Conformal bootstrap and Heterotic string Gepner models

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Abstract

Heterotic string Gepner model in 4-dimensions Hysbrid of

1) the left-moving Superstring obtained from $N=1$ CFT whose additional 6 dimensions are compactified on the product ($M_{\vec{k}}$) of the $N=2$ minimal models of SCFT with the total central charge 9,

and

2) the right-moving Bosonic string, whose additional 22 dimensions are also compactified on the product $N=2$ SCFT M_k, and the remaining 13 dimensions of which form the torus of $E(8)\times SO(10)$.

Such Heterotic string models have
1) $N=1$ Space-time symmetry arising in its left-moving part
and
2) $E(6)$ Gauge symmetry arising in its right-moving part.

These symmetries are necessary for phenomenological reasons. They have been successfully used to derive Grand Unified Theories (GUTs).

Abstract

In Gepner's pioneering work, the requirement that leads to a model having the desired $N = 1$ Spacetime symmetry and $E(6)$ Gauge symmetry was the requirement that spacetime symmetry be compatible with modular invariance.

In our work we show that the requirement for the simultaneous fulfillment of mutual locality of the left-moving vertices of physical states with the space-time symmetry generators

And

of right-moving vertices with generators of $E(6)$-gauge symmetry, which arises after some special reduction

together with the requirement of mutual locality of complete (left-right) vertices of physical states among themselves leads to the same Gepner model.

Plan of the construction

Based on these requirements, we construct the physical states of the theory in three steps.

At the first step, in the left sector we find the generators of $N = 1$ space-time supersymmetry and the set of left-moving physical states that are mutually local with them.

At the second step, in the right sector we find the generators of $E(6)$ symmetry and the set of right-moving physical states that are mutually local with them.

At the third step, we find the products of the left- and right-moving vertices of the states obtained in this way, which are mutually local to each other.

Left-moving sector, N=1 CFT

•It is a product of 4-dimensional $N=1$ CFT for space-time sector with cental charge 6 and $N=1$ CFT for the compact sector realised as a product $N=2$ minimal models with the total cental charge 9.

•4-dimensional $N=1$ CFT of space-time factor is a theory of 4 free bosons $X^{\mu}(z)$ and 4 Majorana fermions $\psi^{\mu}(z)$.

•$N=1$ CFT for the compact sector realised as a product $N=2$ minimal models with the total cental charge 9:

$$M_{\vec{k}}=\prod_{i=1}^{5} M_{k_i},\ \ c_i=\frac{3k_i}{k_i+2},\ \sum_i c_i=9. \qquad (1)$$

•$N=2$ superconformal minimal model contains the primary fields

$$\Phi_{l,q,s}(z),\ \ l=0,...,k,\ \ l+q+s=0 mod\ 2,\ \ s=0,1,2,3,$$
$$\Delta=\frac{l(l+2)-q^2}{4(k+2)}+\frac{s^2}{8},\ \ Q=\frac{q}{k+2}-\frac{s}{2}. \qquad (2)$$

•In NS sector, $s=0,2$, in R sector $s=1,3$.

Left-moving diagonal $N = 1$ Virasoro superalgebra.

•It is a product of 4-dimensional $N = 1$ CFT for space-time sector with central charge 6

$$
\begin{aligned}
&T_{mat}(z) = T_{st}(z) + T_{int}(z), \;\; G_{mat}(z) = G_{st}(z) + G_{int}(z), \\
&T_{st} = -\frac{1}{2}\partial X^{\mu}(z)\partial X_{\mu}(z) - \frac{1}{2}\psi^{\mu}(z)\partial\psi_{\mu}(z), \\
&G_{st}(z) = \partial X^{\mu}\psi_{\mu}(z), \\
&T_{int} = \sum_{i=1}^{5} T_i(z), \;\; G_{int}(z) = \sum_{i=1}^{5}(G_i^{+} + G_i^{-})(z).
\end{aligned} \tag{3}
$$

•The currents $T_i(z)$, $G_i^{\pm}(z)$ together with the $U(1)$ current $J_i(z)$ form $N = 2$ Virasoro superalgebra of the minimal model M_i.

•The $N = 1$ Virasoro superalgebra action is correctly defined on the product of only NS-representations or on the product of only R-representations.

BRST approach

- We use the BRST approach.

$$Q_{BRST} = \oint dz(cT_{mat} + \gamma G_{mat} + \frac{1}{2}(cT_{gh} + \gamma G_{gh})). \qquad (4)$$

- The ghost fields and $N = 1$ Virasoro superalgebra.

$$\beta(z)\gamma(0) = -z^{-1} + ..., \quad b(z)c(0) = z^{-1} + ... \qquad (5)$$

$$\begin{aligned} T_{gh} &= -\partial bc - 2b\partial c - \frac{1}{2}\partial\beta\gamma - \frac{3}{2}\beta\partial\gamma, \\ G_{gh} &= \partial\beta c + \frac{3}{2}\beta\partial c - 2b\gamma. \end{aligned} \qquad (6)$$

- $\beta - \gamma$ space of states is characterized by the vacuum $V_q(z)$, which can be realized as a free scalar field exponent:

$$\begin{aligned} &V_q(z) = \exp(q\phi(z)), \ \phi(z)\phi(0) = -\log(z) + ..., \\ &\beta(z)V_q(0) \sim O(z^q), \ \gamma(z)V_q(0) \sim O(z^{-q}). \end{aligned} \qquad (7)$$

Bosonization of ψ^μ and left-moving vertices

• The left-moving vertex can be written as

$$V^{\vec{l}}_{\vec{\mu}_L} = P_{gh}(\beta,\gamma,b,c)P_{st}(\partial X^\mu,\partial H_a)P(T_i,J_i)\exp\left[q\phi+\imath\lambda^a H_a\right]\Phi^{\vec{l}}_{\vec{l},\vec{q}_L,\vec{s}_L}(z),$$
$$\vec{\mu}_L=(q,\vec{\lambda},\vec{q}_L,\vec{s}_L),\ \ \vec{q}_L=(q^1_L,...,q^5_L),\ \ \vec{s}_L=(s^1_L,...,s^5_L),$$
$$Q^i_L:=\frac{q^i_L}{k_i+2}-\frac{s^i_L}{2}+\ even,\ \ s^i_L=0,1,2,3,\ \ l_i+q^i_L+s^i_L=0 \mod 2. \tag{8}$$

P_{gh}, P_{st}, P_{int} are the polynomials of the corresponding fields and theirs derivations.

• Here we bosonized fermions:

$$H_a(z)H_b(0)=-\delta_{ab}\log(z)+...,\ \ a,b=1,2.$$
$$\frac{1}{\sqrt{2}}(\pm\psi^0+\psi^1)=\exp\left[\pm\imath H_1\right],\ \ \frac{1}{\sqrt{2}}(\psi^2\pm\imath\psi^3)=\exp\left[\pm\imath H_2\right]. \tag{9}$$

The diagonal $N = 1$ Virasoro algebra

•Its action is well defined on the product of only NS - or only R-representations. Using the definition

$$\vec{\mu}_L \cdot \vec{\mu}'_L := -qq' + \vec{\lambda} \cdot \vec{\lambda}' - \frac{1}{2} \sum_i \left(\frac{q_i q'_i}{k_i + 2} - \frac{s_i s'_i}{2} \right) \tag{10}$$

the corresponding restrictions on the vertices that ensure the action of the diagonal $N = 1$ can be rewritten in the form $\vec{\beta}_j \cdot \vec{\mu}_L \in \mathbb{Z}$, where

$$\begin{aligned} \vec{\beta}_1 &:= (1; 1, 0; 0, ..., 0; 0, ..., 0), \\ \vec{\beta}_2 &:= (1; 0, 1; 0, ..., 0; 0, ..., 0), \\ \vec{\beta}_3 &:= (1; 0, 0; 0, ..., 0; 2, 0, 0, 0, 0), ..., \\ \vec{\beta}_7 &:= (1; 0, 0; 0, ..., 0; 0, 0, 0, 0, 2). \end{aligned} \tag{11}$$

Important that the following equations also hold

$$\begin{aligned} &\vec{\beta}_j \cdot \vec{\beta}_0 \in \mathbb{Z}, \ j = 1, ..., 7. \\ &\vec{\beta}_0 := (\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; 1, ..., 1; 1, ..., 1). \end{aligned} \tag{12}$$

Left-moving massless states.

•Among them we find massless spinors. In terms of $H_a(z)$ bosons, the vertices of massless spinors can be written as follows

$$
\begin{aligned}
&S^{\pm}_{\xi,p} = \sum_{\vec{\sigma}} \xi_{\vec{\sigma}}(p) \exp(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{H} \pm \frac{\imath}{2}\sum_i \frac{k_i\phi_i}{\sqrt{k_i(k_i+2)}}) e^{(\imath p_\mu X^\mu)}(z),\\
&\sigma^a = \pm\frac{1}{2},\ \sum_{a=1}^{2}\sigma^a = \pm 1,\ p^2 = 0,\\
&\dot{S}^{\pm}_{\xi,p} = \sum_{\dot{\vec{\sigma}}} \xi_{\dot{\vec{\sigma}}}(p) \exp(-\frac{1}{2}\phi(z) + \imath\dot{\vec{\sigma}}\cdot\vec{H} \pm \frac{\imath}{2}\sum_i \frac{k_i\phi_i(z)}{\sqrt{k_i(k_i+2)}}) e^{(\imath p_\mu X^\mu)}(z),\\
&\dot{\sigma}^a = \pm\frac{1}{2},\ \sum_{a=1}^{2}\dot{\sigma}^a = 0,\ p^2 = 0.
\end{aligned}
\tag{13}
$$

Super-Poincare algebra

•The fields $S^{\pm}_{\xi,p}$ are $SO(1,3)$ Weyl spinors of positive chirality, while $\dot{S}^{\pm}_{\xi,p}$ are $SO(1,3)$ Weyl spinors of negative chirality.

•By setting $p=0$ at these vertices, we obtain currents whose integrals are supergenerators of space-time symmetry

$$
\begin{aligned}
S^{\pm}_{\sigma} &= \exp(-\frac{1}{2}\phi + \imath\sigma^a H_a \pm \frac{\imath}{2}\sum_i \frac{k_i\phi_i}{\sqrt{k_i(k_i+2)}})(z), \\
S^{\pm}_{\dot{\sigma}} &= \exp(-\frac{1}{2}\phi(z) + \imath\dot{\sigma}^a H_a \pm \frac{\imath}{2}\sum_i \frac{k_i\phi_i(z)}{\sqrt{k_i(k_i+2)}})(z).
\end{aligned}
\tag{14}
$$

•The current S^{+}_{σ} is mutually local with $S^{-}_{\dot{\sigma}}$ and S^{-}_{σ} is mutually local with $S^{+}_{\dot{\sigma}}$. We can choose one of two pairs to obtain supercharges that extend Poincaré algebra to $N=1$ Super-Poincaré algebra

$$
\begin{aligned}
&Q^{+}_{\sigma} = \oint dz S^{+}_{\sigma}(z),\ \ Q^{-}_{\dot{\sigma}} = \oint dz S^{-}_{\dot{\sigma}}(z), \\
&[Q^{+}_{\sigma}, Q^{-}_{\dot{\sigma}}]_{+} = (\gamma^{\mu})_{\sigma,\dot{\sigma}} P^{\mu} = (\gamma^{\mu})_{\sigma,\dot{\sigma}} \oint dz \psi^{\mu} e^{-\phi}(z).
\end{aligned}
\tag{15}
$$

Spacetime symmetry and GSO equation in the left sector.

- To obtain supersymmetry, we have to select from vertices

$$V^{\vec{l}}_{\vec{\mu}_L} = P_{gh}(\beta,\gamma,b,c)P_{st}(\partial X^{\mu},\partial H_a)\exp\left[q\phi + \imath\lambda^a H_a\right]\Phi^{\vec{l}}_{\vec{l},\vec{q}_L,\vec{s}_L}(z),$$
$$\vec{\mu_L} = (q;\vec{\lambda};\vec{q}_L;\vec{s}_L),\ \vec{q}_L = (q^1_L,\ldots,q^5_L),\ \vec{s}_L = (s^1_L,\ldots,c^5_L) \tag{16}$$

those that are mutually local with $S^+_{\sigma}, S^-_{\dot{\sigma}}$.

- $S^+_{\sigma}, S^-_{\dot{\sigma}}$ are mutually local with such vertex if

$$\frac{1}{2}q + \sum_a \sigma^a\lambda^a + \frac{1}{2}\sum_{i=1}^{5}\left(\frac{q^i_L}{k_i+2} - \frac{1}{2}s^i_L\right) \in \mathbb{Z}. \tag{17}$$

$$\frac{1}{2}q + \sum_a \sigma^a\lambda^a + \frac{1}{2}\sum_{i=1}^{5} Q^i_L \in \mathbb{Z}. \tag{18}$$

It turns out that it coincides with GSO condition, can be rewritten

$$\vec{\beta}_0\cdot\vec{\mu}_L \in \mathbb{Z},$$
$$\textit{where}\ \vec{\beta}_0 := \left(\frac{1}{2};\frac{1}{2},\frac{1}{2};1,...,1;1,...,1\right). \tag{19}$$

Left-moving massless supermultiplets

•We can also check the fulfillment of the important equality

$$\vec{\beta}_0 \cdot \vec{\beta}_0 = 1$$

.

•The following massless states and their superpartners are mutually local with $S^{+}_{\sigma}, S^{-}_{\dot{\sigma}}$:

•Massless vector field

$$V_{\xi,p} = \xi_{\mu}(p)\psi^{\mu} exp(-\phi) \exp(\imath p_{\mu} X^{\mu})(z), \; \xi_{\mu}(p)p^{\mu} = 0 \tag{20}$$

•Massless scalar field

$$\begin{aligned} V^{c}_{\vec{l},p} &= exp(-\phi)\Phi^{c}_{\vec{l}} \exp(\imath p_{\mu} X^{\mu}(z)), \; \sum_{i=1}^{5} \frac{l_i}{k_i + 2} = 1, \\ V^{a}_{\vec{l},p} &= exp(-\phi(z))\Phi^{a}_{\vec{l}}(z) \exp(\imath p_{\mu} X^{\mu}(z)), \; \sum_{i=1}^{5} \frac{l_i}{k_i + 2} = 1, \end{aligned} \tag{21}$$

where $\Phi^{c}_{\vec{l}}(z)$, $\Phi^{a}_{\vec{l}}(z)$ are chiral and anti-chiral primary states.

•From GSO equation and the requirement of compatibility with $N=1$ Virasoro it follows that the total internal charges $Q_L := \sum_i Q_L^i$ of the selected vertices are integers or half-integers.

•The weight lattice of the algebra $SO(1,3)$, as well as $SO(2n)$, has 4 sublattices. That is weights $\vec{\lambda}$ belong to one of the four conjugacy classes

$$\begin{aligned}
&(0) : (0,0,0,...,0) + \mathit{any\ root};\\
&(V) : (1,0,0,...,0) + \mathit{any\ root};\\
&(S) : (\frac{1}{2},\frac{1}{2},\frac{1}{2},...,\frac{1}{2}) + \mathit{any\ root};\\
&(C) : (-\frac{1}{2},\frac{1}{2},\frac{1}{2},...,\frac{1}{2}) + \mathit{any\ root}.
\end{aligned} \tag{22}$$

•As a result of the requirement of compatibility of the action of the $N=1$ diganal Virasoro algebra and the fulfillment of the GSO equation it follows that in NS sector the weights $\vec{\lambda}$ fall into classes [0] and $[V]$ and in R sector the weights $\vec{\lambda}$ fall into $[S]$ and $[C]$.

•Moreover, we get the following agreement between pictures q, conjugacy classes $\vec{\lambda}$ and the sums of $U(1)$ charges in the compact sector and gives 4 classes of left vertices:

$$\begin{aligned}
&\sum_i Q_L^i \in 2\mathbb{Z}+1 \Rightarrow q=-1, \sum_a \sigma^a\lambda^a = 0 \ mod\ \mathbb{Z}, (\vec{\lambda}\in[0]),\\
&\sum_i Q_L^i \in 2\mathbb{Z} \Rightarrow q=-1, \sum_a \sigma^a\lambda^a = \frac{1}{2} \ mod\ \mathbb{Z}, (\vec{\lambda}\in[V]),\\
&\sum_i Q_L^i \in 2\mathbb{Z}-\frac{1}{2} \Rightarrow q=-\frac{1}{2}, \sum_a \sigma^a\lambda^a = \frac{1}{2} \ mod\ \mathbb{Z}, (\vec{\lambda}\in[S]),\\
&\sum_i Q_L^i \in 2\mathbb{Z}+\frac{1}{2} \Rightarrow q=-\frac{1}{2}, \sum_a \sigma^a\lambda^a = 0 \ mod\ \mathbb{Z}. (\vec{\lambda}\in[C]).
\end{aligned} \tag{23}$$

This completes the selection of the subspace of physical left vertices, consistent with N=1 SUSY.

Right-moving sector, N=0 CFT, $c_{tot} = 26$

- $\bar{X}^{\mu}(\bar{z})$ are bosonic fields in the 4-dimensional space-time with $c = 4$.
- $Y_I(\bar{z})$, $I = 1, ..., 8$ are bosonic fields on the torus of the algebra $E(8)$ with $c = 8$.
- $\bar{H}_{\alpha}(\bar{z})$, $\alpha = 1, ..., 5$ are bosonic fields on the torus of the algebra $SO(10)$ with $c = 5$.
- $M_{\vec{k}}$ is a product of Minimal models M_{k_i} with $c = \sum \frac{3k_i}{k_i+2}$=9.
- Right-moving energy-momentum tensor

$$\bar{T}_{mat}(\bar{z}) = \frac{1}{2}(\eta_{\mu\nu}\bar{\partial}\bar{X}^{\mu}\bar{\partial}\bar{X}^{\nu} + (\bar{\partial}Y_I)^2 + (\bar{\partial}\bar{H}_{\alpha})^2) + \bar{T}_{int}(\bar{z}). \quad (24)$$

- We use the BRST approach introducing right-moving ghosts:

$$\bar{b}(\bar{z})\bar{c}(0) = \bar{z}^{-1} + ... \quad (25)$$

$$\begin{aligned} \bar{Q}_{BRST} &= \oint d\bar{z}\bar{c}(\bar{T}_{mat} + \frac{1}{2}\bar{T}_{gh}), \\ \bar{T}_{gh} &= -\bar{\partial}\bar{b}\bar{c} - 2\bar{b}\bar{\partial}\bar{c}. \end{aligned} \quad (26)$$

Right-moving massless states and $E(8) \oplus SO(10) \oplus U(1)^4$

- $SO(1,3)$ vector $V^\mu(\bar{z}) = \imath\bar{\partial}\bar{X}^\mu(\bar{z})$,
- Currents of $E(8)$ algebra

$$V^I(\bar{z}) = \imath\bar{\partial}\bar{Y}^I(\bar{z}),\ \ I = 1, ..., 8,\ \ V_{\vec{k}}(\bar{z}) = \exp[\imath k_I \bar{Y}^I](\bar{z}),\ \ \vec{k}^2 = 2, \tag{27}$$

$$\vec{k} = \begin{cases} (\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) + \ \textit{permutations}, \\ (\pm \frac{1}{2}, ..., \pm \frac{1}{2}) + \ \textit{permutations},\ \textit{even number of} + \frac{1}{2}. \end{cases} \tag{28}$$

- Currents of $SO(10)$ algebra

$$\begin{aligned} &V^\alpha(\bar{z}) = \imath\bar{\partial}\bar{H}^\alpha(\bar{z}),\ \ \alpha = 1, ..., 5, \\ &V_{\vec{\rho}}(\bar{z}) = \exp[\imath\rho_\alpha \bar{H}^\alpha](\bar{z}),\ \ \rho_\alpha = \pm 1,\ \sum(\rho_\alpha)^2 = 2, \end{aligned} \tag{29}$$

$$\vec{\rho} = (\pm 1, \pm 1, 0, 0, 0) + \ \textit{permutations}. \tag{30}$$

- Currents of $U(1)^4$ algebra

$$I_j(\bar{z}) = \imath\sqrt{\frac{k_j}{k_j + 2}}\bar{\partial}\bar{\phi}_j(\bar{z}) - \frac{k_j}{3(k_j + 2)} J_{int}(\bar{z}),\ \ j = 1, ..., 4. \tag{31}$$

Extension of $SO(10)$ symmetry to $E(6)$

In the right-moving sector there aise also massless $SO(10)$ spinors

$$\Sigma^{\pm}_{\omega}(\bar{z}) = \exp[\imath\omega_{\alpha}\bar{H}^{\alpha}]\exp[\pm\frac{\imath}{2}\sum_{i}\frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}],$$
$$\omega_{\alpha} = \pm\frac{1}{2},\ \sum\omega_{\alpha} = \frac{1}{2}\ mod\ 2\mathbb{Z},$$
$$\Sigma^{\pm}_{\dot{\omega}}(\bar{z}) = \exp[\imath\dot{\omega}_{\alpha}\bar{H}^{\alpha}]\exp[\pm\frac{\imath}{2}\sum_{i}\frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}],$$
$$\dot{\omega}_{\alpha} = \pm\frac{1}{2},\ \sum\dot{\omega}_{\alpha} = -\frac{1}{2}\ mod\ 2\mathbb{Z}. \quad (32)$$

•Σ^{+}_{ω} is mutually local with $\Sigma^{-}_{\dot{\omega}}$. And Σ^{-}_{ω} is mutually local with $\Sigma^{+}_{\dot{\omega}}$.

•45 integrals of $SO(10)$ currents together with 32 integrals of spinor currents $\Sigma^{+}_{\omega}(\bar{z})$, $\Sigma^{-}_{\dot{\omega}}$ and 1 integral $U(1)$ current

$$\bar{J}_{int} = \sum_{i}\sqrt{\frac{k_i}{k_i+2}}\bar{\partial}\bar{\phi}_i(\bar{z}) \quad (33)$$

form the adjoint (78) representation of the algebra $E(6)$.

$E(6)$ algebra.

• One can rewrite the E_6 currents in terms of simple roots of E_6

$$\vec{\alpha}_i = e_1 - e_2, ..., \vec{\alpha}_4 = e_4 - e_5, \vec{\alpha}_5 = e_4 + e_5,$$
$$\vec{\alpha}_6 = -\frac{1}{2}(e_1 + ... + e_5) + \frac{\sqrt{3}}{2} e_6, \tag{34}$$

where e_i are the orthonormal basic vectors in $\mathbb{R}^6$.
So that the Cartan subalgebra currents are

$$h_j(\bar{z}) = \imath \vec{\alpha}_j \cdot \bar{\partial} \vec{\bar{H}}(\bar{z}), \; j = 1, ..., 6, \; where$$
$$\vec{\bar{H}}(\bar{z}) = (\bar{H}_1(\bar{z}), ..., \bar{H}_5(\bar{z}), \bar{H}_6(\bar{z})), \tag{35}$$
$$\bar{H}_6(\bar{z}) = \sum_i \sqrt{\frac{k_i}{3(k_i + 2)}} \bar{\phi}_i(\bar{z}),$$

$$E_j(\bar{z}) = \exp{[\imath \vec{\alpha}_j \vec{\bar{H}}]}(\bar{z}),$$
$$F_j(\bar{z}) = \exp{[-\imath \vec{\alpha}_j \vec{\bar{H}}]}(\bar{z}), \; j = 1, ..., 6. \tag{36}$$

27 of $E(6)$ massless multiplet.

The fundamental 27-representation of the group $E(6)$ consists of a 1-scalar $U(1)$-subalgebra and a 10-vector, as well as a 16-antispinor of the $SO(10)$-subalgebra.

$$\begin{aligned}
&\exp\,[\imath \sum_i \frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}]\bar{\Phi}^{\vec{l}}_{-\vec{l}}(\bar{z}),\\
&\exp\,[\imath v^\alpha \bar{H}_\alpha]\bar{\Phi}^{\vec{l}}_{\vec{l}}(\bar{z}),\\
&\exp\,[\imath\dot{\omega}^\alpha \bar{H}_\alpha + \frac{\imath}{2}\sum_i \frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}]\bar{\Phi}^{\vec{l}}_{\vec{l}}(\bar{z}),\\
&\sum_i \frac{l_i}{k_i+2}=1,\;\; v^\alpha=\pm 1,\;\; \sum_\alpha (v^\alpha)^2=1,\\
&\dot{\omega}^\alpha=\pm\frac{1}{2},\;\; \sum_\alpha \dot{\omega}^\alpha = -\frac{1}{2}\;\; mod\;2\mathbb{Z}.
\end{aligned} \tag{37}$$

The number of generations 27 of $E(6)$ is equal to the number h_{21} of the chiral states of the compact factor given by $M_{\vec{k}}$.

$\bar{27}$ of $E(6)$ massless multiplet.

The fundamental $\bar{27}$ representation of the group $E(6)$ splits into a scalar 1, vector 10 and spinor 16 subalgebras $SO(10)$.

$$
\begin{aligned}
&\exp\left[-\imath\sum_i \frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}\right]\bar{\Phi}^{\vec{l}}_{\vec{l}}(\bar{z}),\\
&\exp\left[\imath v^{\alpha}\bar{H}_{\alpha}\right]\bar{\Phi}^{\vec{l}}_{-\vec{l}}(\bar{z}),\\
&\exp\left[\imath\omega^{\alpha}\bar{H}_{\alpha}-\frac{\imath}{2}\sum_i \frac{k_i\bar{\phi}_i}{\sqrt{k_i(k_i+2)}}\right]\bar{\Phi}^{\vec{l}}_{-\vec{l}}(\bar{z}),\\
&\sum_i \frac{l_i}{k_i+2}=1,\ \ v^{\alpha}=\pm 1,\ \sum_{\alpha}(v^{\alpha})^2=1,\\
&\omega^{\alpha}=\pm\frac{1}{2},\ \sum_{\alpha}\omega^{\alpha}=\frac{1}{2}\ mod\ 2\mathbb{Z}.
\end{aligned}
\tag{38}
$$

The number of generations $\bar{27}$ of $E(6)$ is equal to the number h_{11} of the antichiral states of the compact factor given by the $M_{\vec{k}}$.

•$E(6)$ is considered as a possible gauge group for Grand Unification, which after breaking, give rise to $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model.

•The way to get this is namely through breaking $E(6)$ to $SO(10) \times U(1)$, after this $SO(10)$ to $SU(5) \times U(1)$ and at last $SU(5)$ to $SU(3) \times SU(2) \times U(1)$.

•The adjoint 78 of the $E(6)$ breaks into the adjoint 45, spinor 16 and $\bar{16}$, and a singlet 1 of the $SO(10)$ subalgebra.

•The fundamental representation 27 of the $E(6)$ breaks into a spinor 16, a vector 10 and scalar 1 of the the $SO(10)$ subalgebra.

•Quarks and leptons of each generation of the Standard Model can be placed in 16 of one of the 27 representations.

$E(6)$ symmetry and "GSO" equations

• To obtain a set of states which allows the action of $E(6)$, we have to select vertices that are not only mutually local with $SO(10)$ currents , but also with additional $E(6)$ currents.
The currents $\Sigma^{+}_{\omega}(\bar{z})$, $\Sigma^{-}_{\dot{\omega}}(\bar{z})$ are mutually local with the vertex

$$\begin{aligned}\bar{V}(\bar{z}) = P_{gh}(\bar{b},\bar{c})P_{st}(\bar{\partial}\bar{X}^{\mu})P_{int}(\bar{\partial}\bar{Y}^{I},\bar{\partial}\bar{H}^{\alpha},\bar{T}_i,\bar{J}_i,\bar{G}^{\pm}_i)\\ \exp\left[\imath k_I\bar{Y}^{I}+\imath\Lambda_{\alpha}\bar{H}^{\alpha}\right]\bar{\Phi}^{NS,R}_{\vec{l},\vec{\bar{q}}}(\bar{z}),\end{aligned} \qquad (39)$$

if the following "GSO" equations are satisfied

$$\omega\cdot\Lambda+\frac{1}{2}\sum_i\left(\frac{\bar{q}_i}{k_i+2}-\frac{1}{2}\bar{s}^i_L\right)\in\mathbb{Z}. \qquad (40)$$

These "GSO" equations mean that 6-vector

$$\left(\Lambda,\sum_{i=1}^{5}Q^i_R\right),\ where\ Q^i_R=\frac{\bar{q}_i}{k_i+2}-\frac{1}{2}\bar{s}^i \qquad (41)$$

must be an $E(6)$ weight lattice vector.

$E(6)$ symmetry and "GSO" equations

•From the "GSO" equations in right-moving sector we get that the $SO(10)$ parts of the solutions of the right vertices, that fall into one of the four conjugacy classes, determine the sixth, internal component as follows

$$
\begin{aligned}
\vec{\Lambda} \in [0] &\Rightarrow \sum_{\alpha} u^{\alpha}\Lambda^{\alpha} = 0 \; mod \; 1 \Rightarrow \sum_{i} Q_R^i \in 2\mathbb{Z}, \\
\vec{\Lambda} \in [V] &\Rightarrow \sum_{\alpha} u^{\alpha}\Lambda^{\alpha} = \frac{1}{2} \; mod \; 1 \Rightarrow \sum_{i} Q_R^i \in 2\mathbb{Z} + 1, \\
\vec{\Lambda} \in [S] &\Rightarrow \sum_{\alpha} u^{\alpha}\Lambda^{\alpha} = \frac{1}{4} \; mod \; 1 \Rightarrow \sum_{i} Q_R^i \in 2\mathbb{Z} - \frac{1}{2}, \\
\vec{\Lambda} \in [C] &\Rightarrow \sum_{\alpha} u^{\alpha}\Lambda^{\alpha} = -\frac{1}{4} \; mod \; 1 \Rightarrow \sum_{i} Q_L^i \in 2\mathbb{Z} + \frac{1}{2}.
\end{aligned}
\tag{42}
$$

Mutual locality of complete physical vertices

•Let's start the search for mutually local complete vertices as the following "quasi-diagonal" product of GSO-invariant left-moving and "GSO"-invariant right-moving factors

$$\mathcal{V}^{\vec{I}}_{\vec{\mu}_L,\vec{\mu}_R}(z,\bar{z}) = V^{\vec{I}}_{\vec{\mu}_L}(z) \times \bar{V}^{\vec{I}}_{\vec{\mu}_R}(\bar{z}) =$$
$$P^L_{gh}(\beta,\gamma,b,c)P^L_{st}(\partial X^\mu,\partial H_\alpha)\exp\left[q\phi + \imath\lambda^a H_a\right](z)\Phi_{\vec{I},\vec{q}_L,\vec{s}_L}(z)\times$$
$$\times P^R_{gh}(\bar{b},\bar{c})P^R_{st}(\bar{\partial}\bar{X}^\mu)\exp\left[\imath\epsilon^I\bar{Y}_I + \imath\Lambda^a\bar{H}_a\right](\bar{z})\bar{\Phi}_{\vec{I},\vec{q}_R,\vec{s}_R}(\bar{z}), \tag{43}$$

where we imposed the following "quasi-diagonal" relation on the compact factors

$$\vec{q}_L = \vec{q}_R, \vec{s}_L = \vec{s}_R.$$

•The product of two such vertices, after moving one around the other, receives a complex factor whose monodromy phase is

$$2\pi\imath(-qq' + \vec{\lambda}\cdot\vec{\lambda}' - \vec{\Lambda}\cdot\vec{\Lambda}').$$

Mutual locality of the complete physical vertices

•Since we also assumed that the internal charges in the left and right sectors are the same, this leads to certain correlations between the classes $\vec{\lambda}$ and $\vec{\Lambda}$.

•The reason, as has been shown, is that *GSO* equations and the requirement $N = 1$ to permit Virasoro action lead to a correlation between pictures q, classes of $\vec{\lambda}$ and total internal charges in the left-moving sector.

•The same is correct for classes of $\vec{\Lambda}$ and total internal charges in the riht-moving sector because of "*GSO*" equations.

•In result, we get the four types of quasi-diagonal complete vertices that satisfy these requirements.

$$\sum_i Q^i \in 2\mathbb{Z} \Rightarrow q = -1,\ \lambda \in [V],\ \Lambda \in [0],$$
$$\sum_i Q^i \in 2\mathbb{Z}+1 \Rightarrow q = -1,\ \lambda \in [0],\ \Lambda \in [V],$$
$$\sum_i Q^i \in 2\mathbb{Z}+\frac{1}{2} \Rightarrow q = -\frac{1}{2},\ \lambda \in [C],\ \Lambda \in [C], \tag{44}$$
$$\sum_i Q^i \in 2\mathbb{Z}-\frac{1}{2} \Rightarrow q = \frac{1}{2},\ \lambda \in [S],\ \Lambda \in [S].$$

•After moving one of these vertices around the other, a monodromy phase occurs, which is integer because

$$\vec{\mu}_L \cdot \vec{\mu}'_L - \vec{\mu}_R \cdot \vec{\mu}'_R = -qq' + \vec{\lambda}\cdot\vec{\lambda}' - \vec{\Lambda}\cdot\vec{\Lambda}' \in \mathbb{Z}. \tag{45}$$

•Thus, the vertices are mutually local due to the correlation of the internal charges $\sum Q^i$, $SO(1,3)$ weights λ and $SO(10)$ weights Λ.

•As Gepner showed, such replacement singlet and vector $SO(1,3)$ (for left movers) with vector and singlet $SO(10)$ (for right movers) is necessary to ensure modular invariance of the theory.

Space-time supersymmetry, non-diagonal complete vertices and mutual locality

•The consistency with SUSY also requires to admit nondiagonal complete vertices with $\vec{q}_L \neq \vec{q}_R$ and $\vec{s}_L \neq \vec{s}_R$ since the action of SUSY charges on a diagonal vertex produces superpartners of non-diagonal types.

•The superpartners are generated by $\vec{\beta}_0$ shifts $\vec{\mu}_L \to \vec{\mu}_L + n\vec{\beta}_0$, these non-diagonal vertices obviously satisfy GSO equations.

•The deformations generated by $\vec{\beta}_1$,...,$\vec{\beta}_7$ are also consistent with SUSY action.

•It is easy to check the mutual locality of any pair of complete vertices obtained by this way. Indeed the monodromy phase is given by

$$\begin{aligned} &(\vec{\mu}_L - m^i\vec{\beta}_i)\cdot(\tilde{\vec{\mu}}_L - \tilde{m}^j\vec{\beta}_j) - \vec{\mu}_R\cdot\vec{\mu}_R = \\ &\tilde{m}^i m^j\vec{\beta}_i\cdot\vec{\beta}_j - \tilde{m}^j\vec{\beta}_j\cdot\vec{\mu}_L - m^i\vec{\beta}_i\cdot\tilde{\vec{\mu}}_L \in \mathbb{Z}. \end{aligned} \tag{46}$$

Massless supermultiplets and gauge multiplets

•The last relation means that the set of vectors $\vec{w}^* \equiv \vec{q}_R - \vec{w}$ forms a dual admissible group defining a mirror orbifold.

•It can be shown that applying SUSY operators to this mutually local subset of vertices does not violate mutual locality. In this way we generate supermultiplets of mutually local complete vertices.

•Due to the right "GSO" conditions this set of vertices is also compatible with the action $E(8) \times E(6) \times U(1)^4$ gauge group.

•For phenomenological application, massless states of them are most important.

•These include the $N = 1$ supergravity multiplet, the vector multiplet whose gauge fields transform in the adjoint representation of $E(8) \times E(6)$, 27 and $\bar{27}$ multiplets of $E(6)$, as well as singlets of gauge algebras.

27 and $\bar{27}$ multiplets of algebra $E(6)$

•27 multiplets of algebra $E(6)$ which could include h_{11} generations of quarks/leptons of Standard model

$$\exp[-\phi]\Phi^{\vec{l}}_{\vec{l}}(z) \times \exp[\imath \sum_i \frac{k_i \bar{\phi}_i}{\sqrt{k_i(k_i+2)}}]\bar{\Phi}^{\vec{l}}_{-\vec{l}}(\bar{z}),$$

$$\sum_i \frac{l_i}{k_i+2} = 1, \tag{47}$$

$$+ \ (10+16) \ E(6) \ partners + \ superpartners.$$

•$\bar{27}$ multiplets of algebra $E(6)$ which could include h_{21} anti-generations of quarks/leptons of Standard model

$$\exp[-\phi]\Phi^{\vec{l}}_{\vec{l}}(z) \times \exp[-\imath \sum_i \frac{k_i \bar{\phi}_i}{\sqrt{k_i(k_i+2)}}]\bar{\Phi}^{\vec{l}}_{\vec{l}}(\bar{z}),$$

$$\sum_i \frac{l_i}{k_i+2} = 1, \tag{48}$$

$$+ \ (10+16) \ E(6) \ partners + \ superpartners.$$

Singlets in Quintic case, $k_i = 3$

•In the left-moving sector we have massless chiral or anti-chiral scalars.
The right moving sector contains the next set of massless gauge singlets (dark matter candidates!)

$$W^{c}_{\vec{l},i}(\bar{z}) = \bar{G}^{-}_{i,-\frac{1}{2}}\bar{\Phi}^{c}_{\vec{l}}(\bar{z}),\ \sum_i \frac{l_i}{k_i+2} = 1, \quad (49)$$

$\bar{\Phi}^{c}_{\vec{l}}(\bar{z})$ is a chiral primary field.

$$W^{a}_{\vec{l},i}(\bar{z}) = \bar{G}^{+}_{i,-\frac{1}{2}}\bar{\Phi}^{a}_{\vec{l}}(\bar{z}),\ \sum_i \frac{l_i}{k_i+2} = 1, \quad (50)$$

$\bar{\Phi}^{a}_{\vec{l}}(\bar{z})$ is an anti-chiral primary field.

•The products of these left- and right- moving factors yield a set of 310 complete singlet vertices in the Quintic case

$$\begin{aligned} \mathcal{V}^{c,c}_{\vec{l},i}(z,\bar{z}) &= exp(-\phi(z))\,\bar{G}^{-}_{i,-\frac{1}{2}}\Psi^{c,c}_{\vec{l}}(z,\bar{z})\exp\left(\imath p_\mu X^\mu\right), \\ \mathcal{V}^{a,c}_{\vec{l},i}(z,\bar{z}) &= exp(-\phi(z))\,\bar{G}^{-}_{i,-\frac{1}{2}}\Psi^{a,c}_{\vec{l}}(z,\bar{z})\exp\left(\imath p_\mu X^\mu\right). \end{aligned} \quad (51)$$