

Minimal models of local gauge theories

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Earlier relevant works in collaboration with Glenn Barnich, Konstantin Alkalaev, Alexei Kotov

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Background

- Batalin-Fradkin-Vilkovisky (BFV) and Batalin-Vilkovisky (BV) formalism.
- *Alexandrov, Kontsevich, Schwartz, Zaboronsky (AKSZ)* construction of BV for Lagrangian topological models. Further developments *Cattaneo, Felder, Roytenberg, Reshetikhin, Mnev, Ikeda, ...*
- BV on jet-bundles, local BRST cohomology *Henneaux, Barnich, Brandt, ...*
- Unfolded approach in higher spin gauge theories *M.Vasiliev*
- Geometric approach to PDEs *Vinogradov, Tulczyjew, ...*
- FDA approach to SUGRA *d'Auria, Fre, Castellani, Grassi ...*
- BRST first quantized (cf. L_∞) approach to SFT and gauge fields *Zwiebach; Thorn, Bochicchio, Henneaux, Teitelboim, ...*
- *Fedosov* quantization and its variations

AKSZ construction

(\mathcal{M}, q, ω) - QP-manifold (target space) equipped with:

- \mathbb{Z} -degree (ghost number) $\text{gh}()$
- homological v.f. q , $q^2 = 0$, $\text{gh}(q) = 1$
- (odd)symplectic structure ω , $\text{gh}(\omega) = n - 1$ such that

$$q^2 = 0, \quad L_q \omega = 0$$

It follows: $i_q i_q \omega = 0$ and (locally) $\exists \mathcal{H}$ such that $i_q \omega + d\mathcal{H} = 0$

(\mathcal{X}, d_X, ρ) (source space)

equipped with \mathbb{Z} -degree (ghost number) $\text{gh}()$

homological v.f. d_X and compatible measure ρ

Typically, $\mathcal{X} = T[1]X$, $\dim X = n$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$,
 $d_X = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \dots, n - 1$, and $\rho = 1$

Supermanifold of supermaps: $\hat{\sigma} : T[1]X \rightarrow \mathcal{M}$. ψ^A coordinates on \mathcal{M} . Fields: $\psi^A(x, \theta) := \hat{\sigma}^*(\psi^A)$, $\hat{\sigma} : T[1]X \rightarrow \mathcal{M}$. BV action

$$S_{BV}[\hat{\sigma}] = \int_{T[1]X} (\hat{\sigma}^*(\chi)(d_X) + \hat{\sigma}^*(\mathcal{H})), \quad \text{gh}(S_{BV}) = 0$$

χ is the potential: $\omega = d\chi$. In components:

$$S_{BV} = \int d^n x d^n \theta [\chi_A(\psi(x, \theta)) d_X \psi^A(x, \theta) + \mathcal{H}(\psi(x, \theta))]$$

BV symplectic structure:

$$\bar{\omega} = \int_{T[1]X} \hat{\sigma}^*(\omega_{AB}) \delta\psi^A(x, \theta) \wedge \delta\psi^B(x, \theta), \quad \text{gh}(\bar{\omega}) = -1$$

BV antibracket:

$$(F, G) = \int_{T[1]X} \frac{\delta^R F}{\delta\psi^A(x, \theta)} \omega^{AB}(\psi(x, \theta)) \frac{\delta G}{\delta\psi^B(x, \theta)}, \quad \text{gh}(,) = 1$$

Master equation:

$$(S_{BV}, S_{BV}) = 0 \quad \text{modulo boundary terms}$$

BRST differential $s = (S_{BV}, \cdot)$:

$$s = \int d^n x d^n \theta (d_X \psi^A(x, \theta) + q^A(\psi(x, \theta)) \frac{\delta}{\delta \psi^A(x, \theta)}).$$

Both q and d_X naturally lift to the space of supermaps.

Physical fields: those of vanishing ghost degree

$$\psi^A(x, \theta) = \psi^0{}^A(x) + \psi^1{}^A{}_\mu(x) \theta^\mu + \dots \quad \text{gh}(\psi^k{}^A{}_{\mu_1 \dots \mu_k}) = \text{gh}(\psi^A) - k$$

If $\text{gh}(\psi^A) = k$ with $k \geq 0$ then $\psi^k{}^A{}_{\mu_1 \dots \mu_k}(x)$ is physical. Setting to zero fields of nonzero degree (i.e. restricting to maps) gives the classical action:

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(d_X) + \sigma^*(\mathcal{H}))$$

EL equations of motion:

$$\omega_{AB}(\psi(x, \theta))(d_X \psi^A - q^A) = 0, \quad \Rightarrow \quad (d_X \psi^A(x, \theta) - q^A(\psi(x, \theta))) = 0$$

provided ω_{AB} is invertible.

More invariantly, if $\psi^A(x, \theta) = \sigma^*(\psi^A)$ the equations of motion read as:

$$d_X \sigma^*(\psi^A) = \sigma^*(q\psi^A) \quad \Leftrightarrow \quad d_X \circ \sigma^* = \sigma^* \circ Q$$

so that σ^* is a morphism of respective complexes. Gauge transformations correspond to trivial morphisms:

$$\delta_\epsilon \sigma^* = d_X \circ \epsilon_\sigma^* + \epsilon_\sigma^* \circ q$$

ϵ_σ - gauge parameter. $\epsilon_\sigma^*(fg) = (\epsilon_\sigma^* f)\sigma^*(g) + (-1)^{|f|}\sigma^*(f)\epsilon_\sigma^*(g)$,
i.e. is a vector field along σ .

Example: CS theory,

AKSZ, 1995

Target: $\mathcal{M} = \mathfrak{g}[1]$, q -CE differential, ω – invariant form on \mathfrak{g}
(degree 2 symplectic structure on $\mathfrak{g}[1]$)

Source: $\mathcal{X} = T[1]X$, $\dim X = 3$, $gh()$ – form degree, d_X

$$S_{BV} = \int_X \text{Tr}(AdA + \frac{2}{3}A \wedge A \wedge A) + \text{BV completion}$$

Ghosts and antifields arise as nonzero degree components of a supermap:

$$\hat{\sigma}^*(C) = \overset{0}{C}(x) + A_\mu(x)\theta^\mu + \frac{1}{2}\overset{2}{C}_{\mu\nu}(x)\theta^\mu\theta^\nu + \frac{1}{6}\overset{3}{C}_{\mu\nu\rho}(x)\theta^\mu\theta^\nu\theta^\rho$$

Introducing $C^*, A^{*\mu}$ via $\overset{3}{C}_{\mu\nu\rho}(x) = \epsilon_{\mu\nu\rho}C^*$ and $\overset{2}{C}_{\mu\nu}(x) = \epsilon_{\mu\nu\rho}A^{*\rho}$
the BV symplectic structure

$$\omega_{BV} = \int_X \text{Tr}(\delta A_\mu \wedge \delta A^{*\mu} + \delta C \wedge \delta C^*)$$

Example: 1d AKSZ sigma model

Target: BFV phase space \mathcal{M} equipped with symplectic form ω and BFV-BRST charge $\Omega = c^\alpha T_\alpha + \dots$ such that $\{\Omega, \Omega\} = 0$ and the Hamiltonian $H = H_0 + \dots$ satisfying $\{H, \Omega\} = 0$.
(Generalized) AKSZ action *M.G., Damgaard 1999*

$$S_{BV} = \int dt d\theta (\chi_A d_X \psi^A - \Omega(\psi(t, \theta) - \theta H(\psi(x, \theta)))$$

is a BV extension (*Fisch, Henneaux*) of the Hamiltonian action:

$$S_0 = \int dt (p\dot{q} - H_0 - \lambda^\alpha T_\alpha)$$

Lagrange multipliers λ^α arise as 1-forms associated to BFV ghost variables: $\sigma^*(c^\alpha) = \lambda^\alpha(t)\theta$.

The relation between the BV antibracket and BFV Poisson bracket

$$(\cdot, \cdot)_{BV} = \int dt d\theta \{ \cdot, \cdot \}$$

Explicit realization of the isomorphism of *Barnich, Henneaux 1996*

What we've learned:

- non-diffeo-invariant theories correspond to x^a, θ^a -dependent target structures. Suitable language of fiber bundles or parameterized systems.
- AKSZ unifies BV and BFV. For $X = \Sigma \times \mathbb{R}^1$ taking $T[1]\Sigma$ as a source gives BFV-AKSZ sigma model. *M.G. Barnich 2003; M.G. 2010*. Further developments: *Cattaneo, Mnev, Reshetikhin 2012; Bonechi, Zabzine 2012;*
- More generally, induces (shifted) BV (BFV) on any source manifold. Gives a natural framework to study gauge theories with (asymptotic) boundaries *M.G, Bekaert 2012; Mnev, Schiavina 2019, MG Markov 2023, . . .*

Towards generalized AKSZ

In general, AKSZ equations of motion

$$\omega_{AB}(\psi(x, \theta))(d_X \psi^A(x, \theta) - q^A(\psi(x, \theta))), \quad q^A = q\psi^A.$$

For ω_{AB} invertible, these imply (generalized) zero-curvature and hence the system is topological provided M is finite-dimensional.

What about general local gauge theories? Possible way out is infinite-dimensional \mathcal{M} involving all the curvatures. The idea goes back to unfolded approach of *M.Vasiliev*. General formalism and existence: *Barnich, MG, 2010*

An alternative (with \mathcal{M} finite-dim.): **take ω degenerate so that AKSZ equations of motion kill only part of the curvature.** The first characteristic example is Cartan-Weyl form of Einstein gravity:

Presymplectic AKSZ form of gravity

Target $(\mathfrak{g}[1], q, \omega)$, with \mathfrak{g} Poincare algebra and q its CE differential. Coordinates on $\mathfrak{g}[1]$ in the standard basis ξ^a, ρ^{ab}

$$q\xi^a = \rho^a{}_c \xi^c, \quad q\rho^{ab} = \rho^a{}_c \rho^{cb} + \lambda \xi^a \xi^b,$$

Presymplectic structure:

Alkalaev, M.G. 2013; MG 2016

$$\omega = \epsilon_{abcd} \xi^a d\xi^b d\rho^{cd}, \quad \omega = d\chi$$
$$L_q\omega = 0, \quad d\omega = 0 \quad \Rightarrow \quad i_q\omega + d\mathcal{H} =$$

AKSZ-like action:

$$S[\sigma] = \int_{T[1]X} \sigma^*(\chi)(d_X) + \sigma^*(\mathcal{H}) = \int_{T[1]X} (d_X \gamma^{ab} + \gamma^a{}_c \gamma^{cb}) \epsilon_{abcd} e^c e^d$$

where $e^a = \sigma^*(\xi^a)$ and $\gamma^{ab} = \sigma^*(\rho^{ab})$. Familiar Cartan-Weyl action for GR. Generalization for $n > 4$ and $\Lambda \neq 0$ is obvious.

What about the remaining components of supermaps? Full-scale BV formulation?

General axioms:

Def Pre Q-bundle $\pi : (E, Q) \rightarrow (\mathcal{X}, q)$ \mathbb{Z} -graded manifolds equipped with degree 1 vector fields such that $Q \circ \pi^* = \pi^* \circ q$,
If $Q^2 = 0$ and $q^2 = 0$ one gets \mathbb{Z} -graded version of Q-bundle
Kotov, Strobl 2007.

Def [MG 22, Dneprov, Gritzaenko, MG 23] Weak presymplectic gauge PDE is a pre Q-bundle $\pi : (E, Q) \rightarrow (T[1]X, d_X)$ equipped with presymplectic structure ω , $\text{gh}(\omega) = \dim X - 1$ $d\omega = 0$

$$L_Q \omega \in \mathcal{I}, \quad i_Q i_Q \omega = 0, \quad i_Q L_Q \omega = 0$$

where \mathcal{I} is generated by $\pi^*(\alpha)$ with $\alpha \in \Lambda^{>0}(T[1]X)$, i.e. by $dx, d\theta$

Note that in general $Q^2 \neq 0$! Note that $L_Q \omega \in \mathcal{I}$ implies $i_Q \omega + dH \in \mathcal{I}$ for some $H \in \mathcal{C}^\infty(E)$

Weak presymplectic scalar field

$E = T[1]X \times F$, fiber coordinates:

$$\phi, \phi^a, \quad \text{gh}(\phi) = \text{gh}(\phi^a) = 0$$

$$Qx^a = \theta^a, \quad Q\theta^a = 0, \quad Q\phi = \theta^a \eta_{ab} \phi^b, \quad Q\phi^a = \theta^a V'(\phi)$$

Presymplectic form (cf. *Kijowski, Tulczyjew 1979, Crnkovic, Witten, 1987,...* presymplectic current):

$$\omega = d\chi, \quad \chi = (\theta^a)_a^{n-1} \phi^a d\phi, \quad (\theta)_a^{(n-1)} = (*\theta)_a$$

Note: in general $L_Q\omega \neq 0$ and $Q^2 \neq 0$ but the axioms hold!

$$i_Q\omega + dH \in \mathcal{I} \implies H = -(\theta)^n \left(\frac{1}{2} \phi_a \phi^a + V(\phi) \right)$$

AKSZ-like (aka intrinsic) action: *Schwinger, De Donder-Weyl*

$$S[\phi, \phi^a] = \int_X (dx)^n \left(\phi^a (\partial_a \phi - \frac{1}{2} \phi_a) - V(\phi) \right)$$

Presymplectic AKSZ form of YM:

$E = T[1]X \times F$, fiber coordinates (\mathfrak{g} -valued):

$$C, \quad \text{gh}(C) = 1, \quad F^{a|b}, \quad \text{gh}(F^{a|b}) = 0$$

$$Qx^a = \theta^a, \quad Q\theta^a = 0, \quad QC = -\frac{1}{2}[C, C] + \frac{1}{2}F^{a|b}\theta_a\theta_b, \quad QF^{a|b} = [F^{a|b}, C]$$

Note $Q^2 \neq 0$, in general. Presymplectic structure satisfying $L_Q\omega \in \mathcal{I}$:

Alkalaev, M.G. 2013

$$\omega = d\chi, \quad \chi = (\theta)_{ab}^{(n-2)} \text{Tr} (F^{a|b} dC)$$

AKSZ-like action ($\sigma^*(C) = A_a(x)\theta^a, \sigma^*(F^{a|b}) = F^{a|b}(x)$):

$$S[\sigma] = \int d^n x \text{Tr} \left((\partial_a A_b - \partial_b A_a + [A_a, A_b]) F^{a|b} - \frac{1}{2} (F^{a|b})^2 \right)$$

Features of weak presymplectic gPDEs:

- Almost as good as AKSZ but applies to general local gauge theories
- Encodes a local gauge theory in terms of a finite-dim pre-Q presymplectic manifold. Can be regarded as a minimal model of BV (as we are going to see it arises as a minimal mode of the L_∞ algebra determined by the jet-space BV-BRST differential + descent of the BV symplectic structure)
- Together with minimality condition seems to be an invariant geometrical object underlying local gauge systems. Should be unique modulo suitable equivalence.
- What about full-scale BV? Where does it come from? Existence?

Thm. Let $(E, Q, T[1]X, \omega)$ be a weak presymplectic gauge PDE. Assume that presymplectic form $\bar{\omega}$ induced on $\Gamma_S(E)$ (supersections of $E \rightarrow T[1]X$) is regular. Then, locally,

$$S_{BV}(\hat{\sigma}) = \int_{T[1]X} (\hat{\sigma}^*(\chi)(d_X) + \hat{\sigma}^*(H)), \quad \omega = d\chi$$

defines a local BV system on the symplectic quotient of $\Gamma_S(E)$. The proof is given in terms of $J_S^\infty(E)$. Q, ω induces the usual jet-bundle BV system on the symplectic quotient.

Physical explanation: Shifts along $\ker \bar{\omega}$ are algebraic gauge transf. for S_{BV} . Gauge-fixing them gives BV action satisfying BV master-equation modulo boundary terms. In particular, S_{BV} can be used in the path integral

$$\int_{\tilde{L}} \exp \frac{i}{\hbar} S_{BV}$$

where \tilde{L} also takes into account $\ker \bar{\omega}$. **No need to take the symplectic quotient explicitly**

Example: scalar

Recall: fiber coordinates ϕ, ϕ^a . Coordinates on $\Gamma_S(E)$:

$$\hat{\sigma}^*(\phi) = \overset{0}{\phi}(x) + \overset{1}{\phi_a}(x)\theta^a + \dots$$

$$\hat{\sigma}^*(\phi^a) = \overset{0}{\phi^a}(x) + \overset{1}{\phi_b^a}(x)\theta^b + \dots$$

Presymplectic structure $\omega = (\theta^a)^{n-1} d\phi^a d\phi$ induces on supermaps:

$$\bar{\omega} = \int_X d^n x \left(\delta\overset{0}{\phi} \wedge \delta\overset{1}{\phi_a^a} + \delta\overset{0}{\phi^a} \wedge \delta\overset{1}{\phi_a} \right)$$

All the fields are in the kernel except for:

$$\varphi = \overset{0}{\phi}, \quad \varphi^* = \overset{1}{\phi_a^a}, \quad \varphi^a = \overset{0}{\phi^a}, \quad \varphi_a^* = \overset{1}{\phi_a}$$

Correct set of fields and antifields for the 1st order form of scalar! BV symplectic structure emerged from the **presymplectic current!**

Example: YM

Recall: fiber coordinates $C, F^{a|b}$. Coordinates on $\Gamma_S(E)$:

$$\hat{\sigma}^*(C) = \overset{0}{C}(x) + A_a(x)\theta^a + \frac{1}{2}\overset{2}{C}_{ab}(x)\theta^a\theta^b \dots$$

$$\hat{\sigma}^*(F^{a|b}) = \overset{0}{F}^{a|b}(x) + \overset{1}{F}_c^{a|b}(x)\theta^c + \frac{1}{2}\overset{2}{F}_{cd}^{a|b}(x)\theta^c\theta^d + \dots$$

Presymplectic structure $\omega = \theta_{ab}^{(2)} dF^{ab} dC$ induces on supermaps:

$$\bar{\omega} = \int_X Tr \left(\delta \overset{0}{C} \wedge \delta \overset{2}{F}_{ab}^{a|b} + \delta A_a \wedge \delta \overset{1}{F}_b^{a|b} + \delta \overset{2}{C}_{ab} \wedge \delta \overset{0}{F}^{a|b} \right)$$

All the fields are in the kernel except for:

$$C = \overset{0}{C}, \quad C^* = \overset{2}{F}_{ab}^{a|b}, \quad A_a, \quad A_*^a = \overset{1}{F}_b^{a|b}, \quad F^{a|b}, \quad F_{ab}^* = \overset{2}{C}_{ab}$$

S_{BV} coincides with the standard BV action for YM in the first order formalism.

Where do minimal models come from?

Descent completed BV: $Q = d_h + s$:

Stora; Barnich Brandt Henneaux; Barnich MG; ...

Descent completion of the BV symplectic structure: *Cattaneo*

Mnev Reshetikhin; Sharapov; Mnev Schiavina, MG; ...

$$\overset{n}{\omega} = (dx)^n \omega_{AB}(x, \psi^A) d_V \psi^A d_V \psi^B,$$

$$L_s \overset{n}{\omega} + d_h \overset{n-1}{\omega} = 0, \quad L_s \overset{n-1}{\omega} + d_h \overset{n-2}{\omega} = 0, \quad \dots$$

Taking $\omega = \overset{n}{\omega} + \overset{n-1}{\omega} + \dots + \overset{0}{\omega}$ one finds

$$L_Q \omega = 0, \quad i_Q i_Q \omega = 0$$

Taking minimal model and setting to zero variables from the regular kernel of ω results in the presymplectic minimal model. Derivation of (generalized) AKSZ!

Weak gauge PDEs

MG, Rudinsky 2024

Whats is the analog at the level of equations of motion?

Idea: keep the kernel distribution and forget about the presymplectic structure.

Def. Weak gPDE is a pre- Q -bundle $(E, Q) \rightarrow (T[1]X, d_X)$ equipped with a Q -invariant vertical distribution \mathcal{K} such that $Q^2 \in \mathcal{K}$.
gPDE corresponds to $\mathcal{K} = 0$

Thm. Let $(E, Q, T[1]X, \mathcal{K})$ be a weak gPDE. Assume that prolongation $\bar{\mathcal{K}}$ of \mathcal{K} is regular. Then, at least locally, $J_S^\infty(E)/\bar{\mathcal{K}}$ is a local BV system.

The proof is based on the observation: $\bar{Q}^2 \in \bar{\mathcal{K}} \Rightarrow \bar{Q}^2 f = 0$ for any function f such that $\bar{\mathcal{K}}f = 0$.

Any weak presymplectic gPDE gives weak gPDE by taking $\mathcal{K} = \{V \in \text{Vect}_v(E) : i_V \omega \in \mathcal{I}\}$ and forgetting ω .

Example: self-dual YM

$X = \mathbb{R}^4$ with Euclidean metric and $E \rightarrow T[1]\mathbb{R}^4$, with the fiber being $\mathfrak{g}[1]$, where \mathfrak{g} is a real Lie algebra. Local coordinates on E are: x^a, θ^a, C^A . Useful convention $C = C^A t_A$. The Q -structure is then defined as

$$Q(x^a) = \theta^a, \quad Q(\theta^a) = 0, \quad Q(C) = -\frac{1}{2}[C, C]$$

Distribution \mathcal{K} is generated by:

$$K_A^{(1)ab} = \left(\theta^a \theta^b + \frac{1}{2} \epsilon^{ab}{}_{cd} \theta^c \theta^d \right) \frac{\partial}{\partial C^A}, \quad K_{aA}^{(2)} = \epsilon_{abcd} \theta^b \theta^c \theta^d \frac{\partial}{\partial C^A},$$

Note: $Q^2 = 0$ and $L_Q \mathcal{K} \subset \mathcal{K}$.

Minimal model (in the sense of weak gPDE) of self-dual YM

Example: self-dual YM

Fields parameterizing the quotient $J_S^\infty(E)/\bar{K}$:

$$\overset{0}{C}, \quad A_a \equiv \overset{1}{C}|_a, \quad \mathcal{F}_{ab}^{*-} \equiv \overset{0}{C}|_{ab} - \frac{1}{2}\epsilon_{abcd}\overset{0}{C}|^{cd}.$$

The induced BRST differential s :

$$s(\mathcal{F}_{ab}^{*-}) = -(\mathcal{D}_a A_b - \mathcal{D}_b A_a)^- - [\mathcal{F}_{ab}^{*-}, \bar{C}],$$

$$s(\overset{0}{C}) = -\frac{1}{2}[\overset{0}{C}, \overset{0}{C}], \quad s(A_a) = \mathcal{D}_a \overset{0}{C}$$

where $\mathcal{D}_a = D_a + [A_a, \cdot]$ is the covariant total derivative.

Gives standard BRST complex for self-dual YM.

Conclusions

- (Finite-dimensional) super-geometrical objects underlying local gauge theories. It seems minimal models are canonical.
- Generalization and first principle derivation of the AKSZ construction. Can be considered as an extension of AKSZ to generic local theories.
- Determines a “canonical” first-order realization in terms of the fields taking values in the minimal model. Makes manifest underlying Cartan geometry. Covariant Hamiltonian formalism. Classification?
- Further examples include conformal gravity (*Denprov MG, 2022*), supergravity (*MG Mamekin, to appear*), bigravity (*Gritzaenko MG, to appear*).

- Tool to study geometry underlying a given gauge system. Background fields and background independence can be incorporated in the approach (*MG, Dneprov, to appear*)
- In the case of variational systems unifies Lagrangian BV and Hamiltonian BFV formalism, cf. BV/BFV approach of *Cattaneo et al.*
- Gives a geometrically-invariant approach to study boundary values of gauge fields and asymptotic symmetries *Bekaert, M.G. 2012, MG, Markov 2023*. In particular, Fefferman-Graham construction (and tractor calculus) can be seen as a certain gauge PDE. *Bekaert, M.G. Skvortsov 2017*
- Gives a criterion to characterize local theory in terms of its infinite dimensional equation manifold. Possibly interesting in the HS theory context.