Poisson Gauge Theory and Symplectic Groupoids

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Outline:

- Motivations: Why Noncommutative Field Theory?
- NFT in semiclassical approximation
- Symplectic groupoids as a kinematical arena for NFT
- Poisson Electrodynamics
- Conclusions

Noncommutative Field Theory in Semiclassical Approximation

Noncommutative Field Theory proceeds from the fundamental hypothesis that spacetime coordinates are noncommuting operators rather than real numbers,

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{\pi}^{\mu\nu} \qquad \Rightarrow \qquad \Delta x^\mu \Delta x^\nu \geq |\pi^{\mu\nu}|$$

[W. Heisenberg, 1938; H. S. Snyder, 1947; . . . N. Seiberg & E. Witten, 1999; . . .]

One can further assume these commutation relations to come from the quantization of classical Poisson brackets on the spacetime manifold X:

$$\{x^\mu, x^\nu\} = \pi^{\mu\nu}(x)$$

Poisson Electrodynamics is the semiclassical/low-energy limit of $U(1)$ Noncommutative Gauge Theory where all commutators are replaced by Poisson brackets. The Lie algebra of infinitesimal gauge transformations is postulated in the form

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\{\varepsilon_1, \varepsilon_2\}}$$

[V. G. Kupriyanov, 2021]

Basic Logic Behind Poisson Electrodynamics

In ordinary electrodynamics, **electromagnetic potentials**

$$A = A_\mu(x)\mathrm{d}x^\mu \in \Lambda^1(X)$$

are sections of the cotangent bundle T^*X of the spacetime manifold X:

$$A : X \to T^*X \,.$$

On the other hand, T^*X is the phase space of a point particle in X. The canonical symplectic structure

$$\omega = \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu$$

allows one to define the **strength tensor** of the electromagnetic field as the pull-back

$$F = \mathrm{d}A = A^*(\omega) \in \Lambda^2(X) \,.$$

Phase Space Over a Non-commutative Spacetime

Q: What is the phase space of a point particle living in a noncommutative space-time (X, π)?

It is certainly not T^*X and the naive Poisson brackets

$$\{x^\mu, x^\mu\} = \pi^{\mu\nu}(x)\,, \qquad \{p_\mu, x^\nu\} = \delta^\nu_\mu\,, \qquad \{p_\mu, p_\nu\} = 0$$

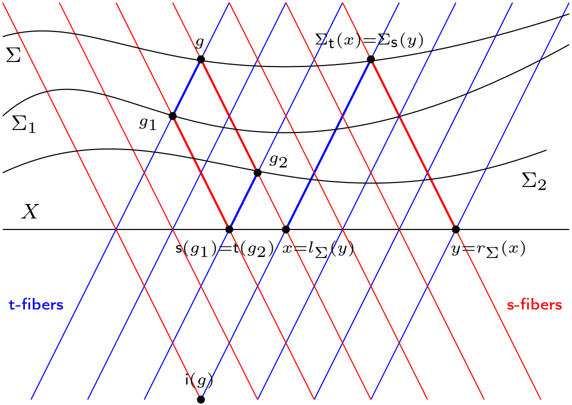
do not work! Mathematically, we need to construct a **symplectic realization** of (X, π), i.e. a symplectic manifold $(\mathcal{G}, \omega)$ together with a Poisson map $p : \mathcal{G} \to X$.

In general, the problem of symplectic realization may have many solutions.

A: $(\mathcal{G}, \omega)$ is a **symplectic groupoid** integrating a given Poisson manifold (X, π).

NB: When a symplectic groupoid exists, it is essentially unique; the corresponding Poisson structure is called **integrable**.

Poisson Groupoids as Phase Spaces



Symplectic groupoid $(\mathcal{G},\omega) \rightrightarrows (X,\pi)$:

- $(\mathcal{G},\omega)$ is a symplectic manifold $(\dim \mathcal{G} = 2\dim X)$
- There are two canonical projections $\mathsf{s},\mathsf{t} : \mathcal{G} \to X$ (source and target):

$$\{\mathsf{s}^* f, \mathsf{s}^* g\}_{\mathcal{G}} = \{f,g\}_X, \ \{\mathsf{t}^* f, \mathsf{t}^* g\}_{\mathcal{G}} = -\{f,g\}_X$$

- The group $\mathscr{B}(\mathcal{G})$ of bisections Σ
- The subgroup $\mathscr{L}(\mathcal{G}) \subset \mathscr{B}(\mathcal{G})$ of Lagrangian bisections $(\omega|_\Sigma = 0)$

The source map $\mathsf{s} : \mathcal{G} \to X$ defines a **symplectic realization** of the Poisson manifold (X,π).

Example: Zero Poisson Brackets

An ordinary(=commutative) spacetime X corresponds to $\pi = 0$. In this case,

$$\mathcal{G} = T^*X \,, \qquad \mathsf{s} = \mathsf{t} = p : T^*X \to X \,,$$

$$\omega = \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu \,.$$

The group of bisections $\mathscr{B}(\mathcal{G}) = \Lambda^1(X)$ is the abelian group with the multiplication law

$$A_1 + A_2 \qquad \forall A_1, A_2 \in \Lambda^1(X) \,.$$

It acts trivially on X.

The subgroup of Lagrangian bisections $\mathscr{L}(\mathcal{G}) \subset \mathscr{B}(\mathcal{G})$ is given by closed 1-forms $A = A_\mu \mathrm{d}x^\mu$:

$$\mathscr{L}(\mathcal{G}) \ni A \qquad \Leftrightarrow \qquad \omega|_{p_\mu = A_\mu} = 0 \qquad \Leftrightarrow \qquad \mathrm{d}A = 0 \,.$$

Example: Linear Poisson Brackets

$$\{x^\mu, x^\nu\} = f^{\mu\nu}_\lambda x^\lambda$$

- f^{ij}_k are structure constants of a Lie algebra $\mathfrak{g}$, so that $x \in \mathfrak{g}^*$.
- $\mathcal{G} = T^*G \simeq G \times \mathfrak{g}^*$, where G is a Lie group integrating $\mathfrak{g}$.
- The source and target maps: $\mathsf{s}(g, x) = x\,, \qquad \mathsf{t}(g, x) = \mathrm{Ad}^*_g x\ .$
- ω is the canonical symplectic structure on T^*G,

$$\omega = \mathrm{d}\theta\,, \qquad \theta = \langle x, g^{-1}\mathrm{d}g\rangle\ .$$

- Bisections: $\Sigma_\mathsf{s} : \mathfrak{g}^* \to G\,, \qquad \Sigma_\mathsf{t} : \mathfrak{g}^* \to G.$

NB: If G is a compact Lie group, then so is the momentum space of the particle !

Postulates of Poisson Electrodynamics

P1: The physical spacetime is a smooth manifold X endowed with a Poisson bivector π.

P2: The phase space of a point charged particle on X is a symplectic groupoid $\mathcal{G} \rightrightarrows X$ integrating the Poisson manifold (X, π).

P3: The configuration space of the electromagnetic field is identified with the group of bisections $\mathscr{B}(\mathcal{G})$ of the symplectic groupoid $\mathcal{G} \rightrightarrows X$.

P4: The gauge group of the electromagnetic field is given by the subgroup of Lagrangian bisections $\mathscr{L}(\mathcal{G}) \subset \mathscr{B}(\mathcal{G})$; the group $\mathscr{L}(\mathcal{G})$ acts on $\mathscr{B}(\mathcal{G})$ by right translations.

[V. G. Kupriyanov, A.A.Sh, and R. J. Szabo, 2024]

Strength Tensors

The **gauge invariant** and **gauge covariant** strength tensors:

$$F^{\mathsf{t}}(\Sigma) = \Sigma_{\mathsf{t}}^{*}\omega\,, \qquad F^{\mathsf{s}}(\Sigma) = \Sigma_{\mathsf{s}}^{*}\omega \qquad \in \Lambda^2(X)\,.$$

Properties:

- $\mathrm{d}F^{\mathsf{s}} = \mathrm{d}F^{\mathsf{t}} = 0$,
- Non-abelian superposition:

$$F^{\mathsf{s}}(\Sigma_1\Sigma_2) = F^{\mathsf{s}}(\Sigma_2) + l_{\Sigma_2}^{*}F^{\mathsf{s}}(\Sigma_1)\,, \qquad F^{\mathsf{t}}(\Sigma_1\Sigma_2) = F^{\mathsf{t}}(\Sigma_1) + r_{\Sigma_1}^{*}F^{\mathsf{t}}(\Sigma_2)\,.$$

  [The generalisation of the usual additivity: $F(A_1 + A_2) = F(A_1) + F(A_2)$.]

- If $\Sigma_2 \in \mathscr{L}(\mathcal{G})$, then $F^{\mathsf{s}}(\Sigma_2) = F^{\mathsf{t}}(\Sigma_2) = 0$ and

$$F^{\mathsf{t}}(\Sigma_1\Sigma_2) = F^{\mathsf{t}}(\Sigma_1)\,, \qquad F^{\mathsf{s}}(\Sigma_1\Sigma_2) = l_{\Sigma_2}^{*}F^{\mathsf{s}}(\Sigma_1)\,.$$

- $F^{\mathsf{s}}(\Sigma) = l_{\Sigma}^{*}F^{\mathsf{t}}(\Sigma)\,, \qquad F^{\mathsf{t}}(\Sigma) = r_{\Sigma}^{*}F^{\mathsf{s}}(\Sigma)\,.$

Action

Gauge invariant and gauge covariant Lagrangians:

$$\mathcal{L}_{\text{inv}} = \mathcal{L}(F^{\mathsf{t}}, g, \pi, \dots) \in \Lambda^{\text{top}}(X) \quad \Leftrightarrow \quad \mathcal{L}_{\text{cov}} = l_\Sigma^* \mathcal{L}_{\text{inv}} = \mathcal{L}(F^{\mathsf{s}}, l_\Sigma^* g, l_\Sigma^* \pi, \dots)$$

Since $l_{\Sigma\Sigma'}^* = l_{\Sigma'}^* l_\Sigma^*$, under the gauge transformations

$$\Sigma \to \Sigma\Sigma' \quad \Rightarrow \quad \mathcal{L}_{\text{cov}} \to l_{\Sigma'}^* \mathcal{L}_{\text{cov}} \qquad \forall \Sigma' \in \mathscr{L}(\mathcal{G})\,.$$

A gauge invariant action:

$$\int_X \mathcal{L}_{\text{cov}} = S_{\text{em}}[\Sigma] = \int_X \mathcal{L}_{\text{inv}}\,.$$

NB: $\mathcal{L}_{\text{cov}}$ is local, while $\mathcal{L}_{\text{inv}}$ is generally nonlocal; they are related by a nonlocal field redefinition.

Minimal Coupling to Matter Fields

Let Φ be a complex field and $\mathcal{L}(\Phi, \partial_\mu \Phi, \ldots)$ be a $U(1)$ invariant Lagrangian.

$$\mathrm{d}F^{\mathsf{t}}(\Sigma) = 0 \qquad \Rightarrow \qquad F^{\mathsf{t}}(\Sigma) = \mathrm{d}\theta(\Sigma)$$

$$F^{\mathsf{t}}(\Sigma\Sigma') = F^{\mathsf{t}}(\Sigma) \qquad \Rightarrow \qquad \theta(\Sigma\Sigma') = \theta(\Sigma) + \mathrm{d}\alpha(\Sigma, \Sigma') \qquad \forall \Sigma' \in \mathscr{L}(\mathcal{G})$$

The minimal coupling is now introduced through the covariant derivative:

$$\partial_\mu \Phi \quad \to \quad D_\mu \Phi = \partial_\mu \Phi + i\theta_\mu(\Sigma)\Phi \,.$$

Then the Lagrangian $\mathcal{L}(\Phi, D_\mu \Phi)$ is invariant under the gauge transformations

$$\Sigma \to \Sigma\Sigma' \,, \qquad \Phi \to e^{-i\alpha(\Sigma,\Sigma')}\Phi \qquad \forall \Sigma' \in \mathscr{L}(\mathcal{G}) \,.$$

In the commutative limit ($\pi \to 0$), this reproduces the minimal coupling and gauge transformations of the ordinary electrodynamics. [A.A.Sh., 2024]

Summary

- In semiclassical/low-energy limit, the noncommutativity of spacetime is controlled by an integrable Poisson structure; the corresponding symplectic groupoid plays the role of the phase space of a point (charged) particle.
- The electromagnetic field is identified with the bisections of the symplectic groupoid; the pull-backs of the symplectic 2-form by bisections give rise to the gauge-invariant and gauge-covariant strength tensors.
- The electromagnetic field admits a minimal coupling to complex matter fields with a good commutative limit.
- In general, the momentum space of a point particle and the target space of the electromagnetic field is a nontrivial (curved) manifold rather than a linear space.