Path integrals in quadratic gravity

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I. Introductory comments

Path integrals in QM and in QFT are considered as the integrals over a Gaussian measure given by a free action $A_0 \sim \int \frac{m\,(x')^2}{2}\,dt$.

In imaginary time, it leads to the Wiener measure

$$w(dx) = \exp\left\{-\frac{1}{2}\int \left(x'(t)\right)^2 dt\right\}\, dx\,.$$

The similar picture takes place in the common models of fundamental interactions in QFT.
The problem is to choose **the true dynamical variable** that define path integrals measure and then to find **the substitution** that transforms the measure into the Wiener one.

II. Schwarzian path integrals

JT dilaton 2D gravity and SYK model (a quantum mechanical model of Majorana fermions with a random all-to-all interaction) lead to an effective theory with the **Schwarzian action**

$$A_{Sch} = -\frac{1}{\sigma^2} \int\limits_{[0,1]} \left[\mathcal{S}ch(\varphi,\, t) + 2\pi^2 \left(\varphi'(t)\right)^2 \right] dt\,,$$

where

$$\mathcal{S}ch(\varphi,\, t) = \left(\frac{\varphi''(t)}{\varphi'(t)}\right)' - \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 .$$

Here $\varphi(t)$ is an orientation preserving ($\varphi'(t) > 0$) diffeomorphism of the interval ($\varphi \in Diff^1_+([0,\, 1])$) .
$\varphi(t)$ plays the role of **the dynamical variable** of the theory.
$-\left(\varphi'(t)\right)^2$ **(wrong sign of the term ?!)**

On the group of diffeomorphisms $Diff_+^1([0,\, 1])$, there exists **a countably-additive measure** formally written as

$$\mu_\sigma(d\varphi) = \exp\left\{ \frac{1}{\sigma^2} \int\limits_{[0,1]} \mathcal{S}ch(\varphi,\, t)\, dt \right\} d\varphi \,.$$

It is generated by the Wiener measure under some special **substitution of variables**.

If we consider a continuous function on the interval [0, 1] $\xi(t)$ satisfying the boundary condition $\xi(0) = 0$ $(\xi \in C_0([0,1]))$, then under the substitution

$$\varphi(t) = \frac{\int\limits_0^t e^{\xi(\tau)} d\tau}{\int\limits_0^1 e^{\xi(\eta)} d\eta}\,, \qquad \xi(t) = \log \varphi'(t) - \log \varphi'(0)\,,$$

the measure $\mu_\sigma(d\varphi)$ on the group $Diff_+^1([0,1])$ turns into **the Wiener measure** $w_\sigma(d\xi)$ on $C_0([0,1])$, and there is **the equality of functional integrals**

$$\int\limits_{Diff_+^1([0,1])} F\left(\varphi, \varphi'\right)\, \mu_\sigma(d\varphi) = \int\limits_{C_0([0,1])} F\left(\varphi(\xi), (\varphi(\xi))'\right)\, w_\sigma(d\xi)\,.$$

Polar decomposition of the Wiener measure

The Wiener measure

$$w_\sigma(dx) = \exp\left\{ -\frac{1}{2\sigma^2} \int_0^1 \left(x'(t)\right)^2 dt \right\} dx \,.$$

is quasi-invariant under the following action of the group of diffeomorphisms $Diff_+^3([0,\,1])$ on $C_+([0,\,1])$:

$$x \mapsto fx \,, \quad (fx)(t) = x\left(f^{-1}(t)\right) \frac{1}{\sqrt{\left(f^{-1}(t)\right)'}} \,,$$

$$x \in C_+([0,\,1]) \,, \quad f \in Diff_+^3([0,\,1]) \,.$$

There is the invariant under the group $Diff_+^3([0, 1])$. It is given by the integral

$$\frac{1}{\rho^2} = \int_0^1 \frac{1}{x^2(t)} dt \,.$$

Define $\varphi \in Diff_+^1([0, 1])$ by the equation

$$\varphi^{-1}(t) = \rho^2 \int_0^t \frac{1}{x^2(\tau)} d\tau \,.$$

Then $x(t)$ is expressed in terms of ρ and $\varphi(t)$

$$x(t) = \rho \, \frac{1}{\sqrt{\left(\varphi^{-1}(t)\right)'}} \, .$$

In this case, we have

$$\int\limits_0^1 x^2(t)dt = \rho^2 \int\limits_0^1 \left(\varphi'(\tau)\right)^2 d\tau \, .$$

Therefore, there is a one-to-one correspondence $(\rho\,,\ \varphi) \leftrightarrow x$, and the space $C_+([0,\,1])$ is stratified into the orbits with different values of the invariant ρ .

Thus, for the Wiener measure on the space $C_+([0, 1])$ the following polar decompositions are valid:

$$w_\sigma(dx) = \mathcal{P}_\sigma(\rho) \; \left(\varphi'(0)\varphi'(1)\right)^{\frac{3}{4}} \; \mu_{\frac{2\sigma}{\rho}}(d\varphi)\, d\rho\,.$$

From the polar decomposition of the Wiener measure, we have the equality for the functional integrals

$$\int_{C_+(S^1)} F(x) \exp\left(-\frac{1}{2\sigma^2}\int_{S^1}\left[(\dot{x})^2 - 2\pi^2 x^2 + \frac{2g}{x^2}\right]dt\right)dx =$$

$$= \int_0^{+\infty} d\rho \exp\left\{-\left(\frac{\sigma^2}{8} + \frac{g}{\sigma^2}\right)\frac{1}{\rho^2}\right\}\times$$

$$\times \int_{Diff_+^1(S^1)} F(x(\rho,\varphi)) \exp\left(\frac{\rho^2}{4\sigma^2}\int_{S^1}\left[\mathcal{S}_\varphi + 2\pi^2(\dot{\varphi}(t))^2\right]\right)d\varphi$$

Therefore we have a transparent connection between conformal quantum mechanics and the Schwarzian theory.

Consider now the Schrödinger equation in imaginary time with the potential $V(q) = \frac{g}{q^2}$ of the form

$$\frac{\partial}{\partial \tau}\psi_g(q,\tau) = \left(\frac{1}{2}\frac{\partial^2}{\partial q^2} - \frac{g}{q^2}\right)\psi_g(q,\tau)$$

with boundary condition

$$\psi_g(q=0,\tau) = 0$$

and initial condition

$$\psi_g(q,0) = \delta(q_0 - q)\,, \qquad q_0 > 0$$

The fundamental solution of the Cauchy problem is

$$\psi(q,\tau) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{(q-q_0)^2}{2t}\right\} - \exp\left\{-\frac{(q+q_0)^2}{2t}\right\}\right].$$

$$w_\sigma(dx) = \exp\left\{-\frac{\sigma^2}{8\rho^2}\right\}\ \frac{e^{\frac{3}{4}\xi(1)}}{\left(\int\limits_0^1 e^{\xi(\tau)} d\tau\right)^{\frac{3}{2}}}\ w_{\frac{2\sigma}{\rho}}(d\xi)\ d\rho\,.$$

Although $x(t)$ and $\xi(\tau)$ are both Wiener processes , the Markov behaviour of $x(t)$ with respect to the time t of "its own world" obviously does not imply its Markov behaviour with respect to the time τ of the "shadow world" , and vice versa.

VV B and ET Sh, Theor. Math. Phys. **200** (2019) 1324, [arXiv:1812.04039].

The proof and the general rules of Schwarzian path integration as well as some properties of Wiener integrals are given in:

VV B and ET Sh, Phys. Part. Nucl. **48** (2017) 267.

VV B and ET Sh, Phys. Rev. **D 96** (2017) 101701(R) , [arXiv:1705.02405 [hep-th]].

VV B and ET Sh, JHEP **11** (2018) 036 , [arXiv:1804.00424].

VV B and ET Sh, Mod. Phys. Lett. A **33** (2018) 1850221 , [arXiv:1806.05605].

VV B and ET Sh, Phys. Part. Nucl. **51** (2020) 424, [arXiv:1912.07841],

VV B and ET Sh, J. Phys. A: Math. Theor. **53** (2020) 485201 [arXiv:1908.10387].

The non-linear non-local substitution of the form

$$y(t) = x(t) + \int_0^t f(x(\tau))d\tau$$

realize the formal connection between the theories with interaction and the free one

$$\int_Y F(x(y)) \exp\left(-\frac{1}{2}\int (\dot{y})^2 dt\right) dy =$$

$$\int_X F(x) \exp\left(-\frac{1}{2}\int \left[\dot{x}^2 + f^2(x(t)) - f'(x)\right]\right) dx$$

In the φ^4 model the substitution takes the form

$$\xi = \varphi + \int \varphi^2(\tau) d\tau$$

and the following equation is valid

$$\int_X F(\varphi) \exp\left\{-\frac{1}{2}\int \left(\dot{\varphi}^2 + \varphi^4 - 2\varphi\right) dt\right\} d\varphi = \int_{C([0,1])} F(\varphi(\xi)) w(d\xi)\,.$$

The space X contains functions with singularities of the form

$$\varphi(t) \sim (t - t_*)^{-1}\,.$$

Consider a model of scalar field with the action

$$A = \frac{1}{2} \int \left((\dot{\varphi})^2 + \lambda^2 e^{2\alpha\varphi} - \alpha\lambda e^{\alpha\varphi} \right) dt \, .$$

By the substitution

$$\xi(t) = \varphi(t) - \varphi(0) + \lambda \int_0^t e^{\alpha\varphi(\tau)} d\tau$$

we turn the theory defined by the action $A(\varphi)$ into that of the free field

$$\int_X F(\varphi) \exp\left(-A(\varphi)\right) d\varphi = \int_{C[0,1]} F(\varphi(\xi)) \exp\left(-\frac{1}{2} \int (\dot{\xi})^2 dt \right) d\xi \, .$$

X is the space of functions that have a singularity of the form $\varphi \sim -\frac{1}{\alpha} \ln t$ at the origin $t = 0$.

III. PI for quantum quadratic gravity

We consider $R + R^2$ theory in the FLRW metric and find **the dynamical variable** $g(\tau)$ that is invariant under the group of diffeomorphisms of the time coordinate.

Euclidean path integrals **not over** the space of metrics $\{\mathcal{G}\}$, as it is usually done, but over the space of continuous functions $\{g(\tau)\}$ related to the conformal factor of the metric.

$$\int F(g) \exp\{-A(g)\}\, dg$$

We study the gravity model with the action

$$A = A_0 + A_1 + A_2 \,,$$

$$A_0 = \Lambda \int d^4x \, \sqrt{-\mathcal{G}} \,, \qquad A_1 = -\frac{\kappa}{6} \int d^4x \, \sqrt{-\mathcal{G}} \, R \,,$$

$$A_2 = \frac{\lambda^2}{72} \int d^4x \sqrt{-\mathcal{G}} \left(R^2 + c_1 \, GB + c_2 \, C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \right) \,,$$

in FLRW metric

$$ds^2 = N^2(\tilde{t})\, d\tilde{t}^2 - a^2(\tilde{t})\, d\vec{x}^2\,, \qquad N(\tilde{t}) > 0\,, \quad a(\tilde{t}) > 0\,.$$

Now the general coordinate invariance of the action is reduced to its invariance under the group of reparametrizations of the time coordinate. We suppose it to be the group of diffeomorphisms of the real semiaxis including zero *Diff* $(\mathbf{R}^+)$.

The two coordinate systems are the most popular. They are the so-called
cosmological coordinate system where

$$N(t) = 1\,,$$

and
conformal coordinate system where

$$N(\tau) = a(\tau)\,,$$

with cosmological time t and conformal time τ being the time variable in the corresponding coordinate system.

We define the action of the diffeomorphisms $\varphi \in \mathit{Diff}\,(\mathbf{R}^+)$ on the functions $N(\tilde{t})$ and $a(\tilde{t})$ as follows:

$$(\varphi\, N)\,(\tilde{t}) = \left(\varphi^{-1}(\tilde{t})\right)'\, N\left(\varphi^{-1}(\tilde{t})\right)\ ; \qquad (\varphi\, a)\,(\tilde{t}) = a\left(\varphi^{-1}(\tilde{t})\right)\ .$$

Instead of the laps and the scale factors, it is convenient to use the functions $f(\tilde{t})$ and $h(\tilde{t})$ defined by the following equations:

$$f^{-1}(\tilde{t}) = \int\limits_0^{\tilde{t}} \frac{N(\tilde{t}_1)}{a(\tilde{t}_1)}\, d\tilde{t}_1 \,, \qquad h(\tilde{t}) = \int\limits_0^{\tilde{t}} N(\tilde{t}_1)\, d\tilde{t}_1 \,,$$

with the transformation rules

$$(\varphi f)(\tilde{t}) = \varphi\left(f(\tilde{t})\right) \equiv (\varphi \circ f)\,(\tilde{t})\,,$$

$$(\varphi h)(\tilde{t}) = h\left(\varphi^{-1}(\tilde{t})\right) \equiv \left(h \circ \varphi^{-1}\right)(\tilde{t})\,.$$

The function

$$g(\tau) = (h \circ f)(\tau) = h(f(\tau))$$

is **invariant** under the diffeomorphisms φ.
It **turns** the conformal coordinate system time τ **to** the cosmological one t

$$t = g(\tau), \qquad \tau = g^{-1}(t).$$

$$Conformal: \quad ds^2 = \left(g'(\tau)\right)^2 \left[d\tau^2 - d\vec{x}^2\right],$$

$$Cosmological: \quad ds^2 = dt^2 - \left(g'\left(g^{-1}(t)\right)\right)^2 d\vec{x}^2.$$

The invariance of the action manifests itself in its **dependence on the only** invariant function g

$$A = A(f,\, h) = A\,(g) = A_0\,(g) + A_1\,(g) + A_2\,(g)\ ,$$

Therefore, every four-dimensional space-time (and the corresponding space-time FLRW metric) is **determined by its proper function** g , and vice versa, every function g **determines the particular four-dimensional space-time.**

We consider the model with the particular relation between the parameters

$$\lambda = \frac{\kappa}{3\sqrt{2\Lambda}},$$

and write down the action of the quadratic gravity model in the form

$$A = \frac{\lambda^2}{2} \int d\tau \left[\left(\frac{g''}{g'} \right)' - \left(\frac{g''}{g'} \right)^2 + \frac{\kappa}{3\lambda^2} (g')^2 \right]^2 .$$

We perform the non-linear non-local substitution

$$\left(\frac{g''}{g'}\right)' - \left(\frac{g''}{g'}\right)^2 + \frac{\kappa}{3\lambda^2}(g')^2 = \frac{1}{\lambda}p'(\tau)$$

and the functional measure of the theory turns into the Wiener measure

$$\mu(dg) \equiv \exp(-A(g))dg = \exp\left(-\int_0^1 \frac{p'^2(\tau)}{2}d\tau\right) \equiv w(dp)\,.$$

Now, the problem is to express $g(\tau)$ through $p(\tau)$ for calculating the average values of the form

$$\langle F(g)\rangle = \int w(dp)F(g_p)\,.$$

This expression can be obtained as power series

$$g_p = \sum_{k=0} g_k(\tau)\lambda^{-k}\,.$$

The first order term is

$$g_1 = \int_0^\tau \frac{1}{1-s}\ln\left(\frac{1-s}{1-\tau}\right)\int_0^s dy\, p(y)\,.$$

The solution of the classical equation of motion

$$\left(\frac{g''}{g'}\right)' - \left(\frac{g''}{g'}\right)^2 + \frac{\kappa}{3\lambda^2}(g')^2 = 0$$

is given by the implicit function

$$\tau = \int_0^{g_{cl}(\tau)} dy \exp\left(-y + \frac{\kappa}{6\lambda^2}y^2\right)$$

and leads to

$$a_{cl}(t) = {g'}_{cl}(g_{cl}^{-1}(t)) = \exp\left(t - \frac{\kappa}{6\lambda^2}t^2\right).$$

We calculate the path integral

$$\langle a(t)\rangle = \int {g_p}'({g_p}^{-1}(t))w(dp)$$

to study the quantum corrections to the classical solution.

The result for the averaged scale factor in the first nontrivial order $\left(\sim \frac{1}{\lambda^2}\right)$ is

$$\langle a(t)\rangle = \left(1 + \frac{1}{\lambda^2}\left[(1-\kappa)\frac{t^2}{6} - \frac{t}{9} + \frac{1}{27}\right]\right) e^t - \frac{e^{-2t}}{27\lambda^2}\,.$$

VV B, VV Ch and ET Sh
Perturbation Theory for Path Integrals in Quadratic Gravity IJMPA (to be published)

We calculated quantum corrections to the classical solution (in model with $\Lambda = 0$) of the form

$$g_{cl} = \frac{\sigma\tau^2}{2}\,, \qquad a_{cl}(t) = \sqrt{2\sigma t}\,,$$

$$\langle a \rangle = \sqrt{2\sigma t}\left\{1 + \frac{1}{\lambda^2}\left[-\frac{59}{63}\left(\frac{2t}{\sigma}\right)^{3/2} + \frac{11}{120}\kappa(2t)^2\right]\right\}.$$

VV B and ET Sh
Path Integrals in Quadratic Gravity JHEP 2022, 112 (2022). [arXiv:2110.06041]

PI for quantum 2D quadratic gravity

The general form of the 2D gravity action up to the terms quadratic in curvature K is

$$\tilde{\mathcal{A}} = c_0 \int \sqrt{\mathcal{G}}\, d^2x + c_1 \int K \sqrt{\mathcal{G}}\, d^2x + c_2 \int K^2 \sqrt{\mathcal{G}}\, d^2x \,.$$

We consider the action restricted to the conformal gauge, where the metric of the 2D surface looks like

$$dl^2 = g(u,\, v)\, \left(du^2 + dv^2\right) = g(z,\, \bar{z})\, dz\, d\bar{z} \qquad \sqrt{\mathcal{G}} = g\,.$$

The Gaussian curvature of the surface is

$$K = -\frac{1}{2g}\,\Delta\,\log g\,,$$

where Δ stands for the Laplacian.

The action is invariant under the complex analytic substitutions. Therefore, we reduce the region of integration to the disc $d\,:\,(|z|\leq 1)\,.$

We consider the specific form of the action

$$A = \frac{\lambda^2}{2} \int\limits_d (K+4)^2 \; g(z,\, \bar{z})\, dz\, d\bar{z} = \frac{\lambda^2}{2} \int\limits_d (\Delta\psi)^2 \;\, dz\, d\bar{z}$$

where

$$\Delta\psi = q\, \Delta \log q + \frac{4}{q}\,, \qquad q = \frac{1}{\sqrt{g}}\,,$$

and study path integrals

$$\int F(\psi)\, \exp\{-A(\psi)\}\, d\psi = \int F(\psi)\, \mu_\lambda(d\psi)$$

over the Gaussian functional measure

$$\mu_\lambda(d\psi) = \frac{\exp\{-A(\psi)\}\, d\psi}{\int \exp\{-A(\psi)\}\, d\psi}\,.$$

The extremum of the action is given by the equation

$$\Delta\psi = 0\,, \qquad q_0\,\Delta\log q_0 + \frac{4}{q_0} = 0\,.$$

We choose the boundary condition corresponding to the Poincare model of the Lobachevsky plane

$$q_0\big|_{|z|=1} = 0\,.$$

The unique solution in the disk $d\,(|z| \leq 1)$ satisfying the boundary condition is

$$q_0 = 1 - z\bar{z}\,.$$

Let us rewrite the action substituting $\psi \to f$ with

$$\Delta\psi = \frac{1}{q_0}\, T^{-1}\, [f]$$

where

$$T^{-1} \equiv \left(q_0^2\,\Delta - 8\right)$$

is the Casimir operator of $SL(2, \mathbb{R})$.

Now the action is written as the integral over the measure

$$\frac{dz\, d\bar{z}}{(1 - z\bar{z})^2}$$

invariant under the action of the group $SL(2, \mathbb{R})$ in the disk

$$A = \frac{\lambda^2}{2} \int\limits_{d} \left(T^{-1}\,[f]\right)^2\, \frac{dz\, d\bar{z}}{q_0^2}\,.$$

Therefore, we obtain the $SL(2, \mathbb{R})$ invariant Gaussian functional measure

$$\mu_\lambda(df) = \frac{\exp\{-A(f)\}\, df}{\int \exp\{-A(f)\}\, df} \,.$$

We reduce path integrals over the measure $\mu_\lambda(df)$ to the products of Wiener integrals.
First, we consider the Fourier series (we use polar coordinates)

$$\Delta\psi(\varrho, \varphi) = x_0(\varrho) + \sum_{n=1}^{\infty} (x_n \cos n\varphi + y_n \sin n\varphi) \; .$$

Now the action is written as

$$A = \frac{\lambda^2}{2} 2\pi \int_0^1 (x_0(\varrho))^2 \; \varrho \, d\varrho +$$

$$+\frac{\lambda^2}{2} \pi \sum_{n=1}^{\infty} \left[\int_0^1 (x_n(\varrho))^2 \; \varrho \, d\varrho + \int_0^1 (y_n(\varrho))^2 \; \varrho \, d\varrho \right] \; .$$

Then, using the relation between ψ and f, we express x_n, y_n in terms of the coefficients of the Fourier series

$$f(\varrho, \varphi) = a_0(\varrho) + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi) \ .$$

Therefore we have

$$\int_0^1 (x_n(\varrho))^2 \; \varrho \, d\varrho = \int_0^{+\infty} \left(U_n'(\tau_n)\right)^2 \; d\tau_n$$

$$U_n(\tau_n) = \varrho^{2n+1} \frac{\left(1+n+(1-n)\varrho^2\right)^2}{(1-\varrho^2)^2} \left(\frac{\left(1-\varrho^2\right) \; a_n(\varrho)}{\varrho^n \; (1+n+(1-n)\varrho^2)} \right)' .$$

$$\tau_n = \int_0^{\varrho} \frac{\varrho_1^{2n+1}}{\left(1-\varrho_1^2\right)^2} \left(\frac{2}{\left(1-\varrho_1^2\right)} + n - 1 \right)^2 \; d\varrho_1 \, .$$

The same equations are valid for the other terms

$$\int_0^1 (y_n(\varrho))^2 \; \varrho \, d\varrho = \int_0^{+\infty} \left(\tilde{U}_n'(\tau_n)\right)^2 \; d\tau_n \, .$$

Now the measure $\mu_\lambda(df)$ is represented as the product of the Wiener measures

$$\mu_\lambda(df) = w_{\frac{1}{\lambda\sqrt{2\pi}}}\,(dU_0)\prod_{n=1}^{\infty} w_{\frac{1}{\lambda\sqrt{\pi}}}\,(dU_n)\;w_{\frac{1}{\lambda\sqrt{\pi}}}\left(d\tilde{U}_n\right)$$

where

$$w_\sigma\,(dU) = \exp\left\{-\frac{1}{2\sigma^2}\int\limits_0^{+\infty}\left(U'(\tau)\right)^2\,d\tau\right\}dU\,.$$

For path integrals with integrands that depend on the modes with definite numbers, the product of the measures is reduced because of the cancelation of the same terms in the nominator and in the denominator.

$SL(2, \mathbb{R})$ invariance of the measure simplifies the calculations.

In particular, due to the $SL(2, \mathbb{R})$ invariance,

$$\langle g(\varrho, \varphi)\rangle_\mu = \int\limits_{C(d)} g(\varrho, \varphi)\, \mu_\lambda(df) = \frac{1}{q_0^2(\varrho)}\, \langle g(0, 0)\rangle_\mu = \frac{1}{q_0^2(\varrho)}\,.$$

Due to the $SL(2, \mathbb{R})$ invariance, the correlation function of the metric

$$\langle g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2)\rangle_\mu = \int\limits_{C(d)} g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2)\ \mu_\lambda(df)$$

can be rewritten as

$$\langle g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2)\rangle_\mu = \frac{q_0^2(\varrho_*)}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)} \int\limits_{C(d)} g(0, 0)\, g(\varrho_*, \varphi_*)\, \mu_\lambda(df)$$

$$= \frac{q_0^2(\varrho_*)}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)}\, \langle g(0, 0)\, g(\varrho_*, \varphi_*)\rangle_\mu \,,$$

where (0, 0) and (ϱ_*, φ_*) are the results of the shift at the Lobachevsky plane of the coordinates (ϱ_1, φ_1) and (ϱ_2, φ_2)

$$\varrho_* = \frac{\sqrt{\varrho_1^2 + \varrho_2^2 - 2\varrho_1\varrho_2 \cos(\varphi_2 - \varphi_1)}}{\sqrt{1 + \varrho_1^2\varrho_2^2 - 4\varrho_1\varrho_2 \cos(\varphi_2 - \varphi_1)}} .$$

In the first nontrivial perturbative order,

$$\langle g(\varrho_1,\, \varphi_1)\, g(\varrho_2,\, \varphi_2)\rangle_\mu$$

$$= \frac{2}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)} \left\{ 1 + \frac{1}{6\pi\lambda^2} \left[-\frac{1}{2}\int\limits_{\varrho_*}^{1} \frac{\log t}{1-t} dt + 2\log 2 + \frac{1}{2} - \frac{1}{1+\varrho_*^2} \right.\right.$$

$$+ \frac{1}{1+\varrho_*^2} \log\left(\frac{1-\varrho_*^2}{\varrho_*}\right) + \log\varrho_* - 2\log(1+\varrho_*^2) - \log\sqrt{1-\varrho_*^2} +$$

$$\left.\left. \log\varrho_* \, \log\sqrt{1-\varrho_*^2} \,\right]\right\} .$$

VV B and ET Sh, **An approach to quantum 2D gravity**,
Phys.Lett.B 836 (2023), 137633 [arXiv:2206.05172].