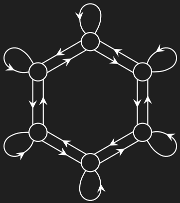
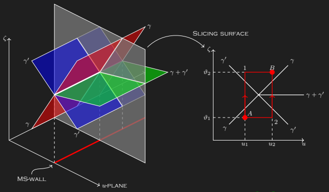
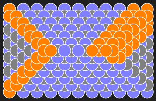
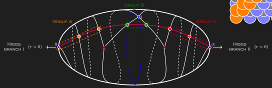








Supersymmetric polynomials and algebro-combinatorial duality

Dmitry Galakhov

Institute for Information Transmission Problems
NRC Kurchatov Institute (ITEP)

Fradkin Conference 2024
Moscow, September 2nd, 2024

Motivation

Based on a series of papers with A.Morozov and N.Tselousov:

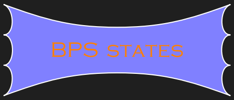
2307.03150, 2402.05920, 2407.03301, 2407.04810

Enum.Geometry

Categorification

New Algebras & Reps

Modularity



Non-perturbative phenomena

Wall-crossing $\longrightarrow$ phase transition

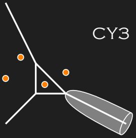
Integrability

Black hole physics

[Nakajima; Kontsevich, Soibelman; Alday, Gaiotto, Tachikawa; Douglass, Moore; Schifman, Vasserot, ...]

UV

Type II **D6/D4/D2/D0** $\longrightarrow$

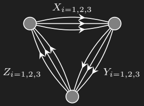


4d $\mathcal{N}=1$ SYM

$\downarrow$

1d $\mathcal{N}=4$ QM

IR



$+ \quad W = \epsilon^{ijk}\,\mathrm{Tr}\, Z_i Y_j X_k$

BPS Algebras

Multiplication:

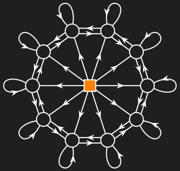
$$\mathbf{m}: \quad \mathcal{H}_\gamma \otimes \mathcal{H}_{\gamma'} \longrightarrow \mathcal{H}_{\gamma+\gamma'}$$

[Harvey-Moore •97] give a physical definition of a BPS algebra through a scattering process:

$$\mathcal{S}\left(\Psi_1 + \Psi_2 \to \mathcal{F}\right) \sim \frac{\langle \mathcal{F} | \mathbf{m}_{\mathrm{HM}}(\Psi_1 \otimes \Psi_2) \rangle}{s - |Z_{\gamma_1+\gamma_2}|^2}$$

[Kontsevich-Soibelman •11] give a mathematical definition of the BPS algebra in terms of equivariant cohomologies of a quiver.

Quiver BPS Algebras



Q_0 • quiver vertices

Q_1 • quiver arrows

Q_2 • superpotential

$a, b \in Q_0$
$|a| = (|a \to a| + 1) \bmod 2$

$I, J \in Q_1$

$$e^{(a)}(z) = \sum_{n \in \mathbb{Z}_{\geqslant 0}} \frac{e_n^{(a)}}{z^n}\,, \qquad f^{(a)}(z) = \sum_{n \in \mathbb{Z}_{\geqslant 0}} \frac{f_n^{(a)}}{z^n}\,, \qquad \psi^{(a)}(z) = \sum_{n \in \mathbb{Z}} \frac{\psi_n^{(a)}}{z^n}\,,$$

$h_I \in \mathbb{C}$ • equiv. weights, flavor charge

Bond factor: $\varphi^{a \Leftarrow b}(u) \equiv \dfrac{\prod_{I \in \{a \to b\}} (u + h_I)}{\prod_{J \in \{b \to a\}} (u - h_J)}$

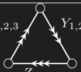
Rational quiver BPS algebra (quiver Yangian)

[Li-Yamazaki •20, Rapcak-Soibelman-Yang-Zhao •18]

$$\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a \Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a \Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\simeq -\delta^{a,b}\,\frac{\psi^{(a)}(z) - \psi^{(a)}(w)}{z-w}\,,
\end{aligned}$$

$\simeq$ • equivalent up to $z^n w^{m \geqslant 0}$, $z^{n \geqslant 0} w^m$

Quiver BPS Algebras II

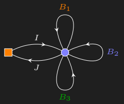
$\mathbb{C}^3$		 $X_{1,2,3}$ $W = \mathrm{Tr}\ X_1 [X_2, X_3]$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_1\right)$
Conifold		$X_{1,2}$  $Y_{1,2}$ $W = \mathrm{Tr}\ (Y_2X_2Y_1X_1 - Y_2X_1Y_1X_2)$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{1\vert 1}\right)$
$xy = z^n w^m$	...	...	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{n\vert m}\right)$
$K_{\mathbb{P}^2}$		$X_{1,2,3}$  $Y_{1,2,3}$ $Z_{1,2,3}$ $W = \mathrm{Tr}\ \epsilon^{ijk} Z_i Y_j Z_k$	$\mathsf{Y}(K_{\mathbb{P}^2})$???

Representations: MacMahon-like, Fock-like, vector-like and more

$\Big[$ $\mathsf{E}_\tau(Q, W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{T}_\beta(Q, W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{Y}(Q, W)$  $\Big] \times \left(\text{toric Calabi-Yau 3-fold}\right)$

D8-D6-D4-D2-D0	D7-D5-D3-D1	D6-D4-D2-D0
$\vartheta_{11}(z\vert\tau)$	$2\sinh(\beta z/2)$	z
3d $\mathcal{N} = 2$	2d $\mathcal{N} = (2,2)$	1d $\mathcal{N} = 4$
Ell. Cohomology	K-theory	Cohomology

(Molten) crystal vacua

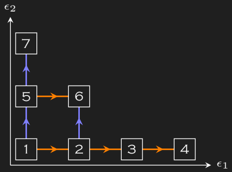


$$, W = \mathrm{Tr}\, B_3\left([B_1, B_2] + IJ\right)$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = r$$

$$[B_1, B_2] + IJ = 0$$

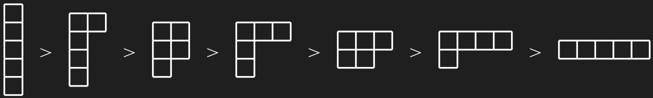
$$[\Phi, B_1] = \epsilon_1 B_1, \quad [\Phi, B_2] = \epsilon_2 B_2, \quad \Phi I = 0, \quad J\Phi = -(\epsilon_1 + \epsilon_2) J$$



$$, \quad I = \begin{pmatrix} \sqrt{7r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi = \mathrm{diag}\left(0, \epsilon_1, 2\epsilon_1, 3\epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2, 2\epsilon_2\right)$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{\frac{6r}{7}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{4r}{7}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{\frac{2r}{7}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3r}{7}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{r} & 0 & 0 \end{pmatrix}$$

Schur, Jack and Macdonald polynomials I



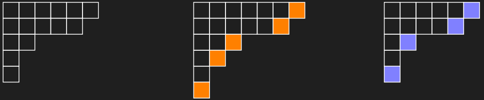
$$m_{\lambda}(x_1, x_2, \ldots, x_N) = \sum_{\alpha \in S_N \cdot \lambda} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N} \qquad (1)$$

1) Upper-triangular decomposition: $\mathsf{P}_{\lambda} = m_{\lambda} + \sum_{\mu < \lambda} K_{\lambda\mu} m_{\mu}$,

2) Orthogonality: $\langle \mathsf{P}_{\lambda}, \mathsf{P}_{\mu} \rangle = 0$ unless $\lambda = \mu$.

Name	P	Norm
Schur	S	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{S}} = \delta_{\lambda\mu} z_{\lambda}$,
Jack	J	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{J}} = \delta_{\lambda\mu} z_{\lambda} \beta^{\ell(\lambda)}$,
Macdonald	M	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{M}} = \delta_{\lambda\mu} z_{\lambda} \times \prod_{x \in \lambda} \frac{1 - q^{2x}}{1 - t^{2x}}$.

Schur, Jack and Macdonald polynomials II

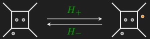


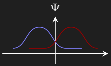
A) λ B) $\mathrm{Add}(\lambda)$ C) $\mathrm{Rem}(\lambda)$

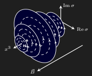
$$\omega_{\square} = x_{\square}\epsilon_1 + y_{\square}\epsilon_2$$

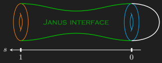
$$e_k|\lambda\rangle = \sum_{\square\in\mathrm{Add}(\lambda)} \mathbf{E}_{\lambda,\lambda+\square}\,\omega_{\square}^k|\lambda+\square\rangle, \quad f_k|\lambda\rangle = \sum_{\square\in\mathrm{Rem}(\lambda)} \mathbf{F}_{\lambda,\lambda-\square}\,\omega_{\square}^k|\lambda-\square\rangle, \quad \psi_{k+m} = [e_k, f_m]$$

$\mathbf{E}$, $\mathbf{F}$ (equiv. int.):









e_k, f_k, ψ_k form an algebra $Y_{\epsilon_1,\epsilon_2}(\widehat{\mathfrak{gl}}_1)$:

$$[\psi_k,\psi_m] = 0, \quad [\psi_{0,1}, e_k] = [\psi_{0,1}, f_k] = 0,$$

$$[\psi_2, e_k] = 2e_k, \quad [\psi_2, f_k] = -2f_k, \quad e_{k+1} = \frac{1}{6}[\psi_3, e_k] + \frac{\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)}{3}\psi_0 e_k, \quad \ldots$$

Heisenberg algebra:

$$p_k = \frac{1}{k!}\mathrm{Ad}_{e_1}^{k-1} e_0, \quad \frac{\partial}{\partial p_k} = \frac{1}{k!}\mathrm{Ad}_{f_1}^{k-1} f_0$$

ALGEBRO-COMBINATORIAL DUALITY

$$|\lambda\rangle = \mathsf{P}_\lambda(p_1, p_2, p_3, \ldots)|\varnothing\rangle$$

HAMILTONIANS = CARTAN OPS ψ_k IN $\mathscr{A}$

ALGEBRA

EIGEN FUNCTIONS
OF HAMILTONIANS
IN AN INTEGRABLE SYSTEM
$H_n \mathrm{Pol}_\lambda = E_\lambda^{(n)} \mathrm{Pol}_\lambda$

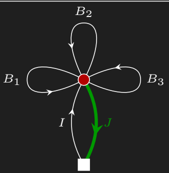
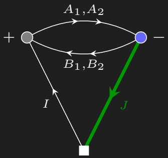
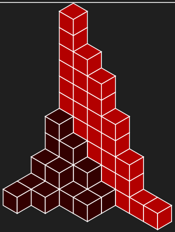
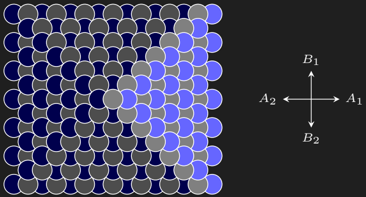
FOCK MODULE
OF AN ALGEBRA $\mathscr{A}$
$\mathscr{A}\big|_{\mathrm{Fock}} \supset$ HEISENBERG TIMES p_k
$\mathrm{Pol}_\lambda(p_k)|\varnothing\rangle = |\lambda\rangle$

COMBINATORICS

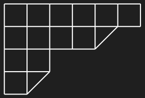
SELF-DUAL
KEROV FUNCTIONS
INVARIANT W.R.T.
PARTITION ORDERING SWITCH

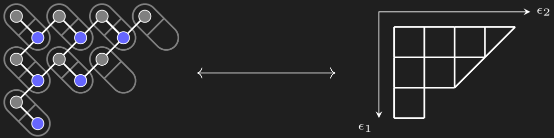
ORTHOGONALITY
+
UPPER-TRIANGULAR DECOMPOSITION
OVER BASIC UNIVERSAL FUNCTIONS

Fock and MacMahon modules

Algebra	$Y(\widehat{\mathfrak{gl}}_1)$	$Y(\widehat{\mathfrak{gl}}_{1\|1})$
4-cycle in CY3		
Quiver		
Superpotential	$W = \mathrm{Tr}\,(B_1[B_2, B_3] + B_3 I J)$	$W = \mathrm{Tr}\,(A_1B_1A_2B_2 - A_1B_2A_2B_1 + A_2 I J)$
Quiver variety	$\partial_{B_3} W = [B_1, B_2] + IJ = 0$	$\partial_{A_2} W = B_2A_1B_1 - B_1A_1B_2 + IJ = 0$
Crystal slice		

Super-partitions and super-Young diagrams

Super-partition : $\left\{4, \frac{7}{2}, 2, 2, \frac{3}{2}, 1\right\} =$ 



$= \{3, 2, \frac{3}{2}, \frac{1}{2}\}$

Generating function: $\sum_{\lambda} q^{2|\lambda|} = \prod_{k=1}^{\infty} \frac{1+q^{2k-1}}{1-q^{2k}}$

For example:

Lvl.1:

$\theta_{\frac{1}{2}}$

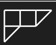
Lvl.2:

p_1

Lvl.3:	
	
$\theta_{\frac{3}{2}}$	$p_1\theta_{\frac{1}{2}}$

Lvl.4:		
		
$\theta_{\frac{3}{2}}\theta_{\frac{1}{2}}$	p_2	p_1^2

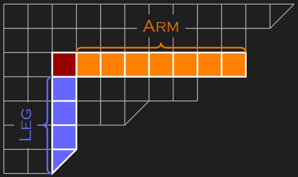
Lvl.5:			
			
$p_2\theta_{\frac{1}{2}}$	$\theta_{\frac{3}{2}}p_1$	$\theta_{\frac{5}{2}}$	$p_1^2\theta_{\frac{1}{2}}$

Lvl.6:				
				
$\theta_{\frac{5}{2}}\theta_{\frac{1}{2}}$	p_2p_1	$\theta_{\frac{3}{2}}p_1\theta_{\frac{1}{2}}$	p_3	p_1^3

Equivariant Euler classes

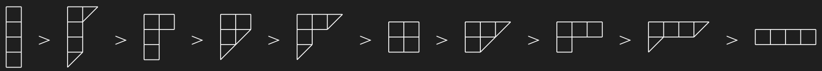
Euler class $\leftrightarrow$ partition function in QFT:

$$\mathrm{Eul}_{\lambda} = \left[\int_{U(\lambda)\in \text{INSTANTON MODULI SPACE}} \mathbb{1}\right]^{-1} = \prod_{\square\in\lambda} v_{\lambda}(\square) = \langle \mathsf{P}_{\lambda}, \mathsf{P}_{\lambda}\rangle$$



Leg parity	Arm parity	$v_{\lambda}(\square)$
EVEN	EVEN	$\left(-\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert+\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert-\epsilon_1\right)\cdot\left(\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert-\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert-\epsilon_2\right)$
ODD	EVEN	$\left(-\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert+\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert\right)\cdot\left(\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert-\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert-\epsilon_2\right)$
EVEN	ODD	$\left(-\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert+\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert-\epsilon_1\right)\cdot\left(\epsilon_1\lvert\mathbf{leg}_{\lambda}(\square)\rvert-\epsilon_2\lvert\mathbf{arm}_{\lambda}(\square)\rvert\right)$
ODD	ODD	1

Combinatorial super-Macdonald polynomials



1) Upper-triangular decomposition: $\mathsf{P}_{\lambda} = m_{\lambda} + \sum_{\mu < \lambda} K_{\lambda\mu}\, m_{\mu}\,,$

2) Orthogonality: $\langle \mathsf{P}_{\lambda}, \mathsf{P}_{\mu} \rangle = 0$ unless $\lambda = \mu\,.$

Name	P	Norm
super-Schur	S	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{S}} = \delta_{\lambda\mu} z_{\lambda}\,,$
super-Jack	J	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{J}} = \delta_{\lambda\mu} z_{\lambda} \beta^{\ell(\lambda^{+})}\,,$
super-Macdonald	M	$\langle P_{\lambda}, P_{\mu} \rangle_{\mathrm{M}} = \delta_{\lambda\mu} z_{\lambda} \times \prod_{x \in \lambda^{+}} \frac{1-q^{2x}}{1-t^{2x}} \times \prod_{y \in \lambda^{-}} q^{2y}\,.$

Two orderings:

For example

- Level 1/2:

$$\mathsf{M}_{\{1/2\}} = \theta_{1/2} .$$

- Level 1:

$$\mathsf{M}_{\{1\}} = p_1 .$$

- Level 3/2:

$$\mathsf{M}_{\{3/2\}} = \frac{q^2\left(t^2-1\right)}{q^2t^2-1} p_1 \theta_{1/2} + \frac{\left(q^2-1\right)}{q^2t^2-1} \theta_{3/2}, \quad \mathsf{M}_{\{1,1/2\}} = p_1\theta_{1/2} - \theta_{3/2} .$$

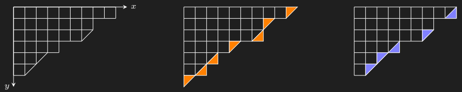
- Level 2:

$$\mathsf{M}_{\{2\}} = \frac{\left(q^2+1\right)\left(t^2-1\right)}{2q^2t^2-2} p_1^2 + \frac{\left(q^2-1\right)\left(t^2+1\right)}{2q^2t^2-2} p_2 ,$$

$$\mathsf{M}_{\{3/2,1/2\}} = \theta_{3/2}\theta_{1/2}, \quad \mathsf{M}_{\{1,1\}} = \frac{1}{2}p_1^2 - \frac{1}{2}p_2 .$$

[Alarie-Vezina, Blondeau-Fournier, Desrosiers, Lapointe, Mathieu, ...]

Algebraic (physical) super-Macdonald polynomials



(a) λ (b) $\mathrm{Add}(\lambda)$ (c) $\mathrm{Rem}(\lambda)$

$$\hat{e}_0^{+}\mathsf{M}_{\lambda} = \sum_{\triangledown\in\mathrm{Add}(\lambda)} \mathbf{E}_{\lambda,\lambda+\triangledown}\mathsf{M}_{\lambda+\triangledown}, \qquad \hat{e}_0^{-}\mathsf{M}_{\lambda} = \sum_{\triangle\in\mathrm{Add}(\lambda)} \mathbf{E}_{\lambda,\lambda+\triangle}\mathsf{M}_{\lambda+\triangle},$$

$$\hat{f}_0^{+}\mathsf{M}_{\lambda} = \sum_{\triangledown\in\mathrm{Rem}(\lambda)} \mathbf{F}_{\lambda,\lambda-\triangledown}\mathsf{M}_{\lambda-\triangledown}, \qquad \hat{f}_0^{-}\mathsf{M}_{\lambda} = \sum_{\triangle\in\mathrm{Rem}(\lambda)} \mathbf{F}_{\lambda,\lambda-\triangle}\mathsf{M}_{\lambda-\triangle}.$$

Super-Heisenberg algebra:

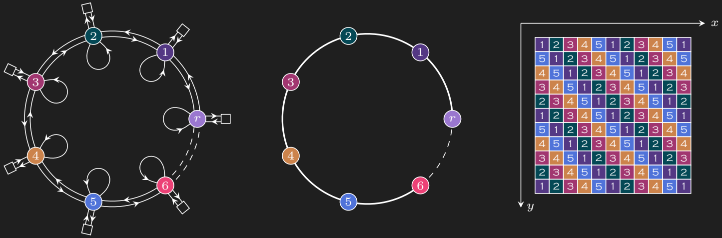
$$\theta_{1/2}\cdot = \hat{e}_0^{+}, \qquad p_1\cdot = \left\{\hat{e}_0^{+},\hat{e}_0^{-}\right\},$$
$$\frac{\partial}{\partial\theta_{1/2}} = \hat{f}_0^{+}, \qquad \frac{\partial}{\partial p_1} = \left\{\hat{f}_0^{+},\hat{f}_0^{-}\right\}.$$

$\mathsf{P}_{\lambda} = m_{\lambda} + \sum_{\mu<\lambda} K_{\lambda\mu}\, m_{\mu}$ - Kostka numbers are functions on super-Young tableaux:

$$,\qquad \begin{aligned}\lambda &= \{7/2, 3, 3/2, 1/2\}\,,\\ \mu &= \{5/2, 2, 2, 3/2, 1/2\}\,,\end{aligned}$$

$$\varnothing \subset \{5/2\} \subset \{3, 3/2\} \subset \{3, 5/2, 1\} \subset \{7/2, 3, 1, 1/2\} \subset \{7/2, 3, 3/2, 1/2\}\,,$$

Uglov polynomials



(A) (B) (C)

$\lambda \diagdown r$	1 (Jack)	2	3	4	5	…
(1)	p_1	p_1	p_1	p_1	p_1	
(2)	$\frac{p_1^2-p_2}{2}$	$\frac{p_1^2-p_2}{2}$	$\frac{p_1^2-p_2}{2}$	$\frac{p_1^2-p_2}{2}$	$\frac{p_1^2-p_2}{2}$	
$(1,1)$	$\frac{\beta p_1^2+p_2}{2}$	$\frac{p_1^2+\beta p_2}{2}$	$\frac{p_1^2+p_2}{2}$	$\frac{p_1^2+p_2}{2}$	$\frac{p_1^2+p_2}{2}$	
(3)	$\frac{p_1^3-3p_2p_1+2p_3}{6}$	$\frac{p_1^3-3p_2p_1+2p_3}{6}$	$\frac{p_1^3-3p_2p_1+2p_3}{6}$	$\frac{p_1^3-3p_2p_1+2p_3}{6}$	$\frac{p_1^3-3p_2p_1+2p_3}{6}$	
$(2,1)$	$\frac{\beta p_1^3-(\beta-1)p_2p_1-p_3}{2\beta+1}$	$\frac{p_1^3-p_3}{3}$	$\frac{p_1^3-p_3}{3}$	$\frac{p_1^3-p_3}{3}$	$\frac{p_1^3-p_3}{3}$	
$(1,1,1)$	$\frac{\beta^2 p_1^3+3\beta p_2p_1+2p_3}{(\beta+1)(\beta+2)}$	$\frac{p_1^3+3\beta p_2p_1+2p_3}{3(\beta+1)}$	$\frac{p_1^3+3p_2p_1+2\beta p_3}{2(\beta+2)}$	$\frac{p_1^3+3p_2p_1+2p_3}{6}$	$\frac{p_1^3+3p_2p_1+2p_3}{6}$	
…						

$$q_1 = e^{\epsilon_1 \hbar + \frac{2\pi i}{r}}, \quad q_2 = e^{\epsilon_2 \hbar - \frac{2\pi i}{r}}, \quad q_3 = e^{-\epsilon_1 \hbar - \epsilon_2 \hbar}, \quad \hbar \to 0$$

$$T_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \;\to\; Y_{\epsilon_1,\epsilon_2}(\widehat{\mathfrak{gl}}_r)$$

Open Problems

- Triangular = Stokes?
- When quiver Yangian (toroidal algebra) can be bosonized?
- Kostka amplitudes?
- $\mathsf{T}(\widehat{\mathfrak{gl}}_{1|1}) \to \mathsf{Y}(\widehat{\mathfrak{gl}}_{r|r})$ super-Uglov polynomials

Thank you for your attention