=\; (x_1 - x_2)^{-2\beta}\;.$$

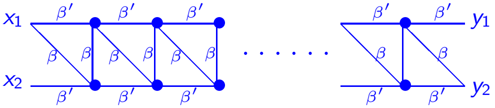
Now we note that $Q^{(\beta)}_{12}$ is the GBO for the planar zig-zag Feynman graphs.
Example for $\hat{Q}^2_{12}$:

$$\langle x_1, x_2|\, \hat{Q}_{12} \underbrace{\quad}_{\int dz_1 dz_2 |z_1, z_2\rangle\langle z_1, z_2|} \hat{Q}_{12}\, |y_1, y_2\rangle \quad =$$

$= \int dz_1 dz_2 \;\; \mathcal{P}_{12}\cdot$ 

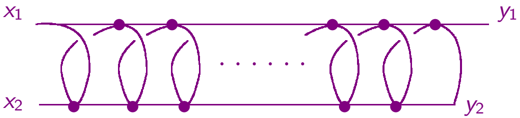
To obtain 2-loop fish diagram we multiply this by $(x_1 - x_2)^{-2\beta}$ and integrate over x_1 and y_2.

for even loops $(2N-2)$



$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

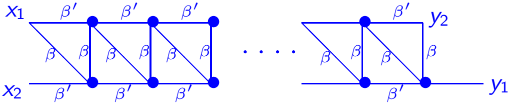
$=$



;

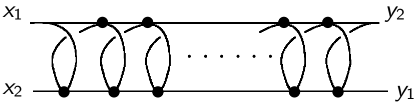
Here we remove the propagator $1/(y_1 - y_2)^{2\beta}$.

for odd loops $(2N-1)$



$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$=$



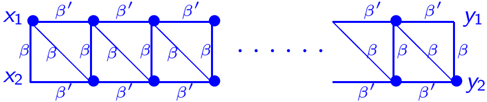
.

The vertices • denote the integration over $\mathbb{R}^D$.

We stress that these Feynman integrals represent the contribution to the 4-point correlation functions in bi-scalar D-dimensional "fishnet" theory [V.Kazakov a.o. (2016,2018)]. For clarity, we present the zig-zag diagrams in the form of the spiral graphs having the cylindrical topology. We also stress that integral kernels, shown in the pictures, in the case $D=4$ and $\beta=1$, contribute to Green's functions of the standard $\phi^4_{D=4}$ field theory.

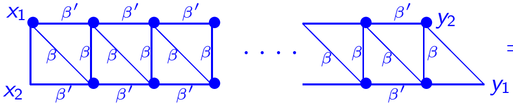
The next important statement is that $Q_{12}^{(\beta)}$ is also the graph building operator for the integrals of the planar zig-zag 2-point Feynman graphs:

1. <u>for even number of loops</u> $2N$



$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} \;;$$

2. <u>for odd number of loops</u> $(2N+1)$

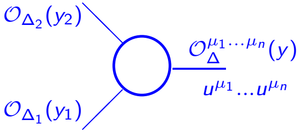


$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} \;.$$

Below we use these representations to evaluate exactly the corresponding classes of 2-point and 4-point Feynman diagrams. For this we need to find eigenvalues and complete set of eigenvectors for GBO $\hat{Q}_{12}^{(\beta)}$.

Remark. The elements $H_\beta := \mathcal{P}_{12}\, \hat{Q}_{12}^{(\beta)} \equiv (\hat{p}_1)^{-2\beta} (\hat{q}_{12})^{-2\beta}$ form a commutative set of operators $[H_\alpha,\ H_\beta] = 0$ $(\forall \alpha, \beta)$.

To find eigenvectors for the graph building operator $Q_{12}^{(\beta)}$ we consider the standard 3-point correlation function (in D-dimensional CFT) of three fields $\mathcal{O}_{\Delta_1}$, $\mathcal{O}_{\Delta_2}$ and $\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}$, where $\mathcal{O}_{\Delta_1}$, $\mathcal{O}_{\Delta_2}$ are scalar fields with conf. dimensions Δ_1, Δ_2, while $\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}$ – (symmetric, traceless and transverse) tensor field with conf. dimension Δ. The conformally invariant expression of this correlation function (up to a normalization) is unique and well known [V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov (1976,1977); E.S.Fradkin , M.Y.Palchik (1978);...]



$$= \; u^{\mu_1}\dots u^{\mu_n}\, \langle \mathcal{O}_{\Delta_1}(y_1)\, \mathcal{O}_{\Delta_2}(y_2)\, \mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}(y)\rangle =$$

$$= \frac{\left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2}\right)^n}{(y_1-y_2)^{2\eta}(y-y_1)^{2\delta}(y-y_2)^{2\rho}} \,,$$

where $u \in \mathbb{C}^D$ such that $(u,u) = u^\mu u^\mu = 0$ and

$$\eta = \frac{1}{2}(\Delta_1+\Delta_2-\Delta+n)\,, \;\; \delta = \frac{1}{2}(\Delta_1+\Delta-\Delta_2-n)\,, \;\; \rho = \frac{1}{2}(\Delta_2+\Delta-\Delta_1-n)\,.$$

We need the special form of the 3-point function (conformal triangle) when parameters $\Delta, \Delta_1, \Delta_2$ are related to two numbers $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$:

$$\Delta_1 = \frac{D}{2} \, , \quad \Delta_2 = \frac{D}{2} - \beta \, , \quad \Delta = D - 2\alpha - \beta + n \, ,$$

so we have for conformal triangle:

$$\langle y_1, y_2 | \Psi^{(n,u)}_{\alpha,\beta}(y) \rangle \ := \ \frac{\left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2} \right)^n}{(y_1 - y_2)^{2\alpha} (y - y_1)^{2\alpha'} (y - y_2)^{2(\alpha+\beta)'}} \, .$$

Proposition 1. *The wave function* $|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = u^{\mu_1} \cdots u^{\mu_n} |\Psi^{\mu_1 \dots \mu_n}_{\alpha,\beta}(y)\rangle$ $(\forall \, \alpha, \beta \in \mathbb{C})$ *is the eigenvector for the graph building operator*

$$\hat{Q}^{(\beta)}_{12} \, |\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = \tau(\alpha, \beta, n) \, |\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle \, ,$$

with the eigenvalue

$$\tau(\alpha, \beta, n) = (-1)^n \, \pi^{D/2} \, \frac{\Gamma(\beta)\Gamma(\alpha)\Gamma\big((\alpha+\beta)' + n\big)}{\Gamma(\beta')\Gamma(\alpha' + n)\Gamma(\alpha + \beta)} \, .$$

The analogous statement, for $D = 4$ and $\beta = 1$, was made by [N.Gromov, V.Kazakov and G.Korchemsky (2018)].

Note that with respect to the standard scalar product in $V_1 \otimes V_2$ the operator $\hat{Q}_{12}^{(\beta)} = \frac{1}{a(\beta)} \mathcal{P}_{12} (\hat{p}_1)^{-2\beta} (\hat{q}_{12})^{-2\beta}$ (for $\beta \in \mathbb{R}$) is Hermitian up to the equivalence transformation:

$$(\hat{Q}_{12}^{(\beta)})^\dagger = \frac{1}{a(\beta)} (\hat{q}_{12})^{-2\beta} (\hat{p}_1)^{-2\beta} \mathcal{P}_{12} = U \hat{Q}_{12}^{(\beta)} U^{-1} ,$$

$$U := \mathcal{P}_{12} (\hat{q}_{12})^{-2\beta} = (\hat{q}_{12})^{-2\beta} \mathcal{P}_{12} .$$

Thus, we modify the scalar product in $V_1 \otimes V_2$

$$\langle \overline{\Psi} | \Phi \rangle := \langle \Psi | U | \Phi \rangle = \int d^4 x_1 d^4 x_2 \frac{\Psi^*(x_2, x_1) \Phi(x_1, x_2)}{(x_1 - x_2)^{2\beta}} ,$$

where $\beta \equiv D - \Delta_1 - \Delta_2$ and with respect to this new scalar product the operator $\hat{Q}_{12}^{(\beta)}$ is Hermitian. Here we introduced the special conjugation

$$\langle \overline{\Psi} | := \langle \Psi | U = \langle \Psi | (\hat{q}_{12})^{-2\beta} \mathcal{P}_{12} ,$$

and operator U plays the role of the metric in $V_1 \otimes V_2$.

Complex parameter α should be also partially fixed.
Indeed, we define conformal dilatation operator

$$\hat{\mathrm{D}} = \frac{i}{2}\sum_{a=1}^{2}(\hat{q}_a\hat{p}_a + \hat{p}_a\hat{q}_a) + \frac{1}{2}(y^\mu\,\partial_{y^\mu} + \partial_{y^\mu}\,y^\mu) - \beta,$$

such that $[\hat{Q}^{(\beta)}_{12},\ \hat{\mathrm{D}}] = 0$ and it is diagonalized simultaneously with $\hat{Q}^{(\beta)}_{12}$:

$$\hat{\mathrm{D}}\;|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = \Big(2\alpha + \beta - \frac{1}{2}D - n\Big)\;|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle\ .$$

For $\beta \in \mathbb{R}$, we obtain $\hat{\mathrm{D}}^\dagger = -U\,\hat{\mathrm{D}}\,U^{-1}$. Thus, operator $\hat{\mathrm{D}}$ is anti-Hermitian with respect to the same new scalar product $\langle\Psi|\,U\,|\Phi\rangle$, and it gives the condition for eigenvalues of $\hat{\mathrm{D}}$:

$$2(\alpha^* + \alpha) = 2n + D - 2\beta \quad\Rightarrow\quad \alpha = \frac{1}{2}\,(n + D/2 - \beta) - i\nu\ , \quad \nu \in \mathbb{R}\ .$$

So, we see that the eigenvalue problem for $\hat{Q}^{(\beta)}_{12}$ is characterized by two real numbers $\beta, \nu \in \mathbb{R}$ and we have $\Delta = \frac{D}{2} + 2i\nu$.

Remarkable fact: under these conditions, the GBO eigenvalue is real

$$\tau(\alpha,\beta,n)=(-1)^n\,\frac{\pi^{D/2}\Gamma(\beta)\,\Gamma(\frac{D}{4}+\frac{n}{2}-\frac{\beta}{2}+i\nu)\,\Gamma(\frac{D}{4}+\frac{n}{2}-\frac{\beta}{2}-i\nu)}{\Gamma(\beta')\,\Gamma(\frac{D}{4}+\frac{n}{2}+\frac{\beta}{2}+i\nu)\,\Gamma(\frac{D}{4}+\frac{n}{2}+\frac{\beta}{2}-i\nu)}\in\mathbb{R}.$$

In view of conditions on α,β, we introduce concise notation

$$|\Psi^{(n,u)}_{\nu,\beta,y}\rangle := |\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = u^{\mu_1}\cdots u^{\mu_n}|\Psi^{\mu_1\dots\mu_n}_{\alpha,\beta}(y)\rangle,$$

$$\Psi^{(n,u)}_{\nu,\beta,y}(x_1\,,x_2) := \langle x_1\,,x_2|\Psi^{(n,u)}_{\nu,\beta,y}\rangle\,.$$

Since the eigenvalue τ is real (it is invariant under the transformation $\nu\to-\nu$), two eigenvectors $|\Psi^{(n,u)}_{\nu,\beta,x}\rangle$ and $|\Psi^{(m,v)}_{\lambda,\beta,y}\rangle$, having different eigenvalues τ (e.g. $n\neq m$ and $\lambda\neq\pm\nu$), should be orthogonal to each other with respect to the new scalar product. Indeed, we have the following orthogonality condition for two conformal triangles (see, e.g., [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)])

$$\overline{\langle\Psi^{(m,v)}_{\lambda,\beta,y}}|\Psi^{(n,u)}_{\nu,\beta,x}\rangle = \int d^Dx_1\,d^Dx_2\,\langle\Psi^{(m,v)}_{\lambda,\beta,y}|U|x_1x_2\rangle\langle x_1x_2|\Psi^{(n,u)}_{\nu,\beta,x}\rangle =$$

$$= \int d^D x_1 \, d^D x_2 \frac{(\Psi^{(m,v)}_{\lambda,\beta,y}(x_2\,,x_1))^* \; \Psi^{(n,u)}_{\nu,\beta,x}(x_1\,,x_2)}{(x_1 - x_2)^{2(D-\Delta_1-\Delta_2)}} =$$

$$= \delta_{nm} C_1(n\,,\nu)\,\delta_{nm}\,\delta(\nu-\lambda)\,\delta^D(x-y)\,(u,v)^n +$$

$$+ C_2(n\,,\nu)\,\delta_{nm}\,\delta(\nu+\lambda)\,\frac{\Big((u,v) - 2\frac{(u,x-y)(v,x-y)}{(x-y)^2}\Big)^n}{(x-y)^{2(D/2+2i\nu)}}, \quad (3)$$

where $(u,v) = u^\mu v^\mu$, $\beta = D - \Delta_1 - \Delta_2 = \Delta_1 - \Delta_2$ and

$$C_1(n\,,\nu) = \frac{(-1)^n \, 2^{1-n} \, \pi^{3D/2+1} \, n! \, \Gamma(2i\nu)\,\Gamma(-2i\nu)}{\Gamma\left(\frac{D}{2}+n\right)\left(\left(\frac{D}{2}+n-1\right)^2 + 4\nu^2\right)\Gamma\left(\frac{D}{2}+2i\nu-1\right)\Gamma\left(\frac{D}{2}-2i\nu-1\right)} \quad (4)$$

We note that the coefficient C_1 is independent on β and plays the important role as the inverse of the Plancherel measure used in the completeness condition (resolution of unity); see below. In contrast to this, the coefficient C_2 in (3) depends on β, but the explicit form for C_2 will not be important for us.

$$C_2(n\,,\nu) = 2\pi^{D+1}\,\frac{n!}{2^n}\,\frac{\Gamma\left(\frac{D}{4}-\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}-i\nu\right)}{\Gamma\left(\frac{D}{4}-\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}+i\nu\right)}\,\frac{\Gamma\left(\frac{D}{4}+\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}-i\nu\right)}{\Gamma\left(\frac{D}{4}+\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}+i\nu\right)}\,\cdot$$

$$\frac{\Gamma(2i\nu)\,\Gamma\left(\frac{D}{2}+2i\nu-1+n\right)}{\Gamma\left(\frac{D}{2}+n-2i\nu\right)\Gamma\left(\frac{D}{2}+2i\nu-1\right)\Gamma\left(\frac{D}{2}+n\right)} \qquad (5)$$

Completeness (or resolution of unity I) for the basis of the eigenfunctions $|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle$ is written as [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)]

$$I = \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n\,,\nu)}\int d^Dx\,|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\overline{\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}}| =$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n\,,\nu)}\int d^Dx\,|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|\,U\,.$$

This is main formula needed to evaluation of ZZDs.

Substitution of this resolution of unity into expressions for zig-zag 4-point Feynman graphs gives (here M is a number of loops)

$$G_4^{(M)}(x_1,x_2;y_1,y_2) = \langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|y_1,y_2\rangle(y_1-y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n,\nu)}\int d^Dx\,\langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|\,U|y_1,y_2\rangle(y_1-y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}d\nu\,\frac{\big(\tau(\alpha,\beta,n)\big)^M}{C_1(n,\nu)}\int d^Dx\,\langle x_1,x_2|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|y_2,y_1\rangle\,, \qquad (6)$$

where the integral over x in the right hand side of (6) is evaluated in terms of conformal blocks [F.A.Dolan, H.Osborn (2001,2004); H.Osborn, A.Petkou (1994)] (in four-dimensional case, this integral was considered in detail by [N. Gromov, V. Kazakov, and G. Korchemsky (2019)]).

Further we use the expression for 2-point zig-zag functions $G_2^{(M)}(x_2,y_1)$

$$G_2^{(M)}(x_2,y_1) = \int d^Dx_1\,d^Dy_2\,\frac{\langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|y_1,y_2\rangle}{(x_1-x_2)^{2\beta}} =$$

and make the same procedure as for 4-point ZZ functions: $G_2^{(M)}(x_2, y_1) =$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n\,,\nu)} \int d^D x_1 d^D y_2\, d^D x\, \frac{\langle x_1, x_2 | \big(\hat{Q}_{12}^{(\beta)}\big)^M | \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} \rangle \langle \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} |\, U | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} d\nu \frac{(\tau(\alpha,\beta,n))^M}{C_1(n\,,\nu)} \int d(x_1, y_2, x) \frac{\langle x_1, x_2 | \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} \rangle \langle \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} | y_2, y_1 \rangle}{(x_1 - x_2)^{2\beta} (y_1 - y_2)^{2\beta}} =$$

$$= \frac{1}{(x_2 - y_1)^{2\beta}} \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+D-2)}{2^n \Gamma(n+D/2-1)} \int_0^{\infty} d\nu\, \frac{\tau^{M+3}(\alpha,\beta,n)}{C_1(n\,,\nu)}\,, \quad (7)$$

where we apply the integral

$$\int d^D x_1 d^D y_2\, d^D x\, \frac{\langle x_1, x_2 | \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} \rangle \langle \Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x} | y_2, y_1 \rangle}{(x_1 - x_2)^{2\beta} (y_1 - y_2)^{2\beta}} =$$

$$= \frac{(-1)^n \Gamma(n+D-2)\Gamma(D/2-1)}{2^n \Gamma(n+D/2-1)\Gamma(D-2)} \frac{\tau^3(\alpha,\beta,n)}{(x_2 - y_1)^{2\beta}}\,. \quad (8)$$

The integral over ν in the right hand side of (7) for $\beta = 1$ and even $D > 2$ can be evaluated explicitly and gives the linear combination of ζ-values with rational coefficients.

To prove Broadhurst and Kreimer conjecture we need to consider the special case $\beta = 1$, $D = 4$. In this case $\alpha = \frac{n+1}{2} - i\nu$ and GBO eigenvalue is simplified

$$\tau(\nu, n) := \tau(\alpha, \beta, n)|_{D=4,\beta=1} = \frac{(-1)^n (2\pi)^2}{(1+n)^2 + 4\nu^2} \,.$$

The coefficient C_1 in the definition of the Plancherel mesure for $\beta = 1$, $D = 4$ is reduced to

$$C_1(n, \nu) = \frac{\pi^5}{2^{n+3}(1+n)\,\nu^2}\,\tau(\nu, n)\,.$$

Finally we substitute $\tau(\nu, n)$, $C_1(n, \nu)$ into (7), integrate over ν and obtain

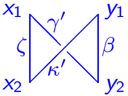
$$G_2(x_2, y_1)|_{D=4,\beta=1} \;=\; \frac{4\pi^{2M}}{(x_2 - y_1)^2}\, C_M \sum_{n=0}^{\infty} (-1)^{n(M+1)} \frac{1}{(n+1)^{2M-1}} \;, \quad (9)$$

where $C_M = \frac{1}{(M+1)}\binom{2M}{M}$ is a Catalan number. The relation (9) is equivalent the Broadhurst and Kreimer formula for the M loop zig-zag diagram (it corresponds to the $(M+1)$ loop contribution to the β-function in $\phi^4_{D=4}$ theory).

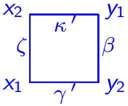
The generalization of the graph building operator is

$$Q_{12}^{(\zeta,\kappa,\gamma)} := \frac{1}{a(\kappa)a(\gamma)}\, \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\,, \qquad \zeta+\beta=\kappa+\gamma\,.$$

We depict the integral kernel of the D-dimensional operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ as follows ($(\kappa' := D/2-\kappa,\ \gamma' := D/2-\gamma)$)



$=$

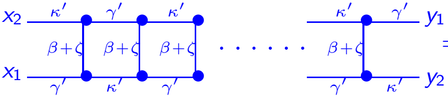


$$= \langle x_1, x_2 | Q_{12}^{(\zeta,\kappa,\gamma)} | y_1, y_2\rangle =$$

$$= \frac{1}{a(\kappa)a(\gamma)} \cdot \langle x_1, x_2 |\, \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\, | y_1, y_2\rangle =$$

$$= \frac{1}{(x_1-x_2)^{2\zeta}(x_2-y_1)^{2\kappa'}\,(x_1-y_2)^{2\gamma'}\,(y_1-y_2)^{2\beta}}\,.$$

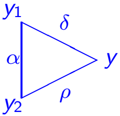
Thus, the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is the GBO for the ladder diagrams



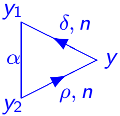
$$= (x_1-x_2)^{2\zeta}\, \langle x_1, x_2 | (\hat{Q}_{12}^{(\zeta,\kappa,\gamma)})^{2N} | y_1, y_2\rangle (y_1-y_2)^{2\beta},$$

Proposition 2. The eigenfunction for the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is given by 3-point correlation function (conformal triangle)

$\langle y_1, y_2 | \Psi^{(n,u)}_{\delta,\rho}(y)\rangle =$



$\cdot \left(\frac{(u, y-y_1)}{(y-y_1)^2} - \frac{(u, y-y_2)}{(y-y_2)^2} \right)^n \equiv$



where we depict the nontrivial rank-n tensor numerator as arrows on the lines (the rank is fixed by indices on the lines: $\rho \to (\rho, n)$, etc) and denote

$$2\alpha = \Delta_1 + \Delta_2 - (\Delta - n)\,, \quad 2\delta = \Delta_1 - \Delta_2 + (\Delta - n)\,, \quad 2\rho = \Delta_2 - \Delta_1 + (\Delta - n)\,,$$

i.e., conformal dimensions $\Delta, \Delta_1, \Delta_2$ are arbitrary parameters in this case. Thus, we have

$$Q_{12}^{(\zeta,\kappa,\gamma)}\, |\Psi^{(n,u)}_{\delta,\rho}(y)\rangle = \bar{\tau}(\kappa, \gamma; \delta, \alpha; n)\, |\Psi^{(n,u)}_{\delta,\rho}(y)\rangle\,.$$

where $\alpha + \rho = \kappa'$, $\alpha + \delta = \gamma'$ and $\bar{\tau}(\kappa, \gamma; \delta, \alpha; n)$ is an eigenvalue

$$\bar{\tau}(\kappa, \gamma; \delta, \alpha; n) = (-1)^n \cdot \tau(\delta', \kappa, n) \cdot \tau(\alpha, \gamma, n)\,,$$

$$\tau(\alpha, \beta, n) = (-1)^n \frac{\pi^{D/2}\Gamma(\beta)\Gamma(\alpha)\Gamma(\alpha' - \beta + n)}{\Gamma(\beta')\Gamma(\alpha' + n)\Gamma(\alpha + \beta)}$$

Remark 1. We introduce new notation $\beta+\zeta=-2u$ and use expressions for α,δ,ρ via conf. dimensions $\Delta_{1,2}$:

$$\beta-\zeta=D-\Delta_1-\Delta_2\;,\quad \gamma-\zeta=D/2-\Delta_1\;,\quad \kappa-\zeta=D/2-\Delta_2\;.$$

In this case the general GBO is equal (up to a normalization factor) to the R-operator [D. Chicherin, S. Derkachov, A. P. Isaev (2013)]

$$R_{\Delta_1\Delta_2}(u)=a(\kappa)a(\gamma)Q_{12}^{(\zeta,\kappa,\gamma)}=$$

$$=\mathcal{P}_{12}\;\hat{q}_{12}^{2(u+\frac{D-\Delta_1-\Delta_2}{2})}\,\hat{p}_1^{2(u+\frac{\Delta_2-\Delta_1}{2})}\;\hat{p}_2^{2(u+\frac{\Delta_1-\Delta_2}{2})}\;\hat{q}_{12}^{2(u+\frac{\Delta_1+\Delta_2-D}{2})}$$

which is conformal invariant solution of the Yang-Baxter equation

$$R_{\Delta_1\Delta_2}(u-v)\,R_{\Delta_1\Delta_3}(u)R_{\Delta_2\Delta_3}(v)=R_{\Delta_2\Delta_3}(v)\,R_{\Delta_1\Delta_3}(u)R_{\Delta_1\Delta_2}(u-v)\;.$$

The operator $R_{\Delta_1\Delta_2}(u)$ intertwines two spaces $V_{\Delta_1}\otimes V_{\Delta_2}\to V_{\Delta_2}\otimes V_{\Delta_1}$, where V_{Δ_i} is the space of scalar conf. fields with dimensions Δ_i. Let we have $V_{\Delta_1}\otimes V_{\Delta_2}=\sum_{\Delta,n}V_{\Delta}^{(n)}$, where $V_{\Delta}^{(n)}$ – is the space of tensor fields. Thus, eigenfunctions of $R_{\Delta_1\Delta_2}(u)=a(\kappa)a(\gamma)Q_{12}^{(\zeta,\kappa,\gamma)}$ should describe the fusion of two scalar conformal fields with dimensions Δ_1, Δ_2 into the composite tensor field with dimension Δ. Thus, conformal triangles are Clebsch-Gordan coefficients which correspond this fusion.

Remark 2. The special case (for $D = 1$ and $\Delta_1 = \Delta_2 \equiv \frac{D}{2} - \xi$) of this R-operator underlies Lipatov's integrable model of the high-energy asymptotics of multicolor QCD. Indeed, we have

$$\mathcal{P}_{12} R_{12}^{(\kappa,\xi)} = \hat{q}_{12}^{2(u+\xi)} \, \hat{p}_1^{2u} \, \hat{p}_2^{2u} \, \hat{q}_{12}^{2(u-\xi)} \quad \overset{u \to 0}{\to} \quad 1 + u \, h_{12}^{(\xi)} + \dots ,$$

$$h_{12}^{(\xi)} = 2 \ln q_{12}^2 + \hat{q}_{12}^{2\xi} \ln(\hat{p}_1^2 \, \hat{p}_2^2) \, \hat{q}_{12}^{-2\xi} ,$$

where $h_{12}^{(\xi)}$ is a local density of the Lipatov's Hamiltonian.

Conclusion.

In this report, we demonstrated:

1.) how the recent progress in the investigations of the multidimensional CFT can be applied, e.g., in the analytical evaluations of massless Feynman diagrams.

2.) We believe that the approach described here gives the universal method of the evaluation of contributions into the special class of correlation functions and critical exponents in various CFT.

3.) We also wonder if it is possible to apply our D-dimensional generalizations to evaluation similar 4-points functions (with fermions) that arise in the generalized "fishnet" model, in double scaling limit of γ-deformed $N = 4$ SYM theory.

4.) Very recent paper M. Kade, M. Staudacher, Supersymmetric brick wall diagrams and the dynamical fishnet, arXiv:2408.05805 [hep-th]. Supersymmetric generalizations of Basso-Dixon fishnet and brick wall diagrams.

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