KPZ scaling from the Krylov space and diagnostics of a quantum chaos

A. Gorsky, Fradkin-100 , September 2, 2024

«KPZ scaling in the Krylov Space»
With S.Nechaev(Paris) and A. Valov(Jerusalem) JHEP (2024)

«Hilbert space geometry and quantum chaos» 2409.xxxxx
With R.Sharipov(Lyublyana), A.Tyutyakina(Paris), V.Gritsev(Amsterdam), A.Polkovnikov(Boston)

Outline of the talk

- Different diagnostics of quantum chaos
- KPZ scaling in the late-time correlators in the Krylov space. Origin and universality. Can this behaviour be diagnostics of integrability or chaos?

  Byproduct lesson for 2d gravity. Double scaling.
- Induced geometry for 2-dimensional parameter space and transition to a quantum chaos

Diagnostics for a quantum chaos

- Late-time behaviour of correlators. Spectral formfactors. Generalization of Lyapunov exponents.OTOC
- Fractal dimensions of eigenfunctions
- Late-time behaviour of Krylov complexity. Asymptotic behaviour of Lanzcos coefficients
- Induced geometry of the parameter space. Response of the system on a perturbation

Historic remarks

– Late-time OTOC -generalization of Larkin-Ovchinnikov approach(65'). Formulation of operator growth picture, scrambling, black hole as chaotic system-Shenker,Stanford, Susskind,Sonner ….

- Fractal dimension diagnostics. Invented for Anderson localization long time ago. Localization of wave functions-integrability, delocalization- chaos. However it turnes out that there are many situations when $0<D_q<1$ — fractal KAM-like phase
First clear-cut example Amini,Khaymovich,Kravtsov (15')

--Krylov space approach, Parker,Cao,Avdoshkin,Saffidi,Altman(19'), Dymarsky-Smolkin In CFT (21'), Krylov complexity, random Krylov chain, Rabinovichi-Barbon (19')

– Take use of response of the system on adiabatic perturbation. Induced metrics on parameter space. Zanardi et al 07-08', Polkovnikov,Gritsev,Liska 13', Polkovnikov et al 19-23' different aspects of the geodesic motion on parameter space

All approaches work well enough but with exceptions in each case

Surprizes with late-time behaviour

Unexpected late-time behaviour for several models. First found
In integrable spin chains (finite) Liubotina, Znidaric,Prosen 19'. Confirmed
later in several other many-body systems

$\langle S^z(x,t)S^z(0,0)\rangle \propto (\Gamma t)^{-\frac{2}{3}})f_{kpz}((\Gamma t)^{-\frac{2}{3}}x)$ $\quad z = 3/2,\, t = x^z$

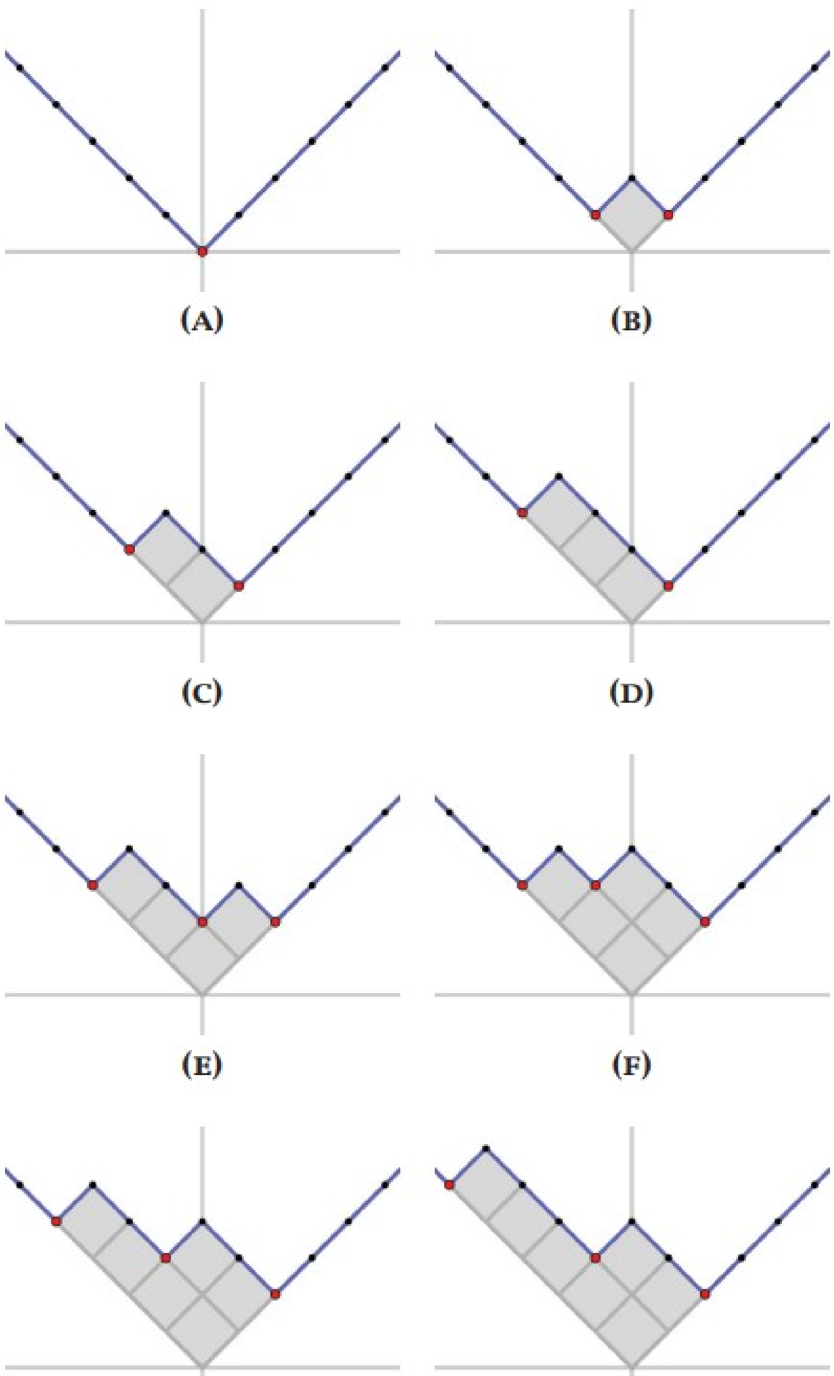
$\langle S_i^z(t)S_i^z(0)\rangle \propto (\Gamma t)^{-\frac{2}{3}}$

Dynamical KPZ critical scaling

What is the origin of this phenomena?
Is it specific for integrable model or takes place for chaotic systems as well?

Seems that for non-integrable deformations works as well.
Several explanations suggested. Goldstone mode for the effective U(1) symmetry
Formation of the giant soliton(in Landau-Lifshitz chain) . No consensus
concerning explanation yet.

Our strategy: 1. consider the Krylov space for the quite generic operators.
2. Take into account the generic behaviour of the Lanzcos coefficients for the
finite system 3. Focus at the Heisenberg time-scale T~dim H



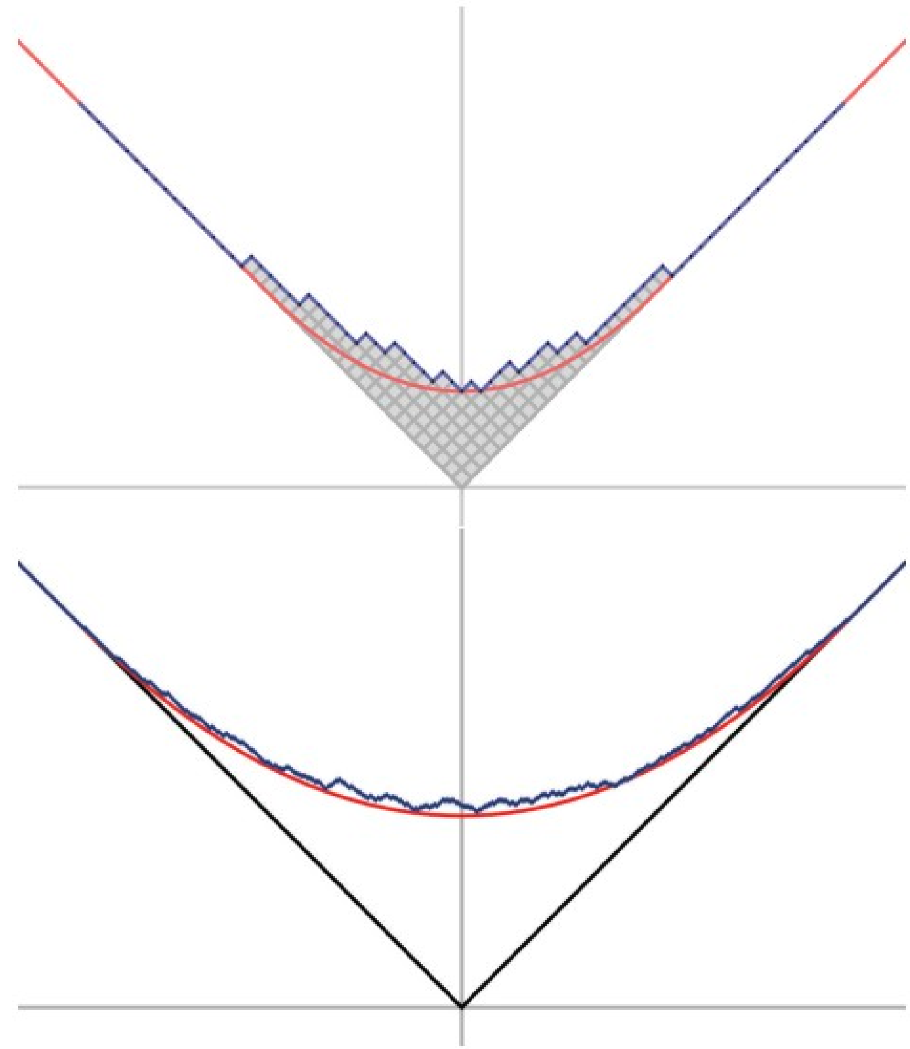


Figure 6. Simulation of the corner growth model. The top shows the model after a medium amount of time and the bottom shows it after a longer amount of time. The blue interface is the simulation, while the red curve is the limiting parabolic shape. The blue curve has vertical fluctuations of order $t^{1/3}$ and decorrelates spatially on distances of order $t^{2/3}$.

-40 0 40 80 120 160 200

KPZ universality class

Kardar-Parisi-Zhang 86'

$$\partial_t h(x,t) = \nu \partial_{xx} h(x,t) - \lambda (\partial_x h(x,t))^2 + \xi$$

$$\partial_t u(x,t) + \partial_x(-\lambda u^2 - \nu \partial_x u - \xi) = 0$$

$$u = \partial_x h,$$

$$Z(x,t) = e^{\lambda h(x,t)}$$

$$\partial_t Z(x,t) = \frac{1}{2}\partial_{xx} Z(x,t) + \xi(x,t) Z(x,t)$$

Partition function of polymer

$$h(x,0) = 0$$

$$h(0,t) \sim vt + (\Gamma t)^{1/3} \chi_{flat}$$

Late time 1-point function

where $\chi_{flat} = \chi_{GOE}$ with the probability distribution

$$P(\chi_{GOE} \leq s) = \det\left(1 - K_{\mathrm{Ai}}(s)\right)$$

and $K_{\mathrm{Ai}}(s)$ in the Fredholm determinant is the Airy kernel.

KPZ universality class

$$h(x,0) = |x| \qquad P\left(\frac{x^2}{2t}h(x,t) - \frac{t}{24} \geq -s\right) = F_t(s)$$

$$F_t(2^{-1/3}t^{1/3}s) \to F_{GUE}(s) \qquad F_{GUE}(s) = \det(I - K_{\mathrm{Ai}}(s)) = \exp\left(-\int_s^\infty (x-s)^2 q^2(x)dx\right)$$

vhere $q(x)$ obeys the Painleve II equation

$$q''(x) = (x + 2q^2(x))q(x)$$

ubject to the boundary condition $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.

$$\langle u(x,t)u(0,0)\rangle \sim (\Gamma t)^{-2/3} f_{kpz}\left((\Gamma t)^{-2/3}x\right)$$

Late time correlator

Tabulated function

Krylov basis. Propeties

$O(t) = e^{iHt}O(0)e^{-iHt}$, $\mathcal{L} = [H, \ldots]$. $\mathcal{L}^n|O\rangle$ Tower of operator for some seed

$$H|K_n\rangle = a_n|K_n\rangle + b_{n-1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

where

$$a_n = \langle K_n|H|K_n\rangle\,, \qquad b_n = \langle A_n|A_n\rangle^{1/2}\,, \quad |A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

Krylov basis $|O_n\rangle = b_n^{-1}|K_n\rangle$

$$L_{nm} = \langle O_n|\mathcal{L}|O_m\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & 0 & \ldots \\ b_1 & a_1 & b_2 & 0 & 0 & \\ 0 & b_2 & a_2 & b_3 & 0 & \\ 0 & 0 & b_3 & a_3 & b_4 & \\ 0 & 0 & 0 & b_4 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Matrix of Lanzcos coefficients

Gram-Scmidt orthogonalization in the operator space. Seed operator O is arbitrary

Krylov basis.Propeties

$$\frac{\partial \phi_n}{\partial t} = -b_{n+1}\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1}$$

$$\phi_n(t) = i^{-n} \langle O_n(0)|O(t)\rangle, \ \phi_n(0) = \delta_{n,0}.$$

Hopping problem on the Krylov chain. Equivalent to the open Toda(Dymarsky A.G 20)

$$C_O(t) = \langle O|e^{i\mathcal{L}t}|O\rangle$$

$$\mu_{2n} = \langle O|\mathcal{L}^{2n}|O\rangle = \frac{d^{2n}}{dt^{2n}}C(t)|_{t=0}; \quad C(t) = \sum_n \frac{\mu_n t^n}{n!}$$

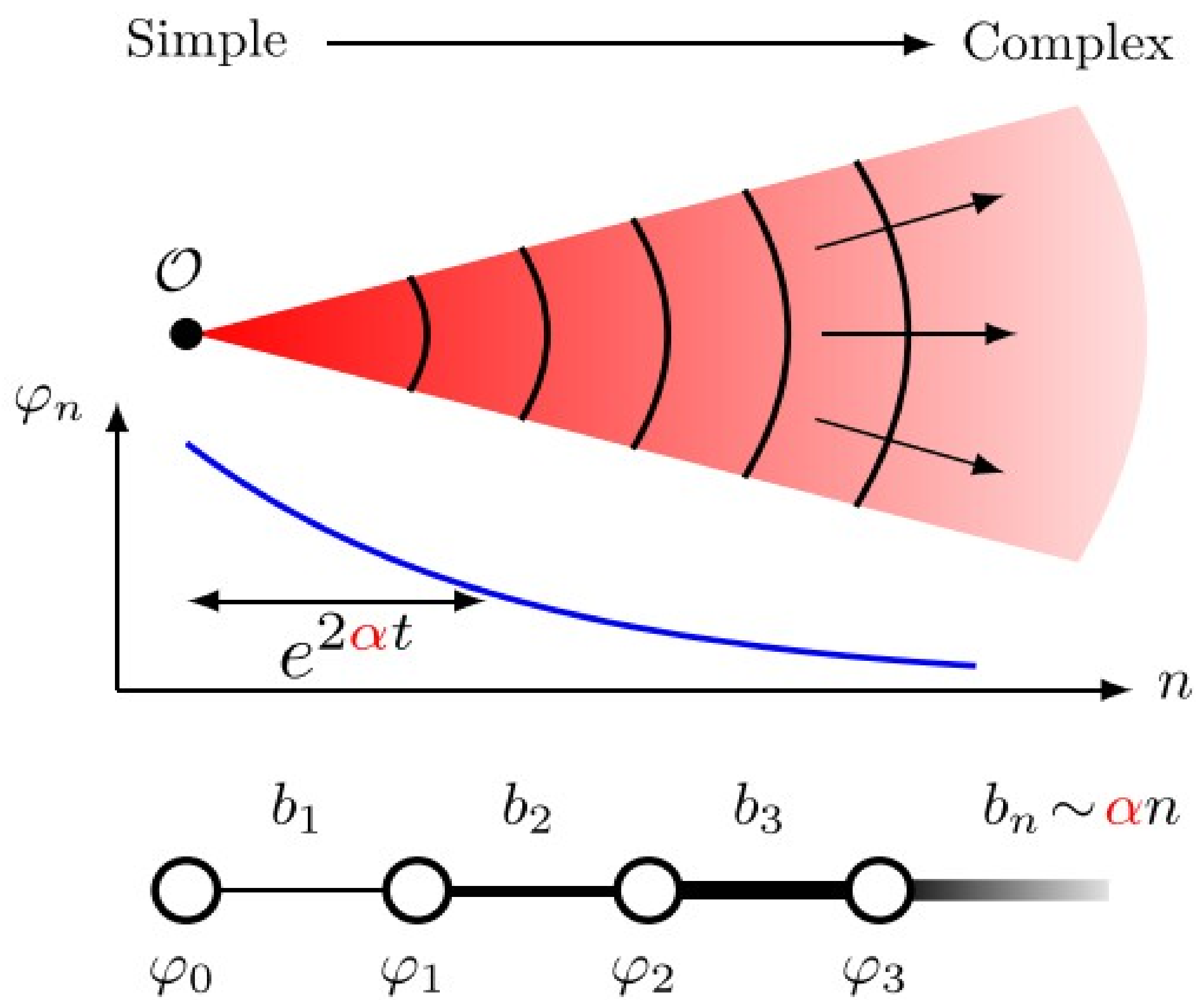
$$\mu_{2n} = \sum_{h_n \in D_n} \prod_k b_{h_k + h_{k+1}/2}$$

Properly defined Dick paths

$$\mathcal{K}(t) = \sum_n n|\phi_n|^2$$

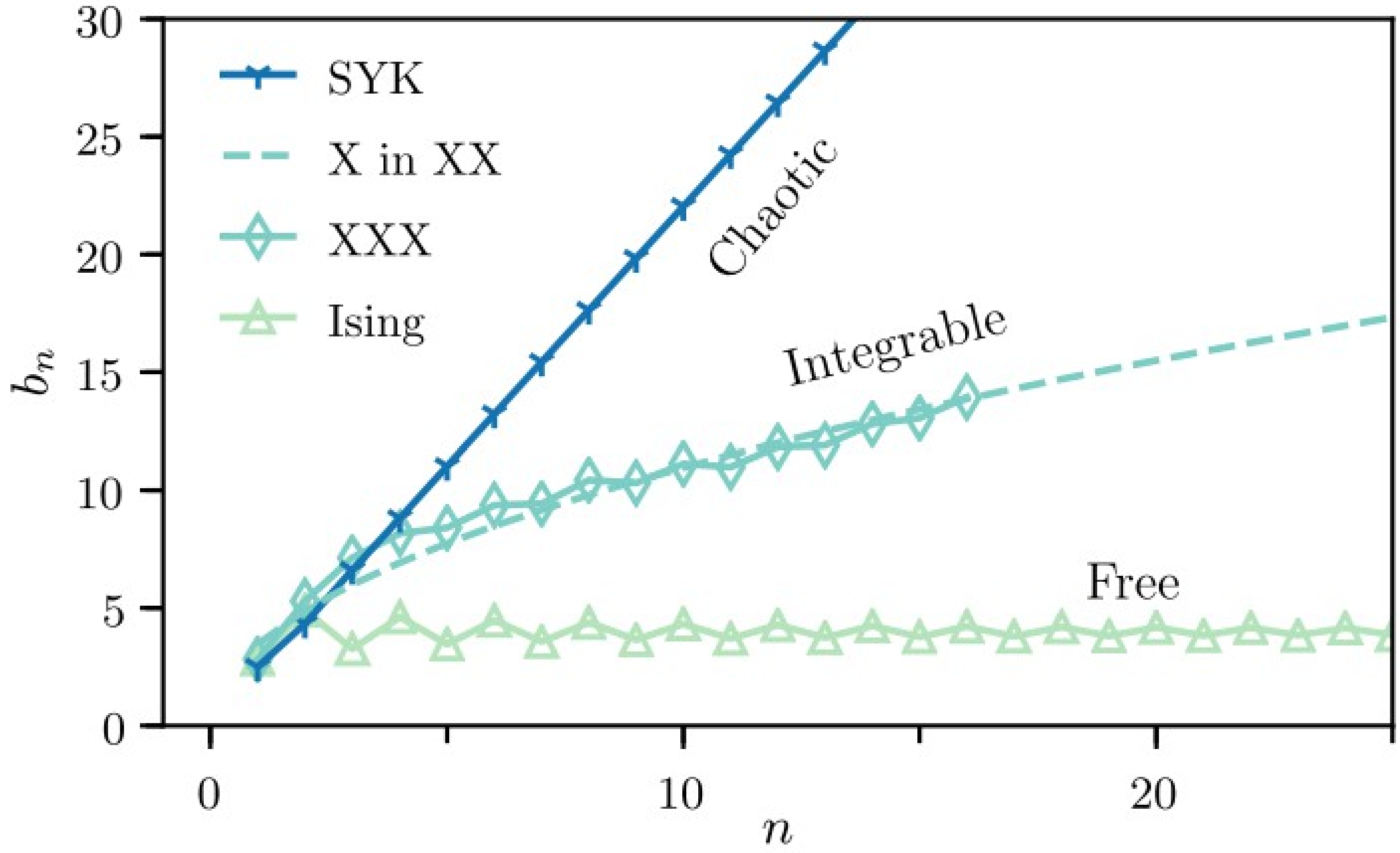
$$\mathcal{K}(t) = e^{\rho t}$$

Krylov complexity   measures the operator spreading



Parker et al 19

Krylov basis properties. Asymptotic behavior of b(n)



Krylov basis in matrix beta-ensemble

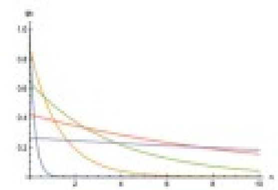
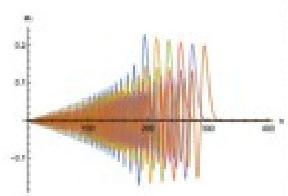
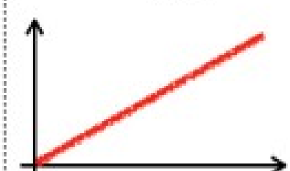
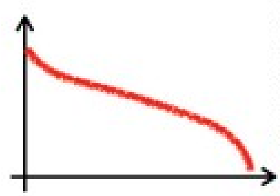
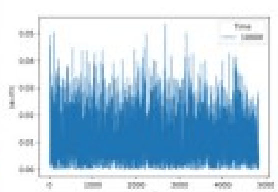
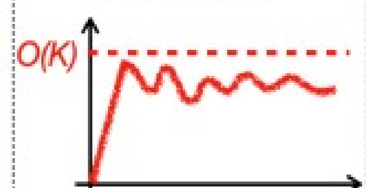
$$p(\lambda_1, \ldots, \lambda_n) = Z_{\beta,N} e^{-\frac{\beta N}{4}\sum_k \lambda_k^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta,$$

Distribution of the eigenvalues

$$H_\beta = \frac{1}{\sqrt{\beta N}} \begin{pmatrix} N(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & N(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix},$$

Balasubramian et al 2022, Dimitru-Edelman 2006

Krylov chains in systems with finite number degrees of freedom

n	Lanczos coefficients	wavefunction	K-complexity	time scales
$1 \ll n < S$	Linear growth in n $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in n $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

Rabinovici,Barbon,Sonner 21-22

Random walks on the Krylov chains

$$K \leq D^2 - D + 1\,,$$

D -dimension of the Hilbert space

K-length of the Krylov chain

The problem of evaluation of autocorrelator of the seed operator and correlators of two operators on the Krylov chain is provided by the evaluation of the transition amplitudes on the Krylov chain with inhomogeneous and in general random hopping

We shall be interested in the transition time of order

Of the length of the chain $T \sim cK$, $c = O(1)$

Nechaev,Valov A.G. 18

Growing Lanczos coefficients

Consider the Krylov chain with the finite UV-like cut-off. Example — finite Gaussian Matrix model. Consider the random walk on such chain at large Euclidean time-Heisenberg.time T~cK, K- length of the chain
Take use some analytic results from the probability theory

Random matrix in Krylov basis

Seems to be overlooked in 2d gravity

$$M_K \sim -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} W'(x)$$

W- Gaussian white noise. Stochastic Airy operator

Ramirez et al 2011

$$N^{1/6}(\lambda_{j,N} - 2\sqrt{N}))_{j=1}^{\infty} \to_{N\to\infty} (\lambda_1 > \lambda_2 \ldots)_{Airy}$$

Asymptotic behavior of tridiagonal matrix near the Spectral edge for any beta.

$$e^{-TM_k} \to \exp\left(-\frac{T}{2} SAO_\beta\right)$$

At large T the spectrum of the «Hamiltonian M» coincides with the spectrum of SAO

Exact result for Gaussian potential

Gorin,Sodin,Borodin 16-18

$$\lim_{N\to\infty} \frac{1}{2}\left(\left(\frac{M_N}{2\sqrt{N}}\right)^{\lceil TN^{2/3}\rceil} + \left(\frac{M_N}{2\sqrt{N}}\right)^{\lceil TN^{2/3}\rceil - 1}\right) = \exp\left(-\frac{T}{2}SAO_\beta\right)$$

$$Z(0,2t^3)e^{\frac{t^3}{12}} = \lim_{K\to\infty} K\left[\left(\frac{M_K}{2\sqrt{K}}\right)^{2[tK^{2/3}]} + \left(\frac{M_K}{2\sqrt{K}}\right)^{2[tK^{2/3}]+1}\right]_{1,1}$$

Return probability
On the Krylov chain
Autocorrelator!

Solution to KPZ equation with wedge initial condition

The KPZ scaling is quite universal for autocorrelators
at the Heisenberg time for systems with finite Hilbert space.
If t^3~K we have identification of Lanzcos time and KPZ time.

Generating functions

Gorin-Borodin 18'

$$E_{KPZ}\left[e^{-uZ(T,0)}e^{\frac{T}{24}}\right] = E_{Airy}\left[\prod_{k=1}^{\infty}\frac{1}{1+u\exp(Ca_k)}\right]$$

$$E_{KPZ}\left[e^{-uZ(T,0)}e^{\frac{T}{24}}\right] = 1+\sum_{k=1}^{\infty}\frac{(-1)^L}{L!}\int_0^{\infty}dx_1\cdots\int_0^{\infty}dx_L\,\det[K_u(x_i,x_j)]_{i,j=1}^{L}$$

where the kernel is as follows

$$K_u(x,x') = \int_{-\infty}^{+\infty}\frac{dy}{1+u^{-1}\exp((T/2)^{1/3}y)}\mathrm{Ai}(x-y)\mathrm{Ai}(x'-y)$$

These exact results are known only for beta=1,2 and Gaussian ensemble. Hence in generic case we have to use numerics for the random walks on the different Krylov chains with different initial and final points on the chain.

Growing Lanzcos's. Numerics

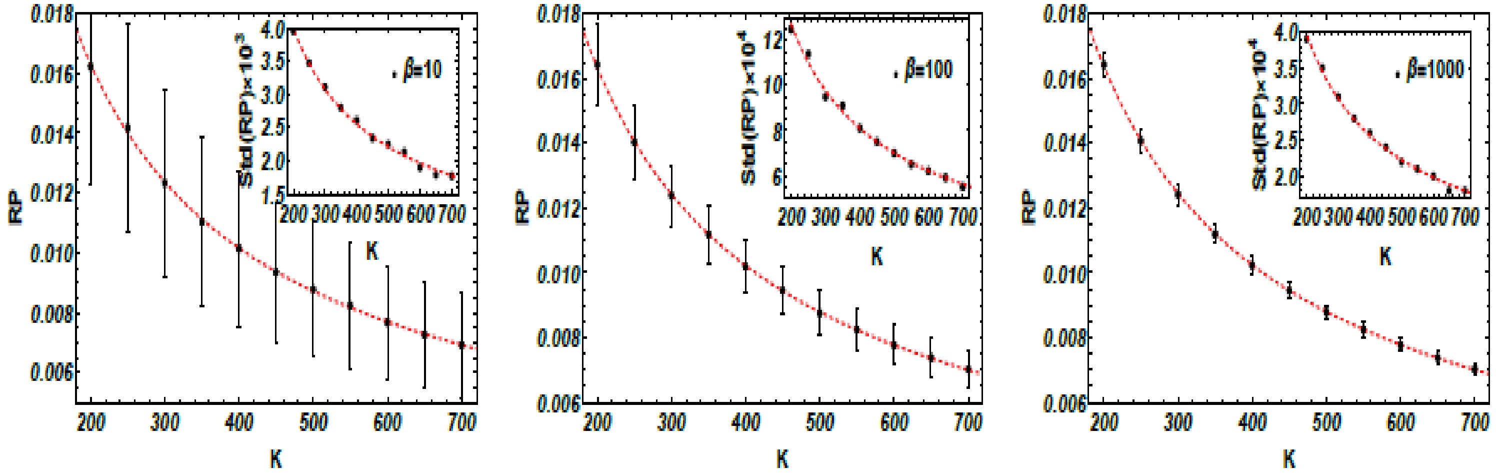


Figure 9. Scaling dependence of the return probability $RP = C(0, K)$ averaged over 1000 realizations of Krylov chains of length $K = 700$ for different β's: black dots and error bars represent mean and standard deviations of the return probabilities; insets show the scaling dependence of the standard deviations. All red dashed curves are proportional to $K^{-2/3}$

KPZ scaling is seen numerically
- both random and deterministic Lanzcos coefficients

--both integrable and chaotic b(n) asymptotics

Descending Lanczos b(n)

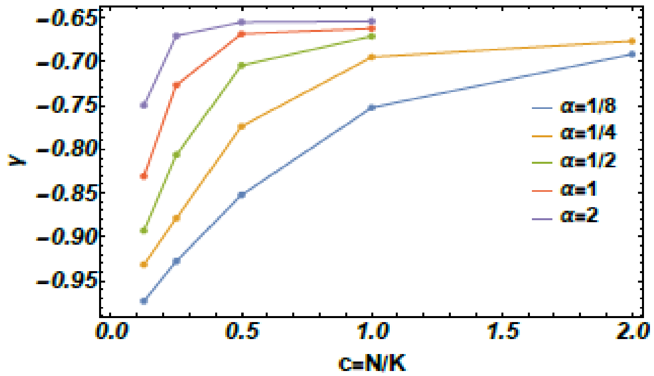


Figure 6. Scaling dependence of the return probability of N-step random walks on a Krylov chain, where the hopping parameters are the Lanczos coefficients $b_n = ((K-n)/K)^{\alpha}$.

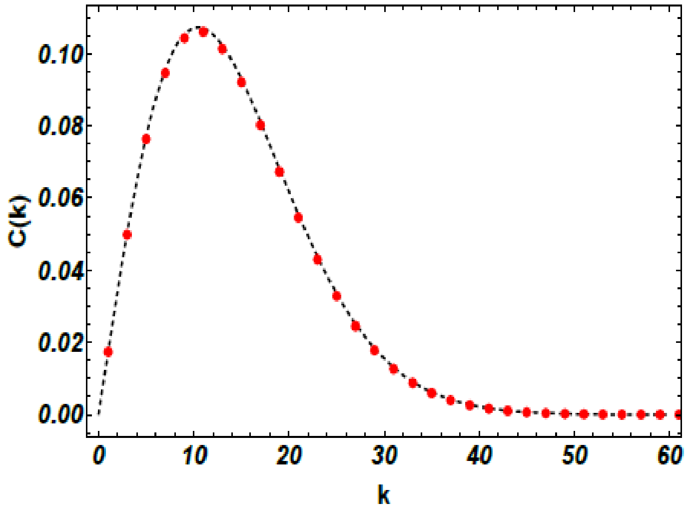


Figure 7. Two-point correlator $C(k,t)$ (red dots) and its approximation $2\alpha^{1/3}K^{-1/3}\mathrm{Ai}[(2\alpha)^{1/3}K^{-1/3}k + a_1]$ (black dashed line) for a Krylov chain, where the hopping parameters $b_k = ((K-k)/K)^{\alpha}$, $\alpha = 1/2$.

3-rd order phase
Transition at
Critical Euclidean time

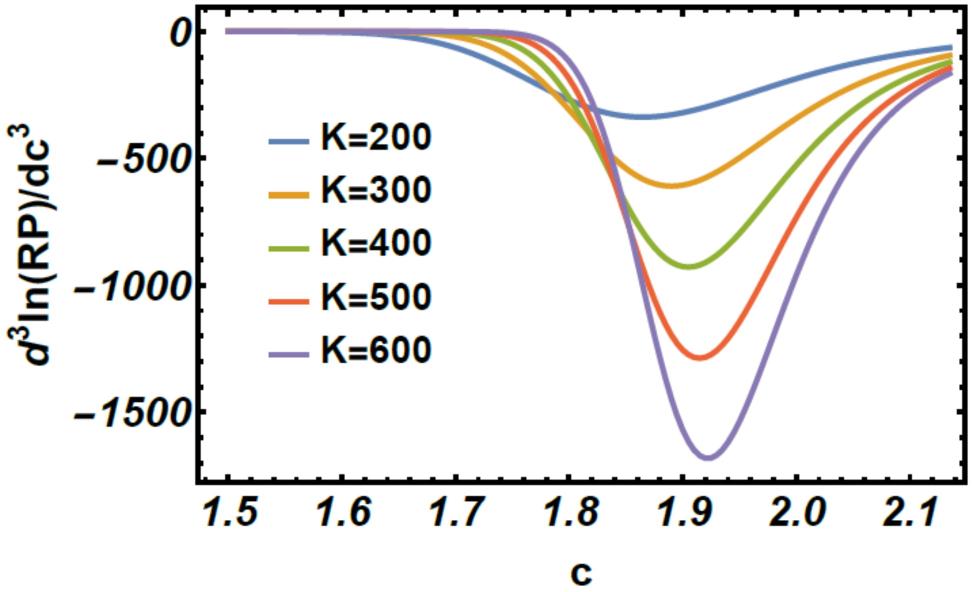


Figure 11. The third derivative of the logarithm of the return probability, $\frac{d^3 \ln C(0,cK)}{dc^3}$, on the Krylov chain of size K with growing Lanczos coefficients $b(n) = \sqrt{n}$.

DQPT in Euclidean
Time !

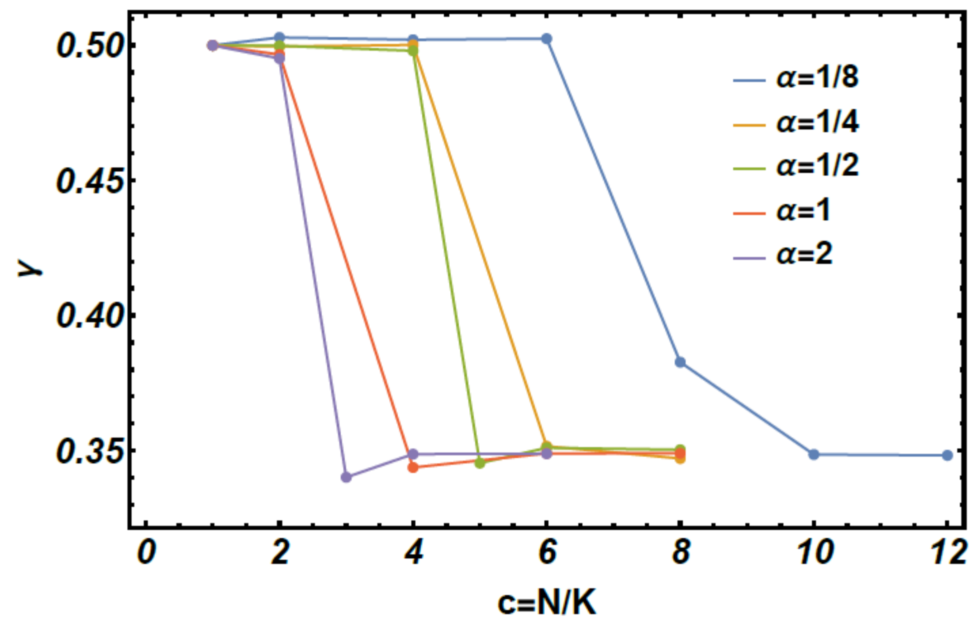


Figure 12. Scaling of fluctuations of the middle point of the N-step Brownian bridge on a Krylov chain of size K, where the hopping parameters are the Lanczos coefficients $b(n) = n^{\alpha}$.

Full Krylov chain for finite system

For finite system we have both regimes Barbon-Rabinovici-Sonner 21'

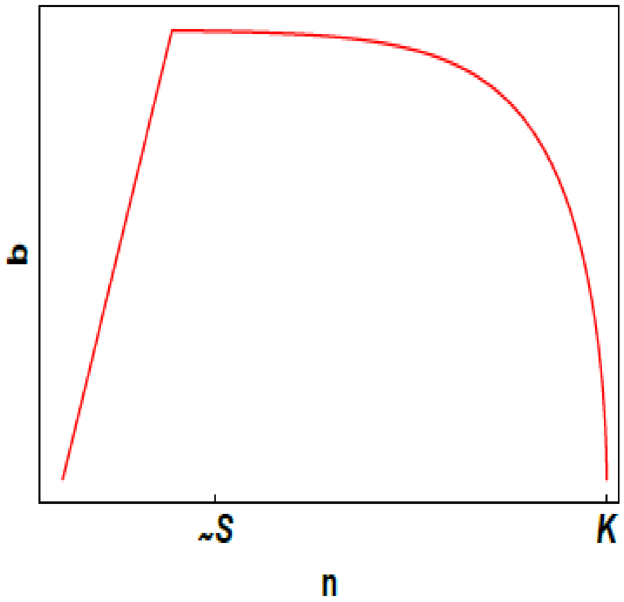


Figure 1. Sketch of the general behaviour of the Lancosz coefficients $b(n)$ for finite chaotic system with S degrees of freedom.

Full Krylov chain for finite system. Gauss-KPZ transition

- There is the Gauss-KPZ transition at the descending part at the Heisenberg time. For random and deterministic b(n), for integrable and chaotic asymptotics
- It is not the 3-rd order phase transition. Crossover
- Possible explanation of experimental data
- Is not useful for chaos diagnostics

2d quantum gravity. Double scaling

Standard double scaling (g-g_c)-> 0 (g-g_c)N= const-> sum over genera

Brezin-Kazakov,Gross-Migdal, Douglas-Shenker 90

N- matrix model with some potential which yields the discretization
Of the random surfaces with possible matter.

$$\mathcal{Z} = \int dM e^{V(M)}$$

Double scaling= spectral edge of M

$$M \to -\hbar^2 \frac{d^2}{dx^2} + u(x)$$

Near the spectral edge

$$u(x) = -x,$$

For gaussian potential =topological (2,1) gravity

The random nature of the matrix M is forgotten

Lesson for 2d quantum gravity. Double scaling.

$$M \to -\hbar^2 \frac{d^2}{dx^2} + x + \frac{1}{\sqrt{\beta}} B' = SAO_\beta$$

We have seen before that at the spectral edge of tridiagonal matrix the randomness survive

If the N dependence is restored — the random term is 1/N suppressed. Any relation with the recent findings?

– Take into account the discreteness of the spectrum even at the spectral edge C.Jonhson (19-22). It reveals the trace of the Trace-Widom distribution which enters the KPZ scaling

--- Assumption that the spectral edge is the point of the new phase transition. The matrix eigenvalue is the control parameter while the spectral density Which is non-analytic at the spectral edge is the order parameter. In this case the double scaling is reconsidered as well
(Sonner, Altland, Verbaarschot … 24)

If we keep the randomness in the double scaling limit what does It mean for the averaging problem in 2d/1d holography?Effect of wormholes?

Saad,Shenker,Stanford 19'

Hilbert space geometry

Sharipov,Tyutyakina,Gritsev,Polkovnikov, A.G. To appear

– Distinguish quantum chaos via response to a perturbation. The quantum tensor quantifies the response- it involves quantum metric(real part) and Berry curvature(imaginary part)

$$g^{(n)}_{\alpha\beta} = \sum_{m \neq n} \frac{\langle n|\partial_\alpha H|m\rangle \langle m|\partial_\beta H|n\rangle}{(E_n - E_m)^2}, \qquad G_{\alpha\beta} = \frac{1}{N}\sum_{n=1}^{N} g^{(n)}_{\alpha\beta},$$

$$ds^2 \equiv 1 - \left|\left\langle n(\vec{\lambda})\middle| n(\vec{\lambda} + d\vec{\lambda})\right\rangle\right|^2.$$

– How the induced geometry of parameter space tells about criticality/integrability/chaoticity?

Several conjectures supported by the examples

1. Quantum metric is singular at quantum critical point,Zanardi et al 07, Gritsev,Polkovnikov,Liska 13'

2. The system selects the geodesic motion in the quantum metric under adiabatic evolution - Sugiura,Clayes,Dymarsky,Polkovnikov 21'

3. Integrable points act as attractors in the parameter space Kim-Polkovnikov 23'

2-dimensional parameter space

Geometry of Interpolation between quantum integrability and quantum chaos with matrix Hamiltonians=finite-dimensional Hilbert space. Combination of analytical and numerical tools

1. Start with the chaotic system and perturb it in chaotic direction

$$H = H_0 + xH_x + yH_y, \qquad \rho(H_a) = e^{-\frac{1}{2}NTr(H_a^2)}, \quad a = 0, x, y.$$

Berry-Schukla (2020) evaluated the metric at the origin- fidelity susceptibility

2. Start with integrable model and perturb it in chaotic direction

$H = \Lambda_0 + xH_x + yH_y,$ where $\Lambda_0 = \text{diag}(\lambda_i)$ with $\rho(\lambda_i) = e^{-\frac{1}{2}\lambda_i^2}$, and H_x and H_y

are drawn from GUE

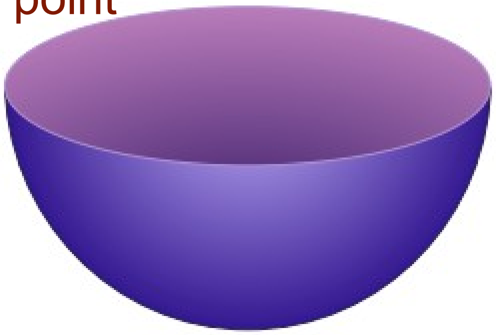
In family of Rozentzweig-Porter models

Independent diagonal matrix — according Berry-Tabor is integrable system.
No interaction between eigenvalues

Chaotic point

$$\left(\frac{dZ}{dr}\right)^2 + \left(\frac{dR}{dr}\right)^2 = \overline{G}_{rr}(r),$$

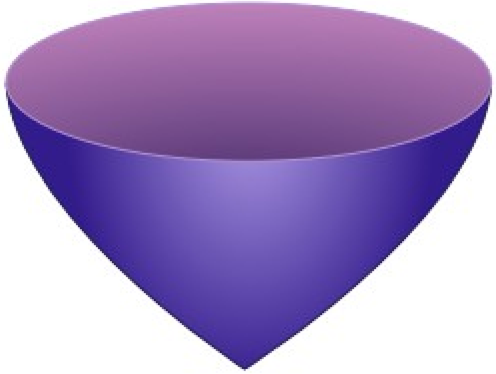
$$R(r)^2 = \overline{G}_{\phi\phi}(r).$$



Exact large N result

$$(Z - \sqrt{N-1})^2 + R^2 = N - 1,$$

Integrable point

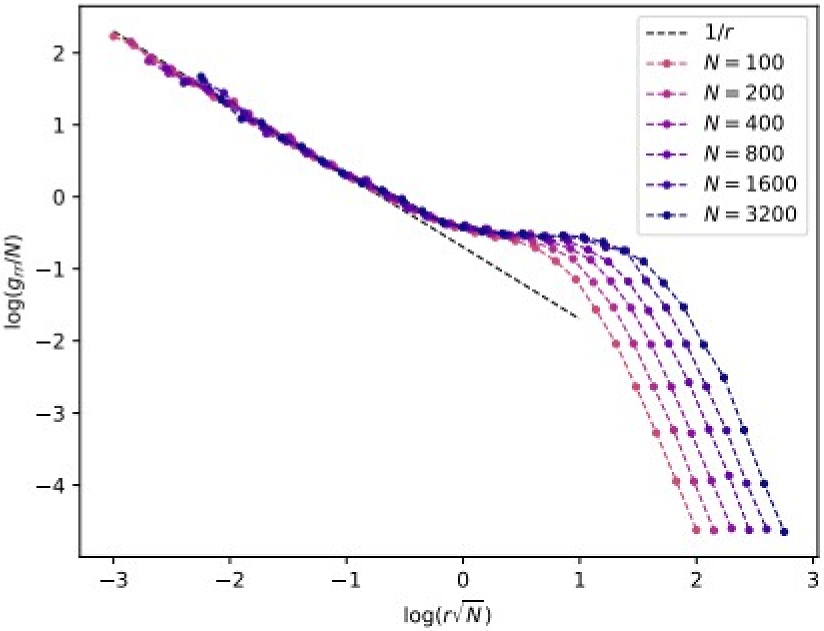


$$\overline{G}_{\phi\phi} = r \frac{1}{2\sqrt{2}} \arctan\left(\frac{\sqrt{2}}{r}\right)$$

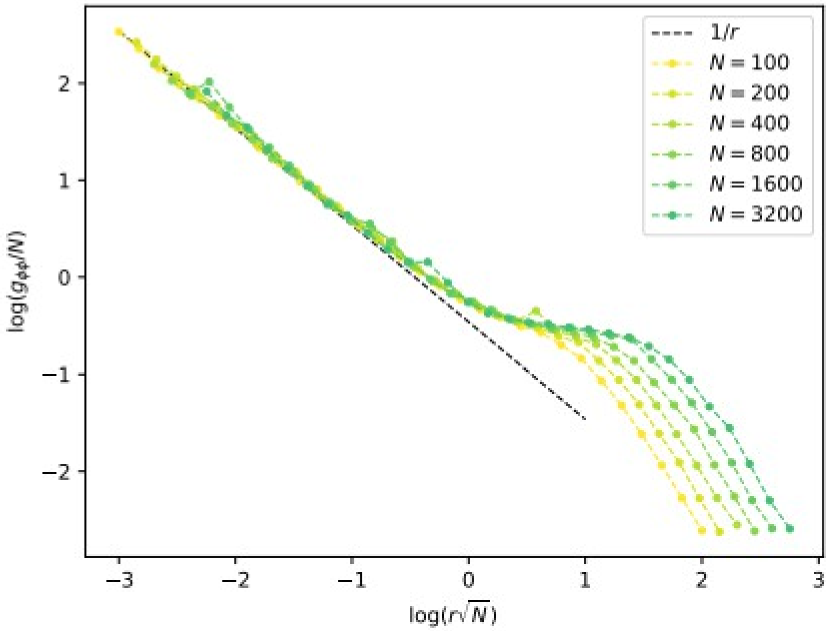
$$\overline{G}_{rr} = \frac{1}{4}\left(\frac{\operatorname{arccot}\left(\frac{r}{\sqrt{2}}\right)}{\sqrt{2}r} - \frac{1}{2+r^2}\right)$$

Exact result for metric in N=2 case

Integrable point at large N



(a) G_{rr} component of QGT as a function of rescaled parameter $r \to r\sqrt{N}$.



(b) G_{pp} component of QGT as a function of rescaled parameter $r \to r\sqrt{N}$.

3 regimes : integrable - KAM-like — chaotic

Universal behaviour 1/r near the integrable point. Precise example of «Integrability is attractive» scenario. r^{-2} behaviour in the chaotic regime

Conclusion

- KPZ scaling at Heisenberg time is universal but exists both for finite integrable and chaotic systems

  Gauss-KPZ crossover for generic systems with finite number degrees of freedom

  Surprizes for the 2d gravity. Double scaling refinement due to the chaotic term. Averaging in holography
- Induced geometry in 2-dimensional parameter space distinguishes the integrability from chaos

Thank you for attention!

Stop the war!