Manifestly supersymmetric effective actions in gauge theories with extended supersymmetry

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Goal

- Brief review of some methods and results on study of quantum structure supersymmetric gauge theories.
- Brief discussion of some aspects of $4D, \mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric theories but the main focus will be on recent results on $6D, \mathcal{N} = (1,0)$ and $6D, \mathcal{N} = (1,1)$ supersymmetric gauge theories.
- Transfer to superfield theories of the techniques developed mainly in quantum gravity: the background field method and the proper time method.

Mainly based on a series of papers published during a few last years in collaboration with E.A. Ivanov, B.M. Merzlikin and K.V. Stepanayatz.

- Supersymmetry is an extension of special relativity symmetry. In its essence, the supersymmetry is a special relativity symmetry extended by the symmetry between bosons and fermions
- From mathematical point of view the relativistic symmetry is expressed in terms of Poincare group with the generators P_m and J_{mn} satisfying the known commutation relations

$$\begin{aligned} [P_r, P_s] &= 0, \\ [J_{rs}, P_m] &= i(\eta_{rm} P_s - \eta_{sm} P_r), \\ [J_{mn}, J_{rs}] &= i(\eta_{mr} J_{ns} - \eta_{ms} J_{nr} + \eta_{ns} J_{mr} - \eta_{nr} J_{ms}) \end{aligned} \quad (1)$$

η_{mn} is the Minkowski metric.

- Extension of special relativity in four dimensions means extension of the Poincare algebra by the generators (supercharges) ${Q^i}_\alpha$, $\bar{Q}_{i\dot{\alpha}}$, $i = 1, 2, ..., \mathcal{N}$
- The relations among the supercharges (Poincare superalgebra) are given in terms of anticommutators

$$\begin{aligned} \{{Q^i}_\alpha, {Q^j}_\beta\} &= \epsilon_{\alpha\beta} Z^{ij} \\ \{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij} \\ \{{Q^i}_\alpha, \bar{Q}_{j\dot{\alpha}}\} &= 2{\delta^i}_j {\sigma^m}_{\alpha\dot{\alpha}} P_m \end{aligned} \quad (2)$$

Z^{ij}, $\bar{Z}_{ij}$ are the central charges; $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ are the invariant tensors of the $SL(2|C)$ group, $\sigma_m = (\sigma_0, \sigma_i)$.

- Supersymmetric field model means a field model invariant under the above superalgebra. Since the supercharges are the spin $s = \frac{1}{2}$ Lorentz group spinors one can expect that any supersymmetric field model must contain both bosonic and fermionic fields

Superspace and superfields

- Minkowski space coordinates x^m have the same tensor structure as the generators of space-time translations P^m. Analogously ones introduce the additional spinor coordinates $\theta^{i\alpha}$ and $\bar{\theta}^{i\dot{\alpha}}$ associated with the supercharges $Q^i{}_\alpha$ and $\bar{Q}_{i\dot{\alpha}}$. The additional coordinates have the fermionic structure as well as the supercharges and anticommute among themselves.
- Manifold parameterized by the commuting (bosonic) coordinates x^m and the anticommuting (fermionic) coordinates $\theta^{i\alpha}$, $\bar{\theta}^{i\dot{\alpha}}$ is called (conventional or general or standard) superspace
- Function defined on superspace is called superfield
- Since the fermionic coordinates are anticommuting, any superfield is no more then polynomial in these coordinates. The coefficients of such a polynomial are the conventionsl bosonic and fermionic fields on Minkowski space. All these coefficients are called the component fields of the superfield.
- Consider the supersymmetry transformations (supertranslations): $\theta^{i\alpha} \to \theta^{i\alpha} + \epsilon^{i\alpha}$, $\bar{\theta}^{i\dot{\alpha}} \to \bar{\theta}^{i\dot{\alpha}} + \bar{\epsilon}^{i\dot{\alpha}}$, $x^m \to x^m + \delta x^m$. The $\epsilon^{i\alpha}$ and $\bar{\epsilon}^{i\dot{\alpha}}$ are the anticommuting transformation parameters, δx^m are expressed in special form through the fermionic coordinates and the anticommuting parameters. The supertranslations define the supersymmetry transformations of the component fields.

Example: $\mathcal{N}=1$ supersymmetric field models

- $\mathcal{N}=1$ superspace: coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.
- Chiral scalar superfield $\Phi(x,\theta,\bar{\theta}) = e^{i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta)$. Component content $\Phi(x,\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + F(x)\theta^2$
- Antichiral scalar superfield $\bar{\Phi}(x,\theta,\bar{\theta}) = e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\bar{\Phi}(x,\bar{\theta})$. Component content $\bar{A}(x,\bar{\theta}) = \bar{A}(x) + \bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x) + \bar{F}(x)\bar{\theta}^2$
- Superfield model (Wess-Zumino model)

$$S[\Phi,\bar{\Phi}] = \int d^4x d^2\theta d^2\bar{\theta}\bar{\Phi}(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta}) + (\int d^4x d^2\theta V(\Phi) + c.c.)$$

$V(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3$. Manifest supersymmetry.

- Component form of Wess-Zumino model

$$S = \int d^4x(-\partial^m\bar{A}\partial_m A - \frac{i}{2}\psi^\alpha{\sigma^m}_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} + \bar{F}F + F(mA + \frac{\lambda}{2}A^2) +$$

$$+\bar{F}(m\bar{A} + \frac{\lambda}{2}\bar{A}^2) - \frac{1}{4}(m+\lambda A)\psi^\alpha\psi_\alpha - \frac{1}{4}(m+\lambda\bar{A})\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}})$$

Non-manifest supersymmetry.

- Real scalar superfield $V(x,\theta,\bar\theta)$. Component content $V(x,\theta,\bar\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) + \theta^2 F(x) + \bar\theta^2\bar F(x) + (\theta\sigma^m\bar\theta)A_m(x) + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \theta^2\bar\theta^2 D(x)$
- Superfield model ($\mathcal{N}=1$ supersymmetric Yang-Mills theory)

$$S_{SYM}[V] = \frac{1}{2g^2}\int d^4x d^2\theta tr(W^\alpha W_\alpha)$$

Superfield V takes the values in Lie algebra of gauge group, $W_\alpha = -\frac{1}{8}\bar D^2(e^{-2V}D_\alpha e^{2V})$, $D^2 = D^\alpha D_\alpha$, $D_\alpha = \partial_\alpha + i(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m$.
Manifest supersymmetry.

- Component form of supersymmetric Yang-Mills theory

$$S_{SYM} = \frac{1}{g^2}\int d^4x tr(-\frac{1}{4}G^{mn}G_{mn} - i\lambda^\alpha\sigma^m_{\alpha\dot\alpha}\nabla_m\bar\lambda^{\dot\alpha} + 2D^2)$$

Non-manifest supersymmetry.

Advantages of superfield formulation

- The SUSY models can be formulated in terms of bosonic and fermionic component fields. Supersymmetry is hidden. It is not very convenient in quantum field theory: supersymmetry algebra is open, very many Feynman diagrams, miraculous cancelations.
- Superfield formulation: manifest SUSY, comparatively small number of supergraphs, origin of miraculous cancelations. Problem: how to formulate the of $\mathcal{N}$-extended SUSY models in terms of unconstrained $\mathcal{N}$-extended superfields. General solution for arbitrary $\mathcal{N}$ is unknown.
- Why we want to get a formulation in terms of unconstrained superfields: **(a)**. In conventional field theory the fields must be the functionally independent arguments of action. Otherwise there is no Lagrangian formulation. **(b)**. Conventional quantum field theory is constructed in terms of unconstrained fields. Manifest supersymmetry, provided by superfield formulation, allows us to control the calculations. Supersymmetry algebra is automatically closed. **(c)**. Superfield formulation provides the simple enough ways to construct the various superinvariants that allows to describe the possible structure of contributions to effective action or to S-matrix.

Power of superfield formulation for $\mathcal{N}=1$ supersymmetric theories-non-renormalization theorem: any L-loop contribution to effective action is written in the form of single integral over $d^4\theta$:

$$\int d^4p_1 \dots d^4p_L d^4\theta \mathcal{F}(p_1, \dots, p_L, \theta).$$

This relation completely determines the possible counterterms in the $\mathcal{N}=1$ supergauge theories up to the coefficients. Complete explanations of miraculous cancelations of possible divergences in component approach.

Problem of $\mathcal{N}=2$ superfield formulation

Superfield Lagrangian formulation of $\mathcal{N}=2$ supersymmetric theories faces the fundamental problems in comparison with $\mathcal{N}=1$ case.

- The simplest $\mathcal{N}=2$ multiplets are the hypermultiplet and vector multiplet.
- On shell the hypermultiplet contains four scalar fields and two spinor fields.
- All these fields are the components of the superfield $q^i(z)$, where $z=(x^m,\theta^i{}_\alpha,\bar{\theta}^{i\dot{\alpha}})$ under constraints

$$D^{(i}_{\alpha}q^{j)}=0,\ \bar{D}^{(i}_{\dot{\alpha}}q^{j)}=0.$$

  The above constraints put all the component fields on free mass shell. The superfield Lagrangian formulation is impossible.
- On shell the vector multiplet contains the vector field, a doublet of spinor fields, and a complex scalar field. All these fields can be include to some superfield satisfying constraints and the off shell Lagrangian formulation in terms of this superfield faces the difficulties.
- Reason of difficulties is that there are too many anticommuting coordinates in $\mathcal{N}=2$ superfields.

Solution has been given in the pioneer works by A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev where the new $\mathcal{N}=2$ superspace has been introduced.

Solution was quite paradoxical. Instead of finding a way to directly reduce the number of fermion coordinates, it was proposed to increase the number of bosonic coordinates. However, after this, the desired reduction in the number of fermion coordinates became possible.

- The conventional $\mathcal{N}=2$ superspace with coordinates $z^M=(x^m,\theta^i_\alpha,\bar\theta^i_{\dot\alpha})$ is extended by the commuting (harmonic) variables $u^\pm_i\ (i=1,2)$ parameterizing two-sphere. $\mathcal{N}=2$ harmonic superspace with coordinates (z,u).
- Supercovariant derivatives D^i_α, $\bar D^i_{\dot\alpha}$ are decomposed into $D^\pm_\alpha=u^\pm_i D^i_\alpha$, $\bar D^\pm_{\dot\alpha}={u_i}^\pm\bar D^i_{\dot\alpha}$, $(D^i_\alpha=\frac{\partial}{\partial\theta^\alpha_i}+...)$.
- The harmonic derivatives are introduces:

$$D^{\pm\pm}=u^{\pm i}\frac{\partial}{\partial u^{\mp i}},\ D^0=u^{+i}\frac{\partial}{\partial u^{+i}}-u^{-i}\frac{\partial}{\partial u^{i}}$$

- Superfields in harmonic superspace are characterized by $U(1)$ charge, $\Phi^{(q)}(z,u)$, $D^0\Phi^{(q)}=q\Phi^{(q)}$.

- Impose the supersymmetric condition of the superfields

$$D^+_\alpha \Phi^{(q)} = 0,\ \ \bar{D}^+_{\dot\alpha} \Phi^{(q)} = 0.$$

  Such superfields depend on coordinates $(x^m_A, \theta^{+\alpha}, \bar\theta^{+\dot\alpha}, u) = (\zeta_A, u)$. Analytic superfields. The number of fermionic coordinates has decreased to their number in $\mathcal{N}=1$ case. The price for this is an increase of bosonic coordinates.
- Construction of off shell $\mathcal{N}=2$ supersymmetric models in terms of analytic superfields.

  Hypermultiplet theory in terms of unconstrained analytic superfield q^+ with action $S_q[q^+]$ and super Yang-Mills theory in terms of unconstrained analytic superfield V^{++}, taking the values in the Lie algebra of gauge group, with $S_{SYM}[V^{++}]$. The dependence on harmonics disappears when moving to the component formulation after using the equations of motion and guage conditions.
- Hypermultiplet coupled to gauge multiplet, action $S_q[q^+, V^{++}] + S_{SYM}[V^{++}]$.
- Practically immediate consequence: divergences are absent beyond one-loop. ($\mathcal{N}=2$ non-renormalization theorem).

- Action of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 2$ superfields

$$S[V^{++}, q^{+}] = -\frac{1}{4g^2} \int d\zeta^{(-4)} trW^2 - \frac{1}{2} \int d\zeta^{(-4)} trq^{+a}(D^{++} + igV^{++})q_a^{+},$$

  where W is superfield strength constructed from V^{++}. The superfields q^{+} and V^{++} take the values in the Lie algebra of gauge group.
- Manifest gauge invariance, manifest $\mathcal{N} = 2$ supersymmetry and hidden $\mathcal{N} = 2$ on-shell supersymmetry rotating W and q^{+}.
- Ultraviolet finite quantum field theoretical model. It is proved very simple. According to $\mathcal{N} = 2$ non-renormalization theorem all the divergences are absent beyond one-loop. One-loop divergences are directly calculated and cancelled out. Theory is anomaly free and conformal invariant at quantum level.

Power of harmonic superspace: non-holomorphic effective potential

- Leading low-energy contribution to $\mathcal{N}=4$ effective action is described by non-holomorphic effective potential $\mathcal{H}(W,\bar{W})$ in the sector of on-shell Abelian background superfield V^{++} (Coulomb branch, $SU(2)$ broken to $U(1)$). Due to gauge invariance the effective action depends only on strengths and has the form

$$\Gamma[W,\bar{W},q^{+}] = \int d^{12}z du[\mathcal{H}(W,\bar{W}) + \mathcal{L}_q(W,\bar{W},q^{+})].$$

- Dimensional consideration: $\mathcal{H}(W,\bar{W}) = \mathcal{H}(\frac{W}{\Lambda},\frac{\bar{W}}{\Lambda})$ with some scale Λ.
- The theory is quantum conformal invariant and hence scale independent

$$\Lambda\frac{d}{d\Lambda}\int d^4x\delta^8\theta\mathcal{H}(\frac{W}{\Lambda},\frac{\bar{W}}{\Lambda}) = 0.$$

General solution is $\mathcal{H}(\frac{W}{\Lambda},\frac{\bar{W}}{\Lambda}) = c\ln\frac{W^2}{\Lambda^2}\ln\frac{\bar{W}^2}{\Lambda^2}$. Constant c is found from quantum field calculations. One can prove that no-holomorphic potential gets neither perturbative non-perturbative contribution beyond one-loop (E. Witten, N. Seiberg, 1995; M. Dine, N. Seiberg, 1997).

Power of harmonic superspace: non-holomorphic effective potential

The complete low-energy action depends on all the fields of $\mathcal{N} = 4$ vector multiplet, that is on V^{++} and q^+. Use the hidden $\mathcal{N} = 2$ supersymmetry. Exact result (I.L.B, E.A. Ivanov, 2002):

$$\Gamma[W, \bar{W}, q^+] = \int d^{12}z du [\mathcal{H}(W, \bar{W}) + \mathcal{L}_q(X)],$$

$$\mathcal{L}_q(X) = c[(X-1)\frac{\ln(1-X}{X} + Li_2(X) - 1],$$

$$X = -\frac{q^{ia}q_{ia}}{W\bar{W}},$$

where $Li_2(X)$ is the Euler dilogarithm. Leading low-energy effective action is exactly found.

Bosonic sector

$$\Gamma \sim \int d^4x \frac{F^4}{|\varphi|^2 + f^2},$$

where the denominator is $SU(4)$ invariant square of scalars from $\mathcal{N} = 4$ vector multiplet.

Study of supersymmetric field theories in various dimensions related to superstring theory.

Specific feature of the superstring theory is existence of so called D-branes which are the $D+1$ dimensional surfaces in the ten-dimensional space-time. In the low-energy limit the D-brane is associated with $D+1$-dimensional extended supersymmetric gauge theory. Therefore, study of low-energy limit of superstring theory can be related to extended supersymmetric field theory in various dimensions.

One can show that $D3$-brane is associated with $D4$, $\mathcal{N}=4$ SYM theory. $D5$-brane is associated with $D6$, $\mathcal{N}=(1,1)$ SYM theory.

Basic Motivations. Study of quantum field models with large number of symmetries

- Explicit symmetries: gauge symmetry, global symmetries, supersymmetries.
- Quantization procedure with preservation of all explicit symmetries.
- Perturbation theory with preservation of all explicit symmetries.
- Hidden (on-shell) symmetries. Preservation of hidden symmetries.
- Divergences, renormalization and effective actions.
- Construction of the new extended supersymmetric invariants as the the finite quantum contributions to effective action

Some problems of higher dimensional supersymmetric gauge theories.

1. Describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings.

2. Description of the interacting multiple $M5$-branes.

- Hypothetic M-theory is characterized by two extended objects: $M2$-brane and $M5$-brane in eleven dimensional space.
- The field description of interacting multiple $M2$-branes is given by Bagger-Lambert-Gustavsson (J. Bagger, N. Lambert, 2007; 2008. A. Gustavsson, 2009) theory which is $3D$, $\mathcal{N} = 8$ supersymmetric gauge theory.
- Lagrangian description of the interacting multiple $M5$-branes is not constructed so far.

3. Problem of miraculous cancelation of some on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop (M.B. Green, J.H. Schwarz, L. Brink, 1982).
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions (P.S. Howe, K.S. Stelle, 1984, 2003; G. Bossard, P.S. Howe, K.S. Stelle, 2009).
- Direct one-loop and two-loop component calculations (mainly in on-shell and in bosonic sector (E.S. Fradkin, A.A. Tseytlin, 1983; N. Marcus, A. Sagnotti, 1984, 1985.)
- Direct calculations of on-shell scattering amplitudes in $6D, \mathcal{N} = (1,1)$ theory up to five loops and in $D8, 10$ theories up to four loops (L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, 2015).

Results: On-shell divergences in maximally extended $6D$ SYM theory start at three loops. One-shell divergences in $8D$ and $10D$ SYM theories start at one loop.

The problems we are dealing with is aimed at studying the off-shell divergence structure. To understand, what is a reason that the divergences at one and two loops are proportional to the classical equations of motion and why, starting from three loops, this is violated.

Preservation of manifest supersymmetry: off-shell superfield formulation. Best formulation for $6D$ supersymmetric gauge theories is a harmonic superspace approach. In the case of $\mathcal{N}=(1,1)$ theory it is provides explicit off-shell $\mathcal{N}=(1,0)$ supersymmetry and hidden on-shell $\mathcal{N}=(0,1)$ supersymmetry.

Preservation of classical gauge invariance in quantum theory: harmonic superfield background field method.

Preservation of explicit gauge invariance and $\mathcal{N}=(1,0)$ supersymmetry at all steps of loop calculations: superfield proper-time methods.

$6D$ superalgebra is described by two independent supercharges. The simplest representations corresponds to $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ supersymmetries. In this sense, the maximally extended rigid supergauge theory is the $\mathcal{N} = (1,1)$ SYM theory.

$\mathcal{N} = (1,0)$ harmonic superspace:
Bosonic (commuting) coordinates x^M, $(M = 0,1,2,3,4,5)$; $u^{\pm i}\,(i = 1,2)$.
Fermionic (anticommuting) coordinates θ^a_i, $(a = 1,2,3,4)$.
Analytic subspace $\zeta = ({x_A}^M,\, \theta^{\pm\, a},\, u^{\pm\, i})$.
$\theta^{\pm\, a} = \theta^a_i u^{\pm\, i}$.
Analytic subspace is closed under the $\mathcal{N} = (1,0)$ supersymmetry.

Basic $\mathcal{N} = (1,0)$ harmonic superfields:

Hypermultiplet is described by analytic superfield $q^+(\zeta)$.
On-shell field contents: scalar field $f^i(x)$ and the spinor field $\psi_a(x)$)

Vector multiplet is described by analytic superfield V^{++}.
On-shell field contents: vector field and spinor field.

Theory of $\mathcal{N} = (1,0)$ non-Abelian vector multiplet coupled to hypermultiplet, (E.I. Ivanov, A.V. Smilga, B.M. Zupnik, Nucl.Phys. B (2005)).

- Action

$$S[V^{++}, q^+] = \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \mathrm{tr} \int d^{14}z \, du_1 \ldots du_n \, \frac{V^{++}(z,u_1) \ldots V^{++}(z,u_n)}{(u_1^+ u_2^+) \ldots (u_n^+ u_1^+)}$$

$$- \int d\zeta^{-4} du \tilde{q}^+ \nabla^{++} q^+$$

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

- Equations of motion

$$\frac{1}{2f^2} F^{++} - i\tilde{q}^+ q^+ = 0 \,, \qquad \nabla^{++} q^+ = 0 \,.$$

$$F^{++} = (D^+)^4 V^{--}, \; D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0$$

$\mathcal{N}=(1,1)$ SYM theory can be formulated in terms of $\mathcal{N}=(1,0)$ harmonic superfields as the $\mathcal{N}=(1,0)$ vector multiplet coupled to hypermultiplet in adjoint representation. The theory is manifestly $\mathcal{N}=(1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N}=(0,1)$ supersymmetry.

- Action

$$S[V^{++},q^{+}]=S_{SYM}[V^{++}]+S_{HYPER}[q^{+},V^{++}]$$

- The action is manifestly $\mathcal{N}=(1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N}=(0,1)$ supersymmetry

$$\delta V^{++}=\epsilon^{+}q^{+},\qquad \delta q^{+}=-(D^{+})^{4}(\epsilon^{-}V^{--})$$

where the transformation parameter $\epsilon^{\pm}_{A}=\epsilon_{aA}\theta^{\pm A}$.

- We start with harmonic superfield formulations of vector multiplet coupled to hypermultiplet.
- Effective action is formulated in the framework of the harmonic superfield background field method. It provides manifest $\mathcal{N}=(1,0)$ supersymmetry and gauge invariance of effective action under the classical gauge transformations.
- Effective action can be calculated on the base of superfield proper-time technique. It provides preservation of manifest $\mathcal{N}=(1,0)$ supersymmetry and manifest gauge invariance at all steps of calculations.
- The effective action can also be calculated perturbatively on the base of Feynman diagrams in superspace (supergraph technique).
- One-loop analysis. We study the model, where the $\mathcal{N}=(1,0)$ vector multiplet interacts with hypermultiplet in the arbitrary representation of the gauge group. Then, we assume in the final result for one-loop divergences, that this representation is adjoint what corresponds to $\mathcal{N}=(1,1)$ SYM theory. Finite one-loop effective action without renormalization.
- Two-loop analysis. All the possible divergences can be listed, using the the superfield power counting and then they can be calculated in the framework of the background field method.

- The superfields V^{++}, q^{+} are splitting into the sum of the background superfields V^{++}, Q^{+} and the quantum superfields v^{++}, q^{+}

$$V^{++} \to V^{++} + f v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The action is expending in a power series in quantum fields. As a result, we obtain the initial action $S[V^{++}, q^{+}]$ as a functional $\tilde{S}[v^{++}, q^{+}; V^{++}, Q^{+}]$ of background superfields and quantum superfields.
- The gauge-fixing function are imposed only on quantum superfiled

$$\mathcal{F}^{(+4)}_{\tau} = D^{++} v^{++}_{\tau} = e^{-ib} (\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \ ,$$

  where $b(z)$ is a background-dependent gauge bridge superfield and τ means τ-frame. In the non-Abelian gauge theory, the gauge-fixing function is background-dependent.
- Faddev-Popov procedure is used. One obtains the effective action $\Gamma[V^{++}, Q^{+}]$ which is gauge invariant under the classical gauge transformations.

Background field method

- The effective action $\Gamma[V^{++},Q^+] = S[V^{++},Q^+] + \bar{\Gamma}[V^{++},Q^+]$ is written in terms of path integral

$$e^{i\bar{\Gamma}[V^{++},Q^+]} = Det^{1/2}\,\widehat{\Box}\int \mathcal{D}v^{++}\,\mathcal{D}q^+\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\; e^{iS_{quant}[v^{++},q^+,\mathbf{b},\mathbf{c},\varphi,V^{++},Q^+]}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S[V^{++}+fv^{++},Q^{+}+q^{+}]-S[V^{++},Q^{+}]-S'[V^{++},Q^{+}](fv^{++},q^{+})+$$

$$+S_{GF}[v^{++},V^{++}] + S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}] + S_{NK}[\varphi,V^{++}].$$

- Gauge fixing term $S_{GF}[v^{++},V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}]$, Nielsen-Kalosh ghost action $S_{NK}[\varphi,V^{++}]$
- Operator $\widehat{\Box}$

$$\widehat{\Box} = \eta^{MN}\nabla_M\nabla_N + W^{+a}\nabla^{-}_{a} + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

One-loop approximation. Only quadratic in quantum fields and ghosts terms are taken into account in the path integral for effective action. It gives after some transformation the one-loop contribution $\Gamma^{(1)}[V^{++}, Q^{+}]$ to effective action in terms of formal functional determinants in analytic subspace of harmonic superspace

$$\Gamma^{(1)}[V^{++}, Q] = \frac{i}{2} Tr_{(2,2)} \ln[\delta^{(2,2)} \widehat{\Box}^{AB} - 2f^2 Q^{+m} (T^A G_{(1,1)} T^B)_m{}^n Q^+_n] -$$

$$- \frac{i}{2} Tr_{(4,0)} \ln \widehat{\Box} - iTr \ln (\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2} Tr \ln (\nabla^{++})^2_{\mathrm{Adj}} + iTr \ln \nabla^{++}_{\mathrm{R}}$$

As usual, $TrlnO = lnDetO$, Tr means the functional trace in analytic subspace and matrix trace.
$(T^A)_m{}^n$ are are generators of the representation for the hypermultiplet.
The $G_{(1,1)}$ is the Green function for the operator ∇^{++}.
Index A numerates the generators, $V^{++} = V^{++A} T^A$. Operator $\widehat{\Box}$ acts on the components V^{++A} as $(\widehat{\Box} V^{++})^A = \widehat{\Box}^{AB} V^{++B}$
Adj and R mean that the corresponding operators are taken in the adjoint representation and in the representation for hypermultiplet.

Manifestly covariant calculation

Calculating the one-loop divergences of superfield functional determinants is carried out in the framework of proper-time technique (superfield version of Schwinger-De Witt technique). Such technique allows us to preserve the manifest gauge invariance and manifest $\mathcal{N} = (1, 0)$ supersymmetry at all steps of calculations.

General scheme of calculations

- Proper-time representation

$$TrlnO \sim Tr \int_0^\infty \frac{d(is)}{(is)^{1+\varepsilon}} e^{isO_1} \delta(1,2)|_{2=1}$$

- Here s is the proper-time parameter and ε is a parameter of dimensional regularization.
- Typically the $\delta(1,2)$ contains $\delta^8(\theta_1 - \theta_2)$, which vanishes at $\theta_1 = \theta_2$
- Typically the operator O contains some number of spinor derivatives D^+_a, D^-_a which act on the Grassmann delta-functions $\delta^8(\theta_1 - \theta_2)$ and can kill them. Non-zero result will be only if all these δ-functions are killed.
- To get divergences, only these terms are taking into account which have the pole $\frac{1}{\varepsilon}$ after integration over proper-time. The other terms generate the finite contributions.

Results of calculations

$$\Gamma^{(1)}_{div}[V^{++},Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} tr \int d\zeta^{(-4)} du (F^{++})^2 -$$

$$-\frac{2if^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du \widetilde{Q}^{+m} (C_2 \delta_m{}^n - C(R)_m{}^n) F^{++} Q^{+}{}_n.$$

- The quantities $C_2, T(R), C(R)$ are defined as follows

$$tr(T^A T^B) = T(R)\delta^{AB}$$

$$tr(T^A_{Adj} T^B_{Adj}) = f^{ACD} f^{BCD} = C_2 \delta^{AB}$$

$$(T^A T^A)_m{}^n = C(R)_m{}^n.$$

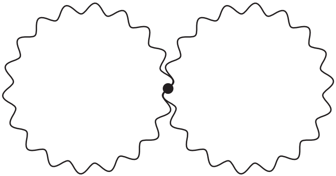
- In $\mathcal{N} = (1,1)$ SYM theory, the hypermultiplet is in the same representation as the vector multiplet. Then $C_2 = T(R) = C(R)$. Then $\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = 0$!

Procedure of calculations: gauge multiplet sector

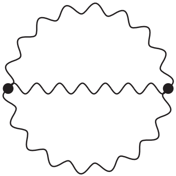
- Two-loop divergences are calculated within background field method and proper-time technique like in one-loop case.
- We begin with only gauge multiplet background.
- Power counting shows that the only possible two-loop divergent contribution in the gauge superfield sector has the following structure

$$\Gamma^{(2)}_{\mathrm{div}}[V^{++}] = a \int d\zeta^{(-4)}\, du\, \mathrm{tr}\left(F^{++}\, \widehat{\Box}\, F^{++}\right)$$

  with some constant a, which diverges after removing a regularization.
- Within background field method, the two-loop contributions to superfield effective action are given by two-loop vacuum harmonic supergraphs with background field dependent lines.
- The background field dependent propagators (lines) are represented by proper-time integrals.
- Constant a in principle should have the following structure $a = \frac{d_1}{\varepsilon} + \frac{d_2}{\varepsilon^2}$ with arbitrary real parameters $d_1\, d_2$.

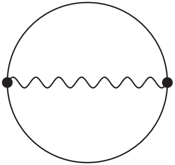


Γ_{I}

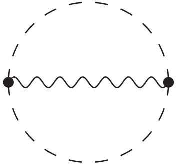


Γ_{II}

Figure: Two-loop Feynman supergraphs with gauge self-interactions vertices.



Γ_{III}



Γ_{IV}

Figure: Two-loop Feynman supergraphs with hypermultiplet and ghosts vertices.

Two-loop divergences

Procedure of calculations

- One can prove that the in the case under consideration the only two-loop divergent contribution comes from the '∞' supergraph.
- Contribution of this supergraph contains the product of two Green functions $G^{(2,2)}(z_1, u_1; z_2, u_2)$ at $z_1 = z_2$.
- Divergent part of such Green function can be calculated and has the form $\sim \frac{1}{\varepsilon} F^{++}$. Therefore $G^{(2,2)}(z_1, u_1; z_2, u_2)|_{z_1=z_2} \sim \frac{1}{\varepsilon} F^{++} + g^{++}$ where g^{++} is some finite functional.
- It means that full two loop contribution of the '∞' supergraph looks like

$$b \int d\zeta^{(-4)}\, du \, (\frac{1}{\varepsilon} F^{++} + g^{++}) \, \widehat{\Box} \, (\frac{1}{\varepsilon} F^{++} + g^{++}).$$

with some constant b. Therefore there are two types of contributions, one containing $\frac{1}{\varepsilon}$ and another one containing $\frac{1}{\varepsilon^2}$.

- The terms with simple pole $\frac{1}{\varepsilon}$ has the form $\sim \frac{1}{\varepsilon} F^{++} \, \widehat{\Box} \, g^{++}$.
- However, the power counting tells us that the two loop divergence has the form $\sim F^{++} \, \widehat{\Box} \, F^{++}$. Therefore, we must assume that $g^{++} = 0$ or $g^{++} \sim F^{++}$.

Results of calculations in gauge multiplet sector

- Further we consider only the case $g^{++} = 0$.
- In this case, the divergent part of two-loop effective action has the form

$$\Gamma^{(2)}_{div} = \frac{8f^2}{(4\pi)^6\varepsilon^2}(C_2)^2 \mathrm{tr}\int d\zeta^{(-4)}\,du\,F^{++}\,\widehat{\Box}\,F^{++},$$

where $F^{++} = 0$ is the classical equation of motion in the case when the hypermultiplet is absent.

- Coefficient c_2 looks like

$$c_2 = \frac{8f^2}{(4\pi)^6\varepsilon^2}(C_2)^2.$$

- Consider the off-shell transformation of the superfield V^{++} in the classical action $V^{++} \to V^{++} - a\,\widehat{\Box}\,F^{++}$.
- The corresponding transformation of the classical action is $\delta S = -a\int d\zeta^{(-4)}\,du\,\mathrm{tr}\,F^{++}\,\widehat{\Box}\,F^{++}$. That allows to cancel completely off-shell the two-loop divergence of the effective action in the gauge multiplet sector.
- Thus, one can state that the theory under consideration is off-shell finite at one- and two-loops (at least in gauge multiplet sector).

Hypermultiplet dependence of the two-loop divergences: indirect analysis.

- The hypermultiplet-dependent contribution to two-loop divergences can be obtained by the straightforward quantum computations of the two-loop effective action taking into account the hypermultiplet background.
- The general form of hypermultiplet dependent divergences can in principle be found without direct calculations, assuming the invariance of the effective action under the hidden $\mathcal{N} = (0,1)$ supersymmetry.
- The result has an extremely simple form

$$\Gamma^{(2)}_{\rm div}[V^{++}, q^{+}] = a \int d\zeta^{(-4)}\, du\, \mathrm{tr}\, E^{++} \,\widehat{\Box}\, E^{++},$$

  where $E^{++} = F^{++} + \frac{i}{2}[q^{+A}, q^{+}_{A}]$ is the left hand side of classical equation of motion for vector multiplet superfield coupled to hypermultiplet.
- Two-loop divergences vanish on-shell as expected.

Hypermultiplet dependence of the two-loop divergences: direct calculations:

$$\Gamma^{(2)}_{\rm div}[V^{++},q^+] = \frac{f^2(C_2)^2}{8(2\pi)^6\epsilon^2}\int d\zeta^{(-4)}\,du\,{\rm tr}\,E^{++}\,\widehat{\Box}\,E^{++}$$

$+$terms proportional to e.o.m for hypermultiplet.

Aspects higher derivative field theories (e.g. review by A.V. Smilga, 2017). Superfield quantum formulation (BIMS, 2020).

The six-dimensional $\mathcal{N}=(1,0)$ supersymmetric higher-derivative gauge theory describes a self-interacting non-Abelian gauge multiplet (E.A. Ivanov, A.V. Smilga, B.M. Zupnik, 2005). Action

$$S_0 = \pm\frac{1}{2g^2}\text{tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,,$$

The action is invariant under the gauge transformation

$$\delta_\lambda V^{\pm\pm} = -D^{\pm\pm}\lambda - i[V^{\pm\pm},\lambda]\,, \qquad \delta_\lambda F^{++} = i[\lambda, F^{++}]\,.$$

with the Hermitian analytic superfield parameter λ
Bosonic component sector

$$S \sim \frac{1}{g^2}\text{tr}\int d^6x\,(\nabla^M F_{MN})^2,$$

F_{MN} is standard Yung-Mills strength. Coupling constant g is dimensionless.

Effective action in higher derivative $6D, \mathcal{N}=(1,0)$ SYM theory

General scheme:

- Background field method
- Power counting. Theory is renormalizable.
- One-loop effective action in terms of functional determinants on superspace.
- Superfield proper-time technique
- One-loop divergences

$$\Gamma^{(1)}_{div} = -\frac{11}{3}\frac{C_2}{(4\pi)^3\varepsilon}\mathrm{tr}\int d\zeta^{(-4)}du(F^{++})^2\,.$$

- Manifestly $\mathcal{N}=(1,0)$ supersymmetric result. Bosonic and fermionic sectors are included.
- Coincides with earlier component calculations (E.A. Ivanov, A.V. Smilga, B.M. Zupnik, 2005; L. Casarin, A.A. Tseytlin, 2019).
- Asymptotic freedom at some sign in classical action. Another sign leads to null-charge problem.

- General procedure for calculation of the divergent and finite parts od effective action in $D4, \mathcal{N}=2$ supergauge theories and successfully applied for studying the low-energy effective action in $D4, \mathcal{N}=2$ SUM theory.
- The six-dimensional $\mathcal{N}=(1,0)$ supersymmetric theory of the non-Abelian vector multiplet coupled to hypermultiplet in the $6D$, $\mathcal{N}=(1,0)$ harmonic superspace was considered.
- Background field method in harmonic superspace was constructed .
- Manifestly supersymmetric and gauge invariant effective action, depending both on vector multiplet and hypermultiplet superfields, was formulated.
- Superficial degree of divergence is evaluated and structure of one- and two-loop counterterms was studied.
- An efficient manifestly gauge invariant and $\mathcal{N}=(1,0)$ supersymmetric technique to calculate the one- and two-loop contributions to effective action was developed. As an application of this technique, we found the one- and two-loop divergences of the theory under consideration.
- The same one-loop divergences have been calculated independently with help of $\mathcal{N}=(1,0)$ supergraphs.
- It is proved that $\mathcal{N}=(1,1)$ SYM theory is one-loop off-shell finite. There is no need to use the equations of motion to prove this property.

- Two-loop divergences of the $6D, \mathcal{N} = (1,1)$ SYM theory were calculated in gauge multiplet sector. The hypermultiplet dependence of two-loop divergences was restore on the base of hidden $\mathcal{N} = (0,1)$ supersymmetry.
- The hypermultiplet dependence of two-loop divergences was explicitly calculated.
- Renormalized higher derivative $6D, \mathcal{N} = (1.0)$ SYM theory in terms of $\mathcal{N} = (1,0)$ superspace was constructed, background field method was developed, one-loop divergences were calculated.
- In fact, general method to study quantum structure of $6D, \mathcal{N} = (1,0)$ supergauge theories was formulated.

- Calculations of the two-loop $\frac{1}{\epsilon}$ divergences.
- Study of the three-loop divergences. Actually, the calculations are already fulfilled, the paper under preparation.

THANK YOU VERY MUCH!