Axiomatic quantum electrodynamics: from causality to convexity of effective action

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ABSTRACT
The theory of the electromagnetic field, specified by the effective action functional, is considered. The causality conditions are imposed in the form of a requirement that the group velocity of propagation of small soft disturbances over an external constant field does not exceed the speed of light in vacuum. It is shown that these conditions lead, in particular, to the conclusion about the positive Gaussian curvature of the surface, which is specified in the local limit by the nonlinear Lagrangian, considered as a function of two invariants of the field

$$S = \int L(z)\mathrm{d}^4 z, \quad L(x) = -\mathfrak{F}(x) + \mathfrak{L}(x)$$

where $\mathfrak{F} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2)$

$\mathfrak{L}(z) = \mathfrak{L}(F_{\alpha\beta}(z))$ depends on invariant combinations of fields and their z-derivatives

In the full theory $\Gamma = \int \mathfrak{L}(z)\mathrm{d}^4 z$ is a nonlocal functional of the field

It carries all information about interaction beetween electromagnetic fields.

Its extremum $\frac{\delta S}{\delta A^{\beta}(y)} = 0$ under the condition $F_{\alpha\beta}(z) = \partial^{\alpha} A_{\beta}(z) - \partial^{\beta} A_{\alpha}(z)$,

yields (nonlinear) equations of motion, whereas higher variation derivatives over vector potentials produce all photon vertices.

In a dynamic theory these objects are subject of calculation

I shall consider the action as *a priory* given and study its properties as these are prescribed by fundamental principles

Local limit of effective action

depends on two field invariants and does not contain their derivatives

$$\mathfrak{L}(z) \quad = \quad \mathfrak{L}(\ F_{\alpha\beta}(z)) \quad = \quad \mathfrak{L}(\mathfrak{F},\mathfrak{G})$$

$$\mathfrak{F} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) \quad , \qquad \mathfrak{G} = - \ \frac{1}{4}\widetilde{F}^{\mu\nu}F_{\mu\nu} = (\mathbf{E}\cdot\mathbf{B}),$$

$$\tilde{F}_{\rho\sigma} = \tfrac{1}{2}\epsilon_{\rho\sigma\lambda\kappa}F^{\lambda\kappa}$$

It governs behavior of soft photons with vanishing momenta

$$k_0 \to 0, \quad \boldsymbol{k} \to 0$$

Causality demands that they should not propagate faster than c=1 against constant $F^{\text{ext}}_{\alpha\beta}$ field background

Group velocities $v_i^{\text{gr}} = \dfrac{\partial k_0}{\partial k_i}, \qquad i = 1, 2, 3$

on mass shells $k_0 = k_0(k_i)$

must be modulo less than unity

Under Lorentz boost, group velocity is added with the velocity **V** of reference frame following the standard relativistic law

$$v'^{\,\mathrm{gr}}_{\|} = v^{\mathrm{gr}}_{\|} \oplus \mathbf{V} \equiv \frac{V + v^{\mathrm{gr}}_{\|}}{1 + V v^{\mathrm{gr}}_{\|}}, \qquad \boldsymbol{v}'^{\,\mathrm{gr}}_{\perp} = \boldsymbol{v}^{\mathrm{gr}}_{\perp} \oplus \mathbf{V} \equiv \frac{\boldsymbol{v}^{\mathrm{gr}}_{\perp}\left(1 - V^2\right)^{1/2}}{1 + V v^{\mathrm{gr}}_{\|}}$$

This allows one to be working in a special frame

A.E. Shabad, Velocity addition and a closed time cycle in Lorentz-noninvariant theories, Teor. Mat. Fiz. 187, 421 (2016) (Theor. and Math. Phys. 187, 813 (2016)), arXiv:1511.08785 (2015)

Photon propagation against the constant background is determined by the second-rank polarization tensor

$$\Pi_{\mu\tau}(x,x') = \left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\tau(x')}\right|_{A=\mathcal{A}_{\rm ext}} \quad \text{where} \quad \Gamma = \int \mathfrak{L}(z)\mathrm{d}^4 z$$

We are communicating the purely geometric result:

Two-dimensional surface, given by the function $Z = \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$

has a nonnegative Hessian

$$\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}^2_{\mathfrak{F}\mathfrak{G}} \geqslant 0.$$

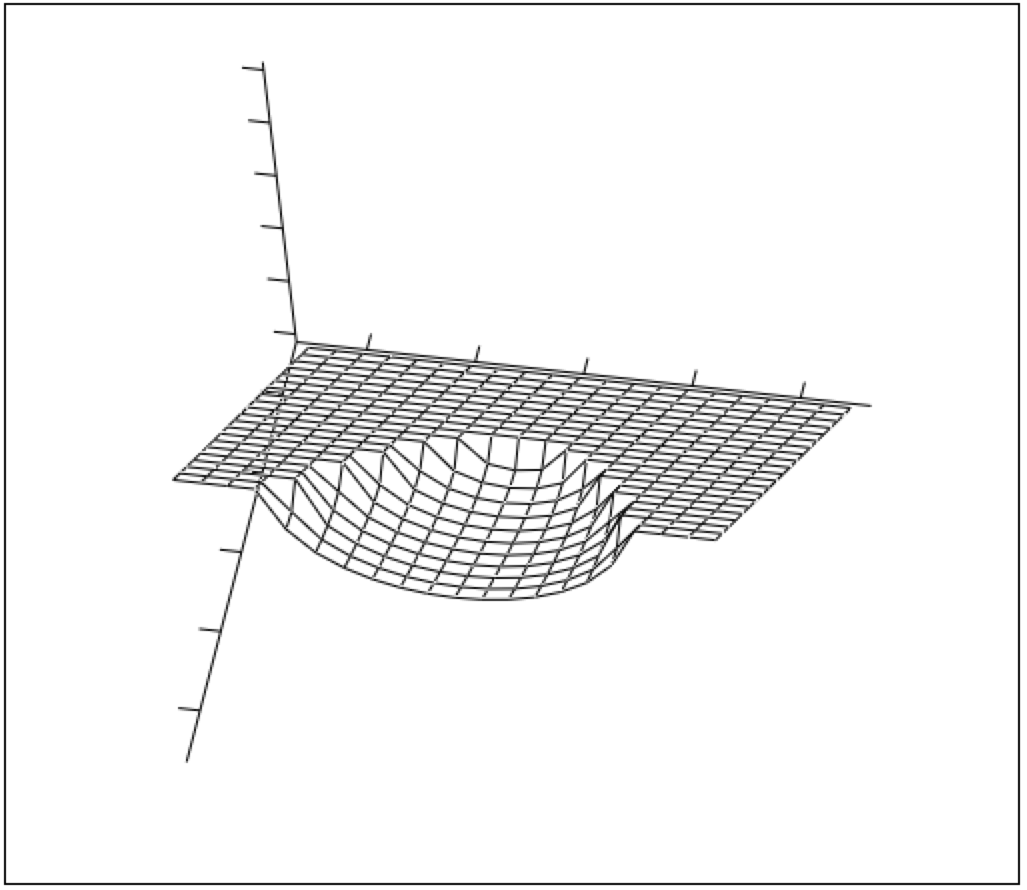
This means that is Gauss curvature is nonnegative.

This surface is tangent to coordinate plane **Z=0** in the origin due to the correspondence principle

$$\mathfrak{L}(0,0) = 0,\ \mathfrak{L}_{\mathfrak{F}}(0,0) = \mathfrak{L}_{\mathfrak{G}}(0,0) = 0$$

More relations are also obtained for field derivatives of Lagrangian.

Example: Born-Infeld Lagrangian



z

$$Z(x,y) := \mathrm{Re}\left[x + 1 - \left(1 + 2 \cdot x - y^2\right)^{\frac{1}{2}}\right]$$

Small deviation from external field

$$A_\mu(x) = \mathcal{A}^{\mathrm{ext}} + a_\mu(x),$$

$$F_{\alpha\beta}(z) = \partial^\alpha A_\beta(z) - \partial^\beta A_\alpha(z)$$

$$F^{\mathrm{ext}}_{\alpha\beta} = \partial^\alpha \mathcal{A}^{\mathrm{ext}}_\beta - \partial^\beta \mathcal{A}^{\mathrm{ext}}_\alpha = Const$$

Linear part of free Maxwell equation

$$[\eta_{\rho\nu}\Box - \partial^\rho \partial^\nu]\, a^\nu(x) + \int d^4x' \Pi_{\rho\nu}(x,x') a^\nu(x') = 0$$

where

$$\Pi_{\mu\tau}(x,x') = \left. \frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\tau(x')} \right|_{A=\mathcal{A}_{\mathrm{ext}}}$$

$$\Gamma = \int \mathfrak{L}(z) \mathrm{d}^4 z$$

Local limit. Lagrangian depends on two field invariants

$$\mathfrak{F} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) \quad , \qquad \mathfrak{G} = \text{ -- } \frac{1}{4} \widetilde{F}^{\mu\nu} F_{\mu\nu} = (\mathbf{E} \cdot \mathbf{B}),$$

$$\tilde{F}_{\rho\sigma} = \tfrac{1}{2} \epsilon_{\rho\sigma\lambda\kappa} F^{\lambda\kappa}$$

$$\frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\tau(y)} = \int \mathrm{d}^4 z \left\{ \frac{\partial \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial \mathfrak{F}(z)} (\eta_{\mu\tau} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\tau}) + \right.$$

$$+ \frac{\partial \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial \mathfrak{G}(z)} \epsilon_{\alpha\mu\beta\tau} +$$

$$+ \frac{\partial^2 \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial (\mathfrak{F}(z))^2} F_{\alpha\mu}(z) F_{\beta\tau}(z) + \frac{\partial^2 \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial (\mathfrak{G}(z))^2} \tilde{F}_{\alpha\mu}(z) \tilde{F}_{\beta\tau}(z) +$$

$$\left. + \frac{\partial^2 \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial \mathfrak{F}(z) \partial \mathfrak{G}(z)} \left[F_{\alpha\mu}(z) \tilde{F}_{\beta\tau}(z) + \tilde{F}_{\alpha\mu}(z) F_{\beta\tau}(z) \right] \right\} \left(\frac{\partial}{\partial z_\alpha} \delta^4(x - z) \right)$$

$$\left(\frac{\partial}{\partial z_\beta} \delta^4(y - z) \right)$$

$$\left(\eta_{\mu\tau}k^2 - k_\mu k_\tau\right) a_\tau(k) + \Pi_{\mu\tau}(k) a_\tau(k) = 0$$

$$k^2 = \mathbf{k}^2 - k_0^2, \qquad F_{\mu\alpha}k_\alpha = (Fk)_\mu$$

$$\begin{aligned}
\Pi_{\mu\tau}(k) &= \Sigma_{i=1}^4 \Psi_{\mu\tau}^{(i)} \Lambda_i \\
\Lambda_1 &= \mathfrak{L}_{\mathfrak{F}},\ \Lambda_2 = \mathfrak{L}_{\mathfrak{F}\mathfrak{F}},\ \Lambda_3 = \mathfrak{L}_{\mathfrak{G}\mathfrak{G}},\ \Lambda_4 = \mathfrak{L}_{\mathfrak{F}\mathfrak{G}} \\
\Psi_{\mu\tau}^{(1)} &= \eta_{\mu\tau}k^2 - k_\mu k_\tau \\
\Psi_{\mu\tau}^{(2)} &= (Fk)_\mu (Fk)_\tau \quad \Psi_{\mu\tau}^{(3)} = \left(\widetilde{F}k\right)_\mu \left(\widetilde{F}k\right)_\tau, \\
\Psi_{\mu\tau}^{(4)} &= (Fk)_\mu \left(\widetilde{F}k\right)_\tau + \left(\widetilde{F}k\right)_\mu (Fk)_\tau
\end{aligned}$$

This polarization tensor is subject of diagonalization

An eigenvector is 4-potential of a normal mode

$$\Pi_{\mu\tau}\ \flat_{\tau}^{(a)} = \varkappa_a\ \flat_{\mu}^{(a)}, \quad a = 1, 2, 3, 4$$

Mode number one is a pure gauge: $\flat_{\mu}^{(4)} = k_{\mu}, \quad \varkappa_4 \quad = \quad 0$

$$a_{\mu}(k) \quad = \quad \Sigma_a^3\ b_{\mu}^{(a)} \mathcal{A}^{(a)}(k)$$

$$\mathcal{A}^{(a)}(k) \quad = \quad \delta\left(k^2 - \varkappa_a(k)\right) w(k)$$

:

Dispersion equations are

$$\varkappa_a(k\tilde{F}^2k, kF^2k, \mathfrak{F}, \mathfrak{G}^2) = k^2, \qquad a = 1, 2, 3.$$

The photon propagator is

$$D_{\mu\tau}(k) = \sum_{a=1}^{4} D_a(k)\, \frac{\flat_\mu^{(a)}\ \flat_\tau^{(a)}}{(\flat^{(a)})^2},$$

$$D_a = \frac{1}{k^2 - \varkappa_a(k)},\ a = 1, 2, 3$$

D_4 Is arbitrary

Eigenvector number one is trivial

$$\flat_\mu^{(1)} = (F^2k)_\mu k^2 - k_\mu(kF^2k) \qquad \varkappa_1 = k^2\mathfrak{L}_{\mathfrak{F}}$$

Remaining two eigenvetors are looked for in the basis of two vectors c+-, such that

$$(c^+c^-) = (c^{\pm}\flat^{(1)}) = (c^{\pm}k) = 0, \quad (c^{\pm})^2 = 1$$

$$c^-_\mu = \frac{\mathcal{B}(Fk)_\mu + \mathcal{E}(\tilde{F}k)_\mu}{(\mathcal{B}^2+\mathcal{E}^2)^{1/2}(k^2\mathcal{E}^2 - kF^2k)^{1/2}}$$

$$c^+_\mu = \mathrm{i}\frac{\mathcal{E}(Fk)_\mu - \mathcal{B}(\tilde{F}k)_\mu}{(\mathcal{B}^2+\mathcal{E}^2)^{1/2}(k^2\mathcal{B}^2 + kF^2k)^{1/2}}$$

where $\mathcal{B} = \sqrt{\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}}, \quad \mathcal{E} = \sqrt{-\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}}$

We have quadratic equation for eigenvalues 2 и 3

$$\varkappa^2 - \varkappa\left(\Pi^{--} + \Pi^{++}\right) - \left(\Pi^{+-}\right)^2 + \Pi^{--}\Pi^{++} = 0$$

wherein $c_\alpha^+ \Pi_{\alpha\beta} c_\beta^+ = \Pi^{++}, \quad c_\alpha^+ \Pi_{\alpha\beta} c_\beta^- = \Pi^{+-}, \quad c_\alpha^- \Pi_{\alpha\beta} c_\beta^- = \Pi^{--}$

Are linear homogeneous functions of invariants kF^2k, k^2 ,
as long as action is local.

Without loss of generality we restrict ourselves to special reference frame where

$$\mathbf{E} \parallel \mathbf{B}$$

$$kF^2k = -B^2k_\perp^2 + E^2(k_3^2 - k_0^2)$$

Dispersion equation

$$\left(k^2\right)^2 - k^2\left(\Pi^{--} + \Pi^{++}\right) - \left(\Pi^{+-}\right)^2 + \Pi^{--}\Pi^{++} = 0$$

L.-h. side is quadratic homogeneous form of

$(k_3^2 - k_0^2)$ и $k_\perp^2$

Quadratic equation for the ratio $\beta = \dfrac{k^2}{k_\perp^2}$

Dispersion equations in the special frame

$$
\begin{aligned}
& \beta^2 \left(1 + \Gamma_2 + \mathfrak{L}_{\mathfrak{F}} \left(\mathfrak{L}_{\mathfrak{F}} - \Gamma_2 + 1\right)\right) - \\
- \quad & \beta \left(\Gamma_2 + \Gamma_1 + \Gamma_2\Gamma_1 - \Gamma_3^2 + \mathfrak{L}_{\mathfrak{F}}\Gamma_2\right) + \Gamma_2\Gamma_1 - \Gamma_3^2 = 0
\end{aligned}
$$

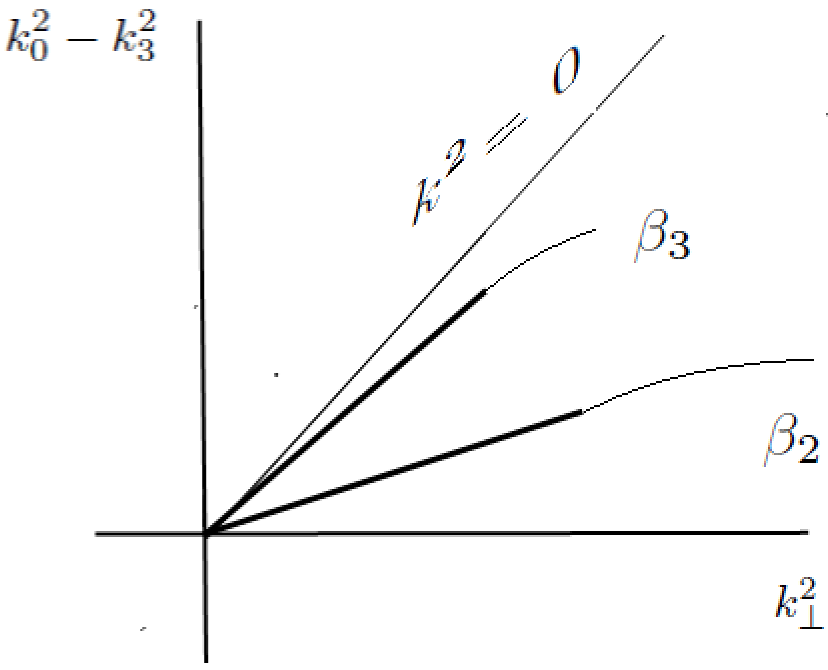
$$
\begin{aligned}
\Gamma_1 &= \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} B^2 + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} E^2 - 2\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}} \\
\Gamma_2 &= \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} E^2 + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} B^2 - 2\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}} \\
\Gamma_3 &= \left(\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\right)\mathfrak{G} - \mathfrak{L}_{\mathfrak{F}\mathfrak{G}} \left(B^2 + E^2\right)
\end{aligned}
$$

$$
\beta = \frac{k^2}{k_\perp^2}
$$

The square trinomial does not contain k,

Both dispersion curves are straight lines

Causality conditions $1 \geqslant \beta \geqslant 0$



$$\beta = \frac{k^2}{k_\perp^2}$$

Slope of dispersion curves $(1-\beta)$

Dispersion curves : $k_0^2 = k_3^2 + k_\perp^2 (1-\beta)$

Group velocity $\mathbf{v}_{gr} = \left(\frac{\partial k_0}{\partial k_3}, \frac{\partial k_0}{\partial k_\perp} \right)$

Causality conditions $\mathbf{v}_{gr}^2 = \frac{k_3^2 + k_\perp^2 (1-\beta)^2}{k_0^2} \leqslant 1$

$$\beta^2 (1 - \mathfrak{L}_{\mathfrak{F}}) (1 - \mathfrak{L}_{\mathfrak{F}} + \Gamma_2) - \beta[(1 - \mathfrak{L}_{\mathfrak{F}}) (\Gamma_2 + \Gamma_1) + \Gamma_2 \Gamma_1 - \Gamma_3^2] + \Gamma_2 \Gamma_1 - \Gamma_3^2 = 0$$

Two conditions for positivity of both roots

As long as $(1 - \mathfrak{L}_{\mathfrak{F}}) (1 - \mathfrak{L}_{\mathfrak{F}} + \Gamma_2)$ > 0, it is required

i) that $\Gamma_2 \Gamma_1 - \Gamma_3^2$ > 0 (product of roots) is positive

$$\Gamma_2 \Gamma_1 - \Gamma_3^2 = \left(E^2 - B^2\right)^2 \left(\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{G}}^2\right) > 0$$

<u>Positivity of Hessian = Positivity of Gauss curvature</u>

ii) that $(1 - \mathfrak{L}_{\mathfrak{F}}) (\Gamma_2 + \Gamma_1) + \Gamma_2 \Gamma_1 - \Gamma_3^2$ >0 (sum of roots is positive

$$(1 - \mathfrak{L}_{\mathfrak{F}}) \left[(\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}) \left(E^2 + B^2\right) - 4\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}\right] + \\ + \left(E^2 - B^2\right)^2 \left(\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{G}}^2\right) > 0$$

In Born-Infeld model it holds: $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} > 0, \quad -4\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}} > 0$


Thanks