Unconstrained all-order higher-spin vertices

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Higher-spin problem

$$S_{HS} = \int \sum_{s} \phi_s \Box_F \phi_s + \sum_{s_1,s_2,s_3} a_{s_1 s_2 s_3} D..D\phi_{s_1} D..D\phi_{s_2} D..D\phi_{s_3} + O(\phi^4)$$

Quadratic action

Fronsdal, Fang and Fronsdal:

$$\delta\phi_{m_1..m_s} = D_{(m_1}\epsilon_{m_2..m_s)}\,, \qquad g^{mn}g^{kl}\phi_{mnkl...} = 0$$

Cubic action

Bengtsson[2], Brink; Berends, Burgers, van Dam; Metsaev; Fradkin, Vasiliev; many more

$$s_1 + s_2 + s_3 - 2\min(s_i) \leq N(D) \leq s_1 + s_2 + s_3$$

No go: Minkowski→ AdS→ higher-spin symmetry

Higher-spin holography

Klebanov, Polyakov; (Sezgin, Sundell; Leigh, Petkou)

$$S_{HS}\Big|_{z\to 0} = O(N)\text{-model}$$

3pt test (Giombi, Yin): OK

Maldacena, Zhiboedov:

$$\langle J_{s_1} J_{s_2} J_{s_3}\rangle = \sin^2\phi \langle J_{s_1} J_{s_2} J_{s_3}\rangle_b + \cos^2\phi \langle J_{s_1} J_{s_2} J_{s_3}\rangle_f + \frac{1}{2}\sin 2\phi \langle J_{s_1} J_{s_2} J_{s_3}\rangle_o$$

In the bulk (Tatarenko, Vasiliev)

Holographic reconstruction (Sleight, Taronna):

$$\langle J_{s_1} J_{s_2} J_{s_3}\rangle \to S^{(3)}_{HS}$$

Locality problem

Quartic reconstruction

Bekaert, Erdmenger, Ponomarev, Sleight:

$$\langle J_{s_1} J_{s_2} J_{s_3} J_{s_4} \rangle \to S^{(4)}_{HS} \quad \text{(nonlocal)}$$

Sleight, Taronna: A type of the nonlocality is

$$\frac{1}{\Box} \quad (?)$$

Gelfond: Direct calculation from Vasiliev equations

(moderate nonlocality)

Locality: state of the art

- HS theory in four dimensions

$$HS_4 = \text{hol} \oplus \text{ahol} \oplus \text{mixed}$$

  Hol: local up to quite higher orders (V.D., Gelfond, Korybut, Vasiliev)
- Hol: generating equations; all-order locality (V.D.)
- Mixed: Moderate nonlocality at quartic order (Gelfond)
- Bosonic HS theory in any dimension: generating equations; all-order locality at the unconstrained level (V.D., Korybut)

Vasiliev's unfolded approach

Stick to the frame-like formalism instead of the metric-like one:

$$\phi_{a_1..a_s} \quad \to \quad \begin{cases} \omega(Y|x) = dx^\mu \omega_\mu(Y|x) & \text{– potentials} \\ C(Y|x) & \text{– curvatures} \end{cases} ,$$

Y^J are the auxiliary generating variables

Nonlinear HS equations of motion:

$$d\omega = -\omega * \omega + \mathcal{V}(\omega, \omega, C) + ... + \mathcal{V}(\omega, \omega, C..C)\,,$$
$$dC = -\omega * C + C * \pi(\omega) + \Upsilon(\omega, C, C) + ... + \Upsilon(\omega, C..C)\,.$$

Consistency

$$d^2 = 0 \quad \to \quad \mathcal{V}(\omega, \omega, C..C)\,, \quad \Upsilon(\omega, C..C) \quad \text{(Vertices)}$$

$$\omega \to \omega + f_\omega(\omega, C..C)\,, \quad C \to C + f_C(C..C) \quad \text{(field redefinition)}$$

- $*$ – associative HS algebra (spanned by Y's)

$$f(Y) * g(Y) = f(Y) e^{i\overleftarrow{\partial}_Y \cdot \overrightarrow{\partial}_Y} g(Y)$$

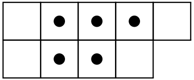
- Lower order vertices

$$\mathcal{V}(\Omega, \Omega, C) \quad \rightarrow \quad \text{Free equations} \qquad (\Omega\text{-vacuum})$$

$$\omega * \omega\,, \quad \omega * C - C * \pi(\omega)\,, \quad \mathcal{V}(\Omega, \omega, C)\,, \quad \Upsilon(\Omega, C, C) \quad \rightarrow \quad \text{Cubic int.}$$

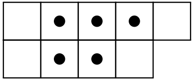
- d-dimensional off-shell bosonic HS algebra:

$$\omega^{a(s-1),\, b(n)} = dx^\mu \omega_\mu^{a(s-1),\, b(n)} =$$



$$, \qquad 0 \leq n \leq s-1$$

$$C^{a(m),\, b(s)} =$$



$$, \qquad m \geq s$$

$$Y^J = (y_\alpha, y_\alpha^a)\,, \quad \alpha = 1, 2\,, \quad a = 0..d-1$$

$$(f * g)(y, \vec{\mathbf{y}}) = \int f\,(y + u, \vec{\mathbf{y}} + \vec{\mathbf{u}})\, g\,(y + v, \vec{\mathbf{y}} + \vec{\mathbf{v}})\, e^{iuv + i\vec{\mathbf{u}}\vec{\mathbf{v}}}$$

Generating interactions (unconstrained)

Off-shell form $Y = (y, \mathbf{y})$

$$d\omega = -\omega * \omega + \mathcal{V}(\omega, \omega, C) + ... + \mathcal{V}(\omega, \omega, C..C)\,,$$
$$dC = -\omega * C + C * \pi(\omega) + \Upsilon(\omega, C, C) + ... + \Upsilon(\omega, C..C)\,.$$

$$\pi f(y, \vec{\mathbf{y}}) = f(-y, \vec{\mathbf{y}})$$

Vasiliev's trick:

$$\omega \to W(z; Y|x) := \omega(Y|x) + W_1(z; Y|x) + W_2(z; Y|x) + \dots\,,$$

$$d_x W + W * W = 0$$

star product is extended to (z, Y) – space

Generating system

V.D.'22; V.D., Korybut'23:

star product:

$$(f * g)(z; Y) = \int f\,(z + u', y + u; \mathbf{y}) \star g\,(z - v, y + v + v'; \mathbf{y})\, e^{iuv + iu'v'} \,,$$

Equations:

$$d_x W + W * W = 0\,,$$

$$d_z W + \{W, \Lambda\}_* + d_x \Lambda = 0\,,$$

$$d_x C + \big(W(z'; y, \vec{\mathbf{y}}) * C - C * W(z'; -y, \vec{\mathbf{y}})\big)\Big|_{z'=-y} = 0$$

$$\Lambda = dz^\alpha z_\alpha \int_0^1 d\tau\, \tau\, C(-\tau z, \vec{\mathbf{y}}) e^{i\tau z y}$$

Comrarison with Vasiliev

Vasiliev equations:

$$
\begin{aligned}
& d_x W + W * W = 0\,, \\
& d_x B + W * B - B * \pi(W) = 0\,, \\
& d_z W + \{W, \Lambda\}_* + d_x \Lambda = 0\,, \\
& d_z \Lambda + \Lambda * \Lambda = i\, dz^\alpha dz_\alpha\, B * \kappa\,, \qquad \kappa = e^{izy} \\
& d_z B + \Lambda * B - B * \pi(\Lambda) = 0\,.
\end{aligned}
$$

star product:

$$
(f * g)(z; Y) = \int f\,(z + u, y + u; \mathbf{y}) \star g\,(z - v, y + v; \mathbf{y})\, e^{iuv}\,,
$$

Perturbative series

Canonical embedding:

$$W=\omega(Y|x)+W^{(1)}(z,Y|x)+W^{(2)}(z,Y|x)+\dots\,,\quad W^{(n)}(0,Y|x)=0\,,\quad \forall n\geq 1$$

Vertices:

$$\mathcal{V}(\omega,\omega,C^n)=-\Big(\sum_{k+m=n}W^{(k)}*W^{(m)}\Big)\Big|_{z=0}$$

$$\Upsilon(\omega,C^n)=-\Big(W^{(n)}(z';y,\vec{\mathbf{y}})*C-C*W^{(n)}(z';-y,\vec{\mathbf{y}})\Big)\Big|_{z'=-y}$$

Source prescription (VD, Gelfond, Korybut, Vasiliev)

$$\omega(y,\vec{\mathbf{y}})=\exp\left(y_\alpha\frac{\partial}{\partial y^t_\alpha}\right)\omega(y^t_\alpha,\vec{\mathbf{y}})\Big|_{y^t_\alpha=0}=e^{-iy_\alpha t^\alpha}\omega(y^t_\alpha,\vec{\mathbf{y}})\Big|_{y^t_\alpha=0}\,,\quad t^\alpha:=i\frac{\partial}{\partial y^t_\alpha}$$

$$C(y,\vec{\mathbf{y}})=\exp\left(y_\alpha\frac{\partial}{\partial y^p_\alpha}\right)C(y^p_\alpha,\vec{\mathbf{y}})\Big|_{y^p_\alpha=0}=e^{-iy_\alpha p^\alpha}C(y^p_\alpha,\vec{\mathbf{y}})\Big|_{y^p_\alpha=0}\,,\quad p^\alpha:=i\frac{\partial}{\partial y^p_\alpha}$$

Master field W:

$$W^{(n)}(z;Y)=\sum_{k=0}^{n}\mathcal{W}^{(k|n)}(z;y|t;p_1..p_n)(\overbrace{C\dots C}^{k}\,\omega\,\overbrace{C\dots C}^{n-k})\Big|_{y^t,y^p=0}$$

$$(\overbrace{C\dots C}^{k}\,\omega\,\overbrace{C\dots C}^{n-k})=\dots\star C\Big(y^{p_k},\vec{\mathbf{y}}\Big)\star\omega\Big(y^t,\vec{\mathbf{y}}\Big)\star C\Big(y^{p_{k+1}},\vec{\mathbf{y}}\Big)\star\dots$$

Vertices

$$\Upsilon(\omega,C^n)=\sum_{k=0}^{n}\Phi_n^{[k]}(y|t;p_1..p_n)\,(\overbrace{C\dots C}^{k}\,\omega\,\overbrace{C\dots C}^{n-k})\Big|_{y^t,y^p=0}$$

$$\mathcal{V}(\omega,\omega,C^n)=\sum_{0\leqslant k_1\leqslant k_2\leqslant n}\Psi_n^{[k_1,k_2]}(y|t_1,t_2;p_1..p_n)\,(\overbrace{C\dots C}^{k_1}\,\omega\,\overbrace{C\dots C}^{k_2-k_1}\,\omega\,\overbrace{C\dots C}^{n-k_2})\Big|_{y^t,y^p=0}$$

W calculation

Iterative equations

$$d_z W^{(n+1)} + \{W^{(n)}, \Lambda\}_* + (d_x \Lambda)^{(n+1)} = 0\,, \quad W^{(0)} = \omega(Y|x)$$

Standard homotopy solution

$$W^{(n+1)} = -\Delta(\{W^{(n)}, \Lambda\}_*)\,, \quad \Delta\big(f_\alpha(z)\, dz^\alpha\big) := z^\alpha \int_0^1 d\tau f_\alpha(\tau z)$$

In terms of sources

$$\begin{cases} \mathcal{W}^{(k|n)} = \Delta\big[\mathcal{W}^{(k|n-1)} * \Lambda' - \Lambda' * \mathcal{W}^{(k-1|n-1)}\big],\ k \in [0, n] \\ \mathcal{W}^{(0|0)} = e^{-iyt} \end{cases}$$

$$\Lambda'(z; y|p) = dz^\alpha\, z_\alpha \int_0^1 d\sigma\, \sigma\, e^{i\sigma z(y+p)} \longrightarrow \Lambda(z; Y) = \Lambda'(z; y|p) C\Big(y^p, \vec{\mathbf{y}}\Big)\Big|_{y^p=0}$$

Vertex Υ

$$\Upsilon(\omega, C^n) \to \Phi_n^{[k]}(y|t; p_1, ..., p_n) = (-)^{k+1}(ty)^{n-1} \int_{\mathcal{D}_n^{[k]}} d\xi d\eta \, e^{-iyP_n(\xi)-itP_n(\eta)+i(ty)\cdot S_n^{[k]}}$$

$$P_n(\zeta) = \sum_{s=1}^{n} \zeta_s p_s \, , \quad S_n^{[k]} = -\sum_{s=1}^{k} \xi_s + \sum_{s=k+1}^{n} \xi_s + \sum_{i<j}^{n} (\xi_i \eta_j - \xi_j \eta_i)$$

$$\mathcal{D}_n^{[k]} = \begin{cases} \eta_1 + ... + \eta_n = 1 \, , \quad \eta_i \geq 0 \, , \\ \xi_1 + ... + \xi_n = 1 \, , \quad \xi_i \geq 0 \, , \\ \eta_i \xi_{i+1} - \eta_{i+1} \xi_i \leqslant 0 \, , \quad i \in [1 \, , \, k-1] \, , \\ \eta_i \xi_{i+1} - \eta_{i+1} \xi_i \geqslant 0 \, , \quad i \in [k+1 \, , \, n-1] \, . \end{cases}$$

Vertex Υ

$$\Upsilon(\omega, C^n) = (-i)^{n-1} \sum_{k=0}^{n} (-)^{k+1} \int_{\mathcal{D}_n^{[k]}} d\xi d\eta \int \frac{d^2u d^2v}{(2\pi)^2} e^{iuv} \times$$

$$\times \left(\prod_{i=1}^{k} \star\, C(\xi_i y + \eta_i v, \vec{\mathbf{y}}) \right) \star (y^\alpha \partial_\alpha)^{n-1} \omega(S_n^{[k]} y + u, \vec{\mathbf{y}}) \star \left(\prod_{j=k+1}^{n} \star\, C(\xi_j y + \eta_j v, \vec{\mathbf{y}}) \right)$$

Geometric interpretation

New variables

$$\mathrm{x}_0 = 0\,, \quad \mathrm{x}_i = \sum_{s=1}^{i} \xi_s\,,$$

$$\mathrm{y}_0 = 0, \quad \mathrm{y}_i = \sum_{s=1}^{i} \eta_s$$

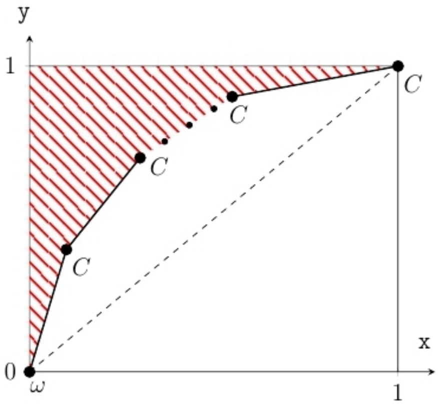
$$\mathrm{x}_i \geq \mathrm{x}_j\,, \quad \mathrm{y}_i \geq \mathrm{y}_j\,, \quad i > j$$

$$\mathrm{x}_i \leq 1\,, \quad \mathrm{y}_i \leq 1\,, \quad \mathrm{x}_n = \mathrm{y}_n = 1$$

Geometric interpretation

ω-left case:

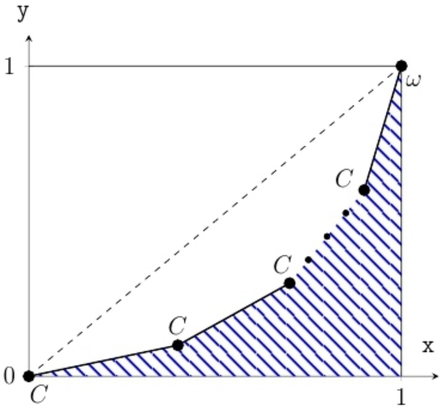
$$\mathcal{D}_n^{[0]}: \qquad \eta_i\xi_{i+1} - \eta_{i+1}\xi_i \geq 0\,, \quad i = 1, \ldots, n-1; \quad \frac{1}{2} S_n^{[0]} = \text{Area}$$



Geometric interpretation

ω-right case:

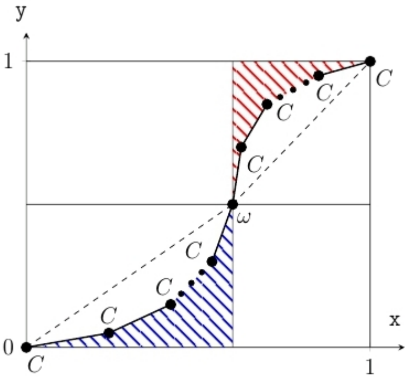
$$\mathcal{D}_n^{[n]}: \qquad \eta_i \xi_{i+1} - \eta_{i+1}\xi_i \leq 0\,, \quad i = 1, \ldots, n-1; \quad \frac{1}{2} S_n^{[n]} = -\text{ Area}$$



Geometric interpretation

ω in the k-th position:

$$\mathcal{D}_n^{[k]} = \begin{cases} \eta_i\xi_{i+1} - \eta_{i+1}\xi_i \leqslant 0\,, & i \in [1\,,\,k-1]\,, \\ \eta_i\xi_{i+1} - \eta_{i+1}\xi_i \geqslant 0\,, & i \in [k+1\,,\,n-1]\,. \end{cases} \qquad \frac{1}{2}S_n^k = \text{Area} - \text{Area}$$



Vertex dualities

Key property of the vertex Υ:

$$\Upsilon(\omega, C^n)\Big|_{y=0} = 0$$

Integrability constraint $d^2 = 0$ mixes vertices Υ and $\mathcal{V}$

$$\begin{aligned}
\mathcal{V}_{\omega\omega C} * C - \omega * \Upsilon_{\omega CC} &\overset{y=0}{=} 0\,, & \mathcal{V}_{\omega C\omega} * C - \omega * \Upsilon_{C\omega C} &\overset{y=0}{=} 0\,,\\
\mathcal{V}_{C\omega\omega} * C - C * \pi(\mathcal{V}_{\omega\omega C}) &\overset{y=0}{=} 0\,, & -\omega * \Upsilon_{CC\omega} - \Upsilon_{\omega CC} * \pi(\omega) &\overset{y=0}{=} 0\,,\\
-\Upsilon_{C\omega C} * \pi(\omega) - C * \pi(\mathcal{V}_{\omega C\omega}) &\overset{y=0}{=} 0\,, & -\Upsilon_{CC\omega} * \pi(\omega) - C * \pi(\mathcal{V}_{C\omega\omega}) &\overset{y=0}{=} 0\,.
\end{aligned}$$

In terms of sources:

$$\Psi_1^{[0,0]}(-p_2|t_1, t_2; p_1) = \Phi_2^{[0]}(t_1|t_2; p_1, p_2) \quad \rightarrow \quad \Psi_1^{[0,0]}(y|t_1, t_2; p) = \Phi_2^{[0]}(t_1|t_2; p, -y)$$

Vertex dualities

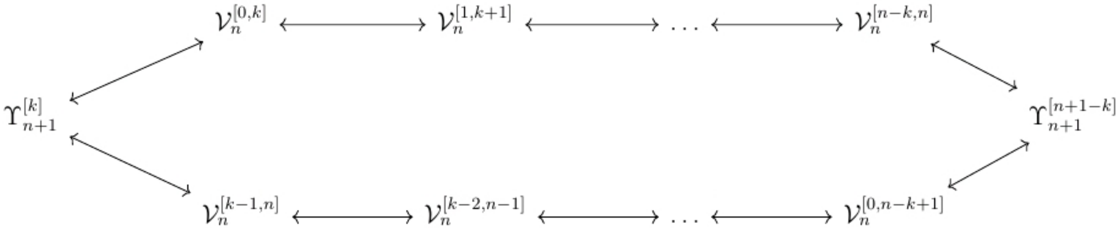
Lower-order example:

$$
\begin{array}{ccc}
\mathcal{V}_{\omega\omega C} & \longleftrightarrow & \mathcal{V}_{C\omega\omega} \\
\updownarrow & & \updownarrow \\
\Upsilon_{\omega CC} & \longleftrightarrow & \Upsilon_{CC\omega}
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \mathcal{V}_{\omega C\omega} & & \\
 & \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow & \\
\Upsilon_{C\omega C} & & & & \Upsilon_{C\omega C} \\
 & \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow & \\
 & & \mathcal{V}_{\omega C\omega} & &
\end{array}
$$

Vertex dualities

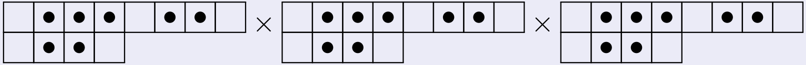
General case:



Interaction structure

Locality

Vertices Υ are spin-local:

$\Upsilon(\omega, C^3) \sim$ 

Finite number of contractions for fixed legs s_i

Vertices $\mathcal{V}$ are spin ultra-local:

$$\mathcal{V}(\omega, \omega, C^n(0, \vec{\mathbf{y}}))$$

Proliferated form and projectively-compact spin locality

$$d_x C_s^k = \sum_{i=0}^{k} \sum_{\vec{n},\vec{s}} \Upsilon_s^i(\omega_{s'}^{n'}, \overbrace{C_{s_0}^{n_0} \dots C_{s_i}^{n_i}}^{i+1}) , \quad C_s := C_s^0 \to C_s^1 \to \dots \to C_s^\infty = C_s^k , \quad k \geq 0$$

Conclusion

- All 'off-shell' bosonic HS vertices are manifestly found
- Vertices appear to be space-time spin local and minimal
- Condition $\Upsilon(\omega, C^n)\Big|_{y=0} = 0$ results in a net of vertex duality relations
- Precise form of the HS vertices reveals an interesting geometric structure: the integration phase space contains convex/concave polygons generalizing the Moyal star product geometric representation
- All-order exact unfolded equation for the Weyl module descendent C_s^k is a polynomial in C

The End