Hamiltonian formalism for hard and soft excitations in a plasma with non-Abelian interaction

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At present, a certain interest is shown in the construction of kinetic description of the new fundamental state of matter: the quark–gluon plasma that consists of asymptotically free quarks, antiquarks, and gluons, which is probably formed during ultrarelativistic heavy ion collisions in the running experimental programs at relativistic heavy ion colliders in USA (RHIC) and in Europe (LHC, CERN). Besides ultrarelativistic hot and dense matter requires its understanding in many problems related to cosmology of the early Universe and astrophysics of compact stars.

In this work, we enlarge the Hamiltonian analysis of dynamics of fermion and boson excitations in the hot QCD-medium at the soft momentum scale carried out in Yu.A. Markov *et al.* (JETP (2020); Int. Mod. Phys. A (2023)) to the hard sector of the quark-gluon plasma excitations (Yu.A. Markov *et al.* Nucl. Phys. A (2024)). Here, we focus our research on the study of the scattering processes of soft boson plasma waves off a hard particle within the real time formalism based on kinetic equations for soft modes.

For this purpose the classical Hamiltonian formalism for systems with distributed parameters is used. It has been systematically developed by V.E. Zakharov (Radiophys. Quantum Electron. (1974)) and presented in detail with many examples of concrete physical systems in comprehensive reviews and in the monograph.

Plan of the talk

- The Lie-Poisson bracket
- Interaction Hamiltonian of plasmons and a hard particle
- Canonical transformation
- Eliminating "nonessential" Hamiltonian $H^{(3)}$
- Effective fourth-order Hamiltonian $\mathcal{H}^{(4)}$
- Kinetic equations for soft gluon excitations
- Equation for the expected value of color charge $\langle \mathcal{Q}^a \rangle$
- Interaction of infinitely narrow packets
- Constructing the exact solution
- Lambert W-function. Time parameterization
- Conclusion

Let us consider the gauge field potential $A^a_\mu(x)$ in the form of the decomposition into plane waves

$$A^a_\mu(x) = \int d\mathbf{k} \left(\frac{Z_l(\mathbf{k})}{2\omega^l_{\mathbf{k}}}\right)^{1/2} \left\{ \epsilon^l_\mu(\mathbf{k})\, a^a_{\mathbf{k}}\, e^{-i\omega^l_{\mathbf{k}} t + i\mathbf{k}\cdot\mathbf{x}} + \epsilon^{*\,l}_\mu(\mathbf{k})\, a^{*\,a}_{\mathbf{k}}\, e^{i\omega^l_{\mathbf{k}} t - i\mathbf{k}\cdot\mathbf{x}} \right\},$$

where $\epsilon^l_\mu(\mathbf{k})$ is the polarization vector of a longitudinal mode ($\mathbf{k}$ is the wave vector) and $\omega^l_{\mathbf{k}}$ is the dispersion relation of the longitudinal mode. We consider the amplitude $a^a_{\mathbf{k}}$ for longitudinal excitations as ordinary (complex) random function. The expectation value of the product of two bosonic amplitudes is

$$\langle a^{*\,a}_{\mathbf{k}}\, a^{a'}_{\mathbf{k}'} \rangle = \mathcal{N}^{aa'}_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}'), \tag{1}$$

where $\mathcal{N}^{aa'}_{\mathbf{k}}$ is the number density of the longitudinal plasma waves. The color indices $a,\, b,\, c,\, \ldots$ run through values $1, 2, \ldots, N_c^2 - 1$ for color $SU(N_c)$ group. For the case of continuous media, we take the following expression as the definition of the Lie-Poisson bracket: (N. Linden *et al.* (1995)):

$$\{F, G\} \equiv \int d\mathbf{k}' \left\{ \frac{\delta F}{\delta a^c_{\mathbf{k}'}} \frac{\delta G}{\delta a^{*c}_{\mathbf{k}'}} - \frac{\delta F}{\delta a^{*c}_{\mathbf{k}'}} \frac{\delta G}{\delta a^c_{\mathbf{k}'}} \right\} + i\, \frac{\partial F}{\partial Q^a} \frac{\partial G}{\partial Q^b}\, f^{abc} Q^c.$$

Here, $Q^a = Q^a(t)$ is the color charge of a hard test particle.

The first term is the standard canonical bracket. In the second term f^{abc} are antisymmetric structure constants of the Lie algebra $\mathfrak{su}(N_c)$. The Hamilton equations for the functions $a^a_{\mathbf{k}}$, $a^{*a}_{\mathbf{k}}$ and Q^a have, correspondingly, the form

$$\frac{\partial a^a_{\mathbf{k}}}{\partial t} = -i\left\{a^a_{\mathbf{k}}, H\right\} \equiv -i\,\frac{\delta H}{\delta a^{*a}_{\mathbf{k}}}, \qquad \frac{\partial a^{*a}_{\mathbf{k}}}{\partial t} = -i\left\{a^{*a}_{\mathbf{k}}, H\right\} \equiv i\,\frac{\delta H}{\delta a^a_{\mathbf{k}}}, \tag{2}$$

$$\frac{dQ^a}{dt} = -i\left\{Q^a, H\right\} = \frac{\partial H}{\partial Q^b}\, f^{abc}Q^c, \quad Q^a|_{t=t_0} = Q^a_0. \tag{3}$$

Here, the function $H = H^{(0)} + H_{int}$ represents a Hamiltonian for the system of plasmons and a hard test particle, $H^{(0)}$ is the Hamiltonian of noninteracting plasmons:

$$H^{(0)} = \int d\mathbf{k}\,(\omega^l_{\mathbf{k}} - \mathbf{v}\cdot\mathbf{k})\, a^{*a}_{\mathbf{k}} a^a_{\mathbf{k}},$$

H_{int} is the interaction Hamiltonian of plasmons and the hard color-charged particle. In the approximation of small amplitudes the interaction Hamiltonian H_{int} can be presented in the form of a formal integro-power series in the bosonic functions $a^a_{\mathbf{k}}$ and $a^{*a}_{\mathbf{k}}$ and in the color charge Q^a:

$$H_{int} = H^{(3)} + H^{(4)} + \dots ,$$

where the third-order interaction Hamiltonian has the following structure:

$$H^{(3)} = \int d\mathbf{k} \left[\phi_{\mathbf{k}}\, a^{a}_{\mathbf{k}}\, Q^{a} + \phi^{*}_{\mathbf{k}}\, a^{*\,a}_{\mathbf{k}}\, Q^{a} \right] \tag{4}$$

$$+ \int d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 \left\{ \mathcal{V}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\, a^{*\,a}_{\mathbf{k}}\, a^{a_1}_{\mathbf{k}_1}\, a^{a_2}_{\mathbf{k}_2} + \mathcal{V}^{\,*\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\, a^{a}_{\mathbf{k}}\, a^{*\,a_1}_{\mathbf{k}_1}\, a^{*\,a_2}_{\mathbf{k}_2} \right\} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$$

$$+ \frac{1}{3} \int d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 \left\{ \mathcal{U}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\, a^{a}_{\mathbf{k}}\, a^{a_1}_{\mathbf{k}_1}\, a^{a_2}_{\mathbf{k}_2} + \mathcal{U}^{\,*\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\, a^{*\,a}_{\mathbf{k}}\, a^{*\,a_1}_{\mathbf{k}_1}\, a^{*\,a_2}_{\mathbf{k}_2} \right\} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)$$

and, correspondingly, the fourth-order interaction Hamiltonian is

$$H^{(4)} = \frac{1}{2} \int d\mathbf{k}\, d\mathbf{k}_1 \left\{ T^{*\,(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}\, a^{a}_{\mathbf{k}}\, a^{a_1}_{\mathbf{k}_1}\, Q^{a_2} + T^{(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}\, a^{*\,a}_{\mathbf{k}}\, a^{*\,a_1}_{\mathbf{k}_1} Q^{a_2} \right\} +$$

$$+\, i \int d\mathbf{k}\, d\mathbf{k}_1\, T^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}\, a^{*\,a}_{\mathbf{k}}\, a^{a_1}_{\mathbf{k}_1}\, Q^{a_2}. \tag{5}$$

We assign to the color charge Q^{a} a degree of nonlinearity of *two*. The vertex functions $\phi_{\mathbf{k}}, \mathcal{V}^{\,a\,a_1\,a_2}_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}, \mathcal{U}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\mathbf{k}_2}$ determine the interaction of a classical color-charged particle with an external gauge field $A^{a}_{\mu}(x)$ and the processes of three-plasmon interaction.

Let us consider the transformation from the normal boson variable $a^{a}_{\mathbf{k}}$ and classical color charge Q^{a} to the new field variable $c^{a}_{\mathbf{k}}$ and color charge $\mathcal{Q}^{a}$:

$$a^{a}_{\mathbf{k}} = a^{a}_{\mathbf{k}}(c^{a}_{\mathbf{k}},\, c^{*\,a}_{\mathbf{k}},\, \mathcal{Q}^{a}), \qquad Q^{\,a} = Q^{\,a}(c^{a}_{\mathbf{k}},\, c^{*\,a}_{\mathbf{k}},\, \mathcal{Q}^{a}). \tag{6}$$

We will demand that the Hamilton equations in terms of the new variables have the form (2) and (3) with the same Hamiltonian H. Straightforward but rather cumbersome calculations result in two systems of integral relations. The first of them has the following form:

$$\int d\mathbf{k}' \left\{ \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{c}_{\mathbf{k}'}} \frac{\delta a^{*b}_{\mathbf{k}''}}{\delta c^{*c}_{\mathbf{k}'}} - \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{*c}_{\mathbf{k}'}} \frac{\delta a^{*b}_{\mathbf{k}''}}{\delta c^{c}_{\mathbf{k}'}} \right\} + i\, \frac{\partial a^{a}_{\mathbf{k}}}{\partial \mathcal{Q}^{c}} \frac{\partial a^{*b}_{\mathbf{k}''}}{\partial \mathcal{Q}^{c'}}\, f^{\,cc'd} \mathcal{Q}^{d} = \delta^{ab} \delta(\mathbf{k} - \mathbf{k}''),$$

$$\int d\mathbf{k}' \left\{ \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{c}_{\mathbf{k}'}} \frac{\delta a^{b}_{\mathbf{k}''}}{\delta c^{*c}_{\mathbf{k}'}} - \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{*c}_{\mathbf{k}'}} \frac{\delta a^{b}_{\mathbf{k}''}}{\delta c^{c}_{\mathbf{k}'}} \right\} + i\, \frac{\partial a^{a}_{\mathbf{k}}}{\partial \mathcal{Q}^{c}} \frac{\partial a^{b}_{\mathbf{k}''}}{\partial \mathcal{Q}^{c'}}\, f^{\,cc'd} \mathcal{Q}^{d} = 0, \tag{7}$$

$$\int d\mathbf{k}' \left\{ \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{c}_{\mathbf{k}'}} \frac{\delta Q^{\,b}}{\delta c^{*c}_{\mathbf{k}'}} - \frac{\delta a^{a}_{\mathbf{k}}}{\delta c^{*c}_{\mathbf{k}'}} \frac{\delta Q^{\,b}}{\delta c^{c}_{\mathbf{k}'}} \right\} + i\, \frac{\partial a^{a}_{\mathbf{k}}}{\partial \mathcal{Q}^{c}} \frac{\partial Q^{\,b}}{\partial \mathcal{Q}^{c'}}\, f^{\,cc'd} \mathcal{Q}^{d} = 0.$$

The second system is written in a similar way.

Let us present the canonical transformations (6) in the form of integro-power series in normal variable $c^a_{\mathbf{k}}$ and in color charge $\mathcal{Q}^a$. In this case the first transformation in (6) up to the terms of the sixth order in $c^a_{\mathbf{k}}$ and $\mathcal{Q}^a$ has the following form:

$$a^a_{\mathbf{k}} = c^a_{\mathbf{k}} + F_{\mathbf{k}}\,\mathcal{Q}^a + \int d\mathbf{k}_1 \Big[\widetilde{V}^{(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}\,c^{*\,a_1}_{\mathbf{k}_1}\,\mathcal{Q}^{a_2} + \widetilde{V}^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}\,c^{a_1}_{\mathbf{k}_1}\,\mathcal{Q}^{a_2}\Big] + \tag{8}$$

$$+\int d\mathbf{k}_1\,d\mathbf{k}_2\Big[V^{(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\,c^{a_1}_{\mathbf{k}_1}\,c^{a_2}_{\mathbf{k}_2} + V^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\,c^{*\,a_1}_{\mathbf{k}_1}\,c^{a_2}_{\mathbf{k}_2} + V^{(3)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}\,c^{*\,a_1}_{\mathbf{k}_1}\,c^{*\,a_2}_{\mathbf{k}_2}\Big] +$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2\Big[W^{(1)aa_1a_2a_3}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} c^{a_1}_{\mathbf{k}_1} c^{a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3} + W^{(2)aa_1a_2a_3}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} c^{*a_1}_{\mathbf{k}_1} c^{a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3} + W^{(3)aa_1a_2a_3}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} c^{*a_1}_{\mathbf{k}_1} c^{*a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3}\Big].$$

Similarly, the most common power-series expansion for the transformation (6) up to the terms of the sixth order is

$$Q^a = \mathcal{Q}^a + \int d\mathbf{k}_1\,\Big[M^{\,a\,a_1\,a_2}_{\mathbf{k}_1}\,c^{a_1}_{\mathbf{k}_1}\,\mathcal{Q}^{a_2} + M^{*\,a\,a_1\,a_2}_{\mathbf{k}_1}\,c^{*\,a_1}_{\mathbf{k}_1}\,\mathcal{Q}^{a_2}\Big] \tag{9}$$

$$+\int d\mathbf{k}_1\,d\mathbf{k}_2\Big[M^{(1)\,a\,a_1\,a_2\,a_3}_{\mathbf{k}_1,\,\mathbf{k}_2}\,c^{a_1}_{\mathbf{k}_1}\,c^{a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3} + M^{(2)\,a\,a_1\,a_2\,a_3}_{\mathbf{k}_1,\,\mathbf{k}_2}\,c^{*\,a_1}_{\mathbf{k}_1}\,c^{a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3}$$

$$+\,M^{*\,(1)\,a\,a_1\,a_2\,a_3}_{\mathbf{k}_1,\,\mathbf{k}_2}\,c^{*\,a_1}_{\mathbf{k}_1}\,c^{*\,a_2}_{\mathbf{k}_2}\,\mathcal{Q}^{a_3}\Big] + \ldots\,.$$

Substituting the expansions (8) and (9) into a system of the canonicity conditions (7), we obtain rather nontrivial integral relations connecting various coefficient functions among themselves.

Here, we have provided only algebraic relations for the lowest second-order coefficient functions:

$$V^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} = -2V^{*\,(1)\,a_2\,a_1\,a}_{\mathbf{k}_2,\,\mathbf{k}_1,\,\mathbf{k}}, \quad V^{(3)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} = V^{(3)\,a_1\,a\,a_2}_{\mathbf{k}_1,\,\mathbf{k},\,\mathbf{k}_2}, \qquad (10)$$

$$M^{\,a\,a_1\,a_2}_{\mathbf{k}} + iF^{*}_{\mathbf{k}} f^{\,a\,a_1\,a_2} = 0, \qquad (11)$$

$$\widetilde{V}^{(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1} - \widetilde{V}^{(1)\,a_1\,a\,a_2}_{\mathbf{k}_1,\,\mathbf{k}} - iF_{\mathbf{k}}F_{\mathbf{k}_1} f^{\,a\,a_1\,a_2} = 0, \qquad (12)$$

$$\widetilde{V}^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1} + \widetilde{V}^{*\,(2)\,a_1\,a\,a_2}_{\mathbf{k}_1,\,\mathbf{k}} + iF_{\mathbf{k}}F^{*}_{\mathbf{k}_1} f^{\,a\,a_1\,a_2} = 0. \qquad (13)$$

The next step in constructing the effective theory is the procedure of eliminating the third-order interaction Hamiltonian $H^{(3)}$, Eq. (4), upon switching from the original bosonic function $a^a_{\mathbf{k}}$ and the color charge Q^a to the new function $c^a_{\mathbf{k}}$ and color charge $\mathcal{Q}^a$ as a result of the canonical transformations.

To eliminate the third-order interaction Hamiltonian $H^{(3)}$, we substitute the expansions (8) and (9) into the free-field Hamiltonian $H^{(0)}$ and keep only the terms that have a degree of nonlinearity of two or three in the new variables $c^{\,a}_{\mathbf{k}}$ and $\mathcal{Q}^a$. Then in the third-order Hamiltonian $H^{(3)}$, Eq. (4), we perform the simple replacement of variables: $a^a_{\mathbf{k}} \rightarrow c^a_{\mathbf{k}}$ and $Q^a \rightarrow \mathcal{Q}^a$. We add the expression thus obtained to the expression that follows from the free-field Hamiltonian $H^{(0)}$, and collect similar terms.

Eliminating "nonessential" Hamiltonian $H^{(3)}$

From the requirement of excluding third-order terms in the Hamiltonian $H^{(3)}$, Eq. (4) containing the vertex functions $\phi_{\mathbf{k}}$ and $\phi^{*}_{\mathbf{k}}$, we obtain an explicit form of the coefficient function $F_{\mathbf{k}}$ in the canonical transformation of the normal variable $a^{a}_{\mathbf{k}}$, Eq. (8), in terms of the vertex function $\phi_{\mathbf{k}}$:

$$F_{\mathbf{k}} = -\frac{\phi^{*}_{\mathbf{k}}}{\omega^{l}_{\mathbf{k}} - \mathbf{v}\cdot\mathbf{k}}\,. \tag{14}$$

The relation (14) has a meaning due to the absence of linear Landau damping. Making use of (14), from (11) we immediately find an explicit form of the coefficient function $M^{\,a\,a_1\,a_2}_{\mathbf{k}}$ entering into the canonical transformation of color charge Q^{a}:

$$M^{\,a\,a_1\,a_2}_{\mathbf{k}} = i f^{\,a\,a_1\,a_2}\,\frac{\phi_{\mathbf{k}}}{\omega^{l}_{\mathbf{k}} - \mathbf{v}\cdot\mathbf{k}}. \tag{15}$$

Furthermore, the requirement to exclude third-order terms in the Hamiltonian $H^{(3)}$, Eq. (4), containing the vertex functions $\mathcal{V}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\mathbf{k}_2}$ and $\mathcal{U}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}$, leads to the already known expressions (Yu. Markov *et al.* (2020)) for the coefficient functions $V^{\,(1,3)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}$ in the canonical transformation for $a^{a}_{\mathbf{k}}$:

$$\begin{aligned} V^{\,(1)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} &= -\frac{\mathcal{V}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}}{\omega^{l}_{\mathbf{k}} - \omega^{l}_{\mathbf{k}_1} - \omega^{l}_{\mathbf{k}_2}}\,\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2),\\ V^{\,(3)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2} &= -\frac{\mathcal{U}^{*\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}}{\omega^{l}_{\mathbf{k}} + \omega^{l}_{\mathbf{k}_1} + \omega^{l}_{\mathbf{k}_2}}\,\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2). \end{aligned} \tag{16}$$

4. Effective fourth-order Hamiltonian $\mathcal{H}^{(4)}$

After eliminating the Hamiltonian $H^{(3)}$ we obtain the effective fourth-order Hamiltonian $\mathcal{H}^{(4)}$ describing the elastic scattering process of plasmon off a hard color-charged particle:

$$\mathcal{H}^{(4)} = i\int d\mathbf{k}_1 d\mathbf{k}_2\, \mathcal{T}^{\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2}\, c^{*\,a_1}_{\mathbf{k}_1} c^{\,a_2}_{\mathbf{k}_2}\, \mathcal{Q}^{\,a}. \tag{17}$$

Here the *complete effective amplitude* $\mathcal{T}^{\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2}$ can be represented as the sum of two contributions

$$\mathcal{T}^{\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} = -i\left[\omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1 - \mathbf{k}_2)\right] \widetilde{V}^{\,(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2} + \widetilde{T}^{\,(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}, \tag{18}$$

where, in turn, the effective amplitude $\widetilde{T}^{\,(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$ has the following structure:

$$\widetilde{T}^{\,(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2} = T^{\,(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} \; + \tag{19}$$

$$+ f^{\,a_1\,a_2\,a}\left\{\frac{\phi^{\,*}_{\mathbf{k}_1}\phi_{\,\mathbf{k}_2}}{\omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot\mathbf{k}_2} + 2i\left(\frac{\mathcal{V}_{\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_1-\mathbf{k}_2}\,\phi^{\,*}_{\mathbf{k}_1-\mathbf{k}_2}}{\omega^l_{\mathbf{k}_1-\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1-\mathbf{k}_2)} - \frac{\mathcal{V}^{\,*}_{\mathbf{k}_2,\,\mathbf{k}_1,\,\mathbf{k}_2-\mathbf{k}_1}\,\phi_{\,\mathbf{k}_2-\mathbf{k}_1}}{\omega^l_{\mathbf{k}_2-\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k}_2-\mathbf{k}_1)}\right)\right\}.$$

The first term on the right-hand side of (18) has the resonance factor

$$\Delta\omega_{\mathbf{k}_1,\mathbf{k}_2} \equiv \omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1 - \mathbf{k}_2),$$

which in fact represents a consequence of the momentum and energy conservation laws in the scattering process under investigation. In the case of $\Delta\omega_{\mathbf{k}_1,\mathbf{k}_2} \neq 0$ the problem of determining the coefficient function $\widetilde{V}^{\,(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$ arises.

Effective fourth-order Hamiltonian $\mathcal{H}^{(4)}$

Figure 1 gives the diagrammatic interpretation of different terms in curly brackets in the effective amplitude (19).

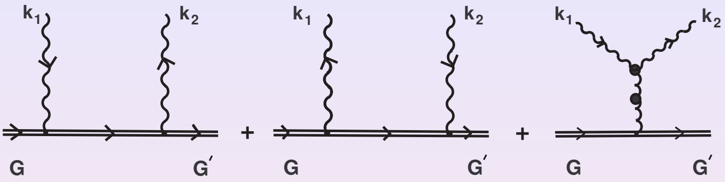

Рис.: The effective amplitude $\widetilde{T}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$ for the elastic scattering process of plasmon off a hard color particle. The blob stands for HTL resummation and the double line denotes the hard particle

The first two graphs represent the Compton scattering of soft boson excitations off a hard test particle induced by the first term in curly brackets of the expression (19). The incoming and outgoing wave lines in fig. 1 correspond to the normal variables $c^{a_1}_{\mathbf{k}_1}$ and $c^{*\,a_2}_{\mathbf{k}_2}$, respectively, and the horizontal double line between two interaction vertices corresponds to the "propagator" of the hard particle

$$1/(\omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot\mathbf{k}_1).$$

The interaction vertices correspond to the functions $\phi^*_{\mathbf{k}_1}$ or $\phi_{\mathbf{k}_2}$. The remaining graph is connected with the interaction of hard particle with plasmons through the three-plasmon vertex function $\mathcal{V}^{\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1,\,\mathbf{k}_2}$ with intermediate "virtual" oscillation to which the factor $1/(\omega^l_{\mathbf{k}_1-\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1-\mathbf{k}_2))$ in (19) corresponds.

Note that this factor can also be written in a slightly different form

$$\frac{1}{\omega^l_{\mathbf{k}_1-\mathbf{k}_2} - \omega^l_{\mathbf{k}_1} + \omega^l_{\mathbf{k}_2}},$$

if the resonance frequency difference $\Delta\omega_{\mathbf{k}_1,\mathbf{k}_2}$ is exactly zero. The last expression represents (up to a multiplier) an approximation of the effective (retarded) gluon propagator ${}^*\widetilde{\mathcal{D}}_{\mu\nu}(k)$ at the plasmon pole $\omega \sim \omega^l_{\mathbf{k}}$.

Finally, the first term $T^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2}$ on the right-hand side of (19) must be associated with the process of direct interaction of two plasmons with a hard particle, as shown in fig. 2.

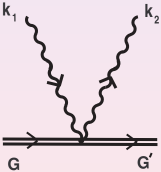

Рис.: Direct interaction of two plasmons with a hard particle

In the particular physical system under consideration, such interaction is forbidden, and therefore we should assume

$$T^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} \equiv 0.$$

5. The coefficient function $\widetilde{V}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$

The coefficient function $\widetilde{V}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$ must satisfy the canonicity condition (13):

$$\widetilde{V}^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} + \widetilde{V}^{*\,(2)\,a\,a_2\,a_1}_{\mathbf{k}_2,\,\mathbf{k}_1} = -i\,f^{\,a\,a_1\,a_2} F_{\mathbf{k}_1} F^{*}_{\mathbf{k}_2}$$

or, if we factorize color and momentum dependence $\widetilde{V}^{(2)aa_1a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} = f^{\,aa_1a_2}\,\widetilde{V}^{(2)}_{\mathbf{k}_1,\mathbf{k}_2}$:

$$\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2} - \widetilde{V}^{*\,(2)}_{\mathbf{k}_2,\,\mathbf{k}_1} = -i\,F_{\mathbf{k}_1} F^{*}_{\mathbf{k}_2}. \tag{20}$$

The last relation can be considered as a functional equation for the coefficient function $\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}$. We can find its solution by writing down the general solution of the associated homogeneous equation and the particular solution of the nonhomogeneous one.

Searching for a partial solution in the form $\alpha\,F_{\mathbf{k}_1} F^{*}_{\mathbf{k}_2}$, where α is some complex number and passing then from the coefficient function $F_{\mathbf{k}}$ to the vertex function $\phi_{\mathbf{k}}$ by the rule (14), we get from (20)

$$\Big(\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}\Big)_{\rm inhom} = \left(\mathrm{Re}\,\alpha - \frac{i}{2}\right) \frac{\phi^{*}_{\mathbf{k}_1}\,\phi_{\mathbf{k}_2}}{\big(\omega^{l}_{\mathbf{k}_1} - \mathbf{v}\cdot\mathbf{k}_1\big)\big(\omega^{l}_{\mathbf{k}_2} - \mathbf{v}\cdot\mathbf{k}_2\big)},$$

where $\mathrm{Re}\,\alpha$ is an arbitrary numerical parameter.

The coefficient function $\widetilde{V}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$

We take the general solution of the homogeneous equation in the following form:

$$\left(\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}\right)_{\rm hom} = \Lambda^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2} + \frac{\mathcal{V}_{\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_1-\mathbf{k}_2}\,\phi^{*}_{\mathbf{k}_1-\mathbf{k}_2}}{\left(\omega^l_{\mathbf{k}_1}-\omega^l_{\mathbf{k}_2}-\omega^l_{\mathbf{k}_1-\mathbf{k}_2}\right)\left(\omega^l_{\mathbf{k}_1-\mathbf{k}_2}-\mathbf{v}\cdot(\mathbf{k}_1-\mathbf{k}_2)\right)}$$

$$+\frac{\mathcal{V}^{*}_{\mathbf{k}_2,\,\mathbf{k}_1,\,\mathbf{k}_2-\mathbf{k}_1}\,\phi_{\mathbf{k}_2-\mathbf{k}_1}}{\left(\omega^l_{\mathbf{k}_2}-\omega^l_{\mathbf{k}_1}-\omega^l_{\mathbf{k}_2-\mathbf{k}_1}\right)\left(\omega^l_{\mathbf{k}_2-\mathbf{k}_1}-\mathbf{v}\cdot(\mathbf{k}_2-\mathbf{k}_1)\right)},$$

where $\Lambda^{(2)}_{\mathbf{k}_1,\mathbf{k}_2}$ is an arbitrary function satisfying the condition: $\Lambda^{(2)}_{\mathbf{k}_1,\mathbf{k}_2}=\Lambda^{*(2)}_{\mathbf{k}_2,\mathbf{k}_1}$.
Summing the partial and general solutions and assuming for the sake of definiteness

$$\mathrm{Re}\,\alpha\equiv 0,\qquad \Lambda^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}\equiv 0,$$

we obtain the required function:

$$\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2} = -\frac{i}{2}\,\frac{\phi^{*}_{\mathbf{k}_1}\,\phi_{\mathbf{k}_2}}{\left(\omega^l_{\mathbf{k}_1}-\mathbf{v}\cdot\mathbf{k}_1\right)\left(\omega^l_{\mathbf{k}_2}-\mathbf{v}\cdot\mathbf{k}_2\right)}+ \tag{21}$$

$$+\frac{\mathcal{V}_{\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_1-\mathbf{k}_2}\,\phi^{*}_{\mathbf{k}_1-\mathbf{k}_2}}{\left(\omega^l_{\mathbf{k}_1}-\omega^l_{\mathbf{k}_2}-\omega^l_{\mathbf{k}_1-\mathbf{k}_2}\right)\left(\omega^l_{\mathbf{k}_1-\mathbf{k}_2}-\mathbf{v}\cdot(\mathbf{k}_1-\mathbf{k}_2)\right)}+$$

$$+\frac{\mathcal{V}^{*}_{\mathbf{k}_2,\,\mathbf{k}_1,\,\mathbf{k}_2-\mathbf{k}_1}\,\phi_{\mathbf{k}_2-\mathbf{k}_1}}{\left(\omega^l_{\mathbf{k}_2}-\omega^l_{\mathbf{k}_1}-\omega^l_{\mathbf{k}_2-\mathbf{k}_1}\right)\left(\omega^l_{\mathbf{k}_2-\mathbf{k}_1}-\mathbf{v}\cdot(\mathbf{k}_2-\mathbf{k}_1)\right)}.$$

We need to consider in more detail the practical implication of the coefficient function $\widetilde{V}^{(2)}$ for the Hamilton formalism under consideration when the resonance frequency difference

$$\Delta\omega_{\mathbf{k}_1,\mathbf{k}_2} \equiv \omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1 - \mathbf{k}_2),$$

is different from zero. We recall for this purpose that the function $T^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1}$ in initial fourth-order interaction Hamiltonian $H^{(4)}$, satisfies the requirement of reality of this Hamiltonian

$$T^{(2)\,a\,a_1\,a_2}_{\mathbf{k},\,\mathbf{k}_1} = -T^{*\,(2)\,a_1\,a\,a_2}_{\mathbf{k}_1,\,\mathbf{k}}. \tag{22}$$

Let us now consider the effective amplitude $\widetilde{T}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$, which is defined by the expression (19). If one does not use the resonance condition

$$\omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1 - \mathbf{k}_2) = 0, \tag{23}$$

then it is not difficult to verify that in contrast to (22), we have

$$\widetilde{T}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2} \neq -\widetilde{T}^{*\,(2)\,a_2\,a_1\,a}_{\mathbf{k}_2,\,\mathbf{k}_1}.$$

The complete effective amplitude $\mathcal{T}^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2}$

Substituting the functions $\widetilde{V}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}$ and $\widetilde{T}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}$, Eq. (21) and (19), into (18) and performing simple algebraic transformations, we define an explicit form of the *complete effective amplitude* $\mathcal{T}^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} = f^{\,a\,a_1\,a_2}\,\mathcal{T}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2}$, where

$$\mathcal{T}^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2} = T^{(2)}_{\mathbf{k}_1,\,\mathbf{k}_2} + \frac{1}{2}\left(\frac{1}{\omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot\mathbf{k}_1} + \frac{1}{\omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot\mathbf{k}_2}\right)\phi^*_{\mathbf{k}_1}\,\phi_{\mathbf{k}_2} \qquad (24)$$

$$+\, i\left[\left(\frac{1}{\omega^l_{\mathbf{k}_1-\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1-\mathbf{k}_2)} + \frac{1}{\omega^l_{\mathbf{k}_1-\mathbf{k}_2} - \omega^l_{\mathbf{k}_1} + \omega^l_{\mathbf{k}_2}}\right)\mathcal{V}_{\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_1-\mathbf{k}_2}\,\phi^*_{\mathbf{k}_1-\mathbf{k}_2}\right.$$

$$\left. -\left(\frac{1}{\omega^l_{\mathbf{k}_2-\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k}_2-\mathbf{k}_1)} + \frac{1}{\omega^l_{\mathbf{k}_2-\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} + \omega^l_{\mathbf{k}_1}}\right)\mathcal{V}^*_{\mathbf{k}_2,\,\mathbf{k}_1,\,\mathbf{k}_2-\mathbf{k}_1}\,\phi_{\mathbf{k}_2-\mathbf{k}_1}\right].$$

The presented form of the complete effective amplitude $\mathcal{T}^{(2)}$ (24) makes the validity of the requirement of reality of the Hamiltonian $\mathcal{H}^{(4)}$ practically obvious

$$\mathcal{T}^{(2)\,a\,a_1\,a_2}_{\mathbf{k}_1,\,\mathbf{k}_2} = -\mathcal{T}^{*\,(2)\,a\,a_2\,a_1}_{\mathbf{k}_2,\,\mathbf{k}_1}.$$

Thus, the role of the coefficient function $\widetilde{V}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$ is reduced to the total symmetrization of the effective amplitude $\widetilde{T}^{(2)\,a_1\,a_2\,a}_{\mathbf{k}_1,\,\mathbf{k}_2}$. This involves the fulfillment of the necessary symmetry condition without any use of the resonance condition (23).

Now we turn to the construction of a kinetic equation describing the elastic scattering process of a plasmon off a hard color particle. As the interaction Hamiltonian we consider the effective Hamiltonian $\mathcal{H}^{(4)}$ (17). The equations of motion for the bosonic normal variables $c^{a'}_{\mathbf{k}'}$ and $c^{*\,a}_{\mathbf{k}}$, and the color charge $\mathcal{Q}^a$ are defined by the corresponding Hamilton equations (2) and (3). For soft Bose-excitations we find

$$\frac{\partial c^{a'}_{\mathbf{k}'}}{\partial t} = -i\left\{c^{a'}_{\mathbf{k}'}, \mathcal{H}^{(0)} + \mathcal{H}^{(4)}\right\} = \tag{25}$$

$$= -i\left(\omega^l_{\mathbf{k}'} - \mathbf{v}\cdot\mathbf{k}'\right)c^{a'}_{\mathbf{k}'} + \int d\mathbf{k}_1\, \mathcal{T}^{\,a'\,a_1\,d}_{\,\mathbf{k}',\,\mathbf{k}_1}\, c^{a_1}_{\mathbf{k}_1}\, \mathcal{Q}^{\,d}$$

and the corresponding equation for the conjugate boson normal amplitude $c^{*\,a}_{\mathbf{k}}$. For the classical color charge we get

$$\frac{d\mathcal{Q}^{\,d}}{dt} = -i\left\{\mathcal{Q}^{\,d}, \mathcal{H}^{(0)}+\mathcal{H}^{(4)}\right\} = \tag{26}$$

$$= \frac{\partial\left(\mathcal{H}^{(0)} + \mathcal{H}^{(4)}\right)}{\partial \mathcal{Q}^{d'}}\, f^{\,dd'e}\, \mathcal{Q}^{e} = i f^{\,dd'e}\int d\mathbf{k}_1\, d\mathbf{k}_2\, \mathcal{T}^{\,a_1\,a_2\,d'}_{\,\mathbf{k}_1,\,\mathbf{k}_2}\, c^{*\,a_1}_{\mathbf{k}_1} c^{a_2}_{\mathbf{k}_2}\, \mathcal{Q}^{\,e},$$

where the free Hamiltonian $\mathcal{H}^{(0)}$ in the term of new variables is of the form

$$\mathcal{H}^{(0)} = \int d\mathbf{k}\left(\omega^l_{\mathbf{k}} - \mathbf{v}\cdot\mathbf{k}\right) c^{*\,a}_{\mathbf{k}}\, c^{a}_{\mathbf{k}}.$$

Kinetic equation for the plasmon number density $\mathcal{N}_{\mathbf{k}}^{aa'}$ are defined by employing the Hamilton equations (25) and (26), and the definition (1):

$$\frac{\partial \mathcal{N}_{\mathbf{k}}^{aa'}}{\partial t} = -i\left\{\mathcal{T}_{\mathbf{k},\,\mathbf{k}}\left(\mathcal{N}_{\mathbf{k}}T^{e}\right)^{aa'} - \mathcal{T}^{*}_{\mathbf{k},\,\mathbf{k}}\left(T^{e}\mathcal{N}_{\mathbf{k}}\right)^{aa'}\right\}\langle\mathcal{Q}^{e}\rangle - \tag{27}$$

$$-\int d\mathbf{k}_{1}\left|\mathcal{T}_{\mathbf{k},\,\mathbf{k}_{1}}\right|^{2}\times$$

$$\left\{\frac{1}{\Delta\omega_{\mathbf{k},\,\mathbf{k}_{1}}+i0}\left((\mathcal{N}_{\mathbf{k}}T^{d'}\mathcal{N}_{\mathbf{k}_{1}}T^{d})^{aa'}f^{dd'e}\langle\mathcal{Q}^{e}\rangle+i\left[(\mathcal{N}_{\mathbf{k}}T^{e}T^{d})^{aa'}-(T^{e}\mathcal{N}_{\mathbf{k}_{1}}T^{d})^{aa'}\right]\langle\mathcal{Q}^{d}\rangle\langle\mathcal{Q}^{e}\rangle\right)\right.$$

$$\left.-\frac{1}{\Delta\omega_{\mathbf{k},\,\mathbf{k}_{1}}-i0}\left((T^{d'}\mathcal{N}_{\mathbf{k}_{1}}T^{d}\mathcal{N}_{\mathbf{k}})^{aa'}f^{dd'e}\langle\mathcal{Q}^{e}\rangle+i\left[(T^{e}T^{d}\mathcal{N}_{\mathbf{k}})^{aa'}-(T^{e}\mathcal{N}_{\mathbf{k}_{1}}T^{d})^{aa'}\right]\langle\mathcal{Q}^{d}\rangle\langle\mathcal{Q}^{e}\rangle\right)\right\}$$

Let us consider the following color decomposition of the matrix function $\mathcal{N}_{\mathbf{k}}^{aa'}$:

$$\mathcal{N}_{\mathbf{k}}^{aa'} = \delta^{aa'}N_{\mathbf{k}}^{l} + (T^{c})^{aa'}\langle\mathcal{Q}^{c}\rangle W_{\mathbf{k}}^{l}. \tag{28}$$

We define the kinetic equations for the colorless and color parts of the plasmon number density, i.e. for the scalar functions $N_{\mathbf{k}}^{l}$ and $W_{\mathbf{k}}^{l}$.

Using the color expansion (28) and the formulae for the traces of the product of two and three color matrices T^{a} in the adjoint representation of the Lie algebra $\mathfrak{su}(N_c)$ (H. Haber (2021)), we find first moment about color for the equation (27):

$$d_A \frac{\partial N^l_{\mathbf{k}}}{\partial t} = 2 N_c\, \mathfrak{q}_2(t) \big(\mathrm{Im}\, \mathcal{T}_{\mathbf{k},\mathbf{k}}\big) W^l_{\mathbf{k}} \tag{29}$$

$$- N_c\, \mathfrak{q}_2(t) \int d\mathbf{k}_1 \left|\mathcal{T}_{\mathbf{k},\mathbf{k}_1}\right|^2 \Big\{ \big(N^l_{\mathbf{k}} - N^l_{\mathbf{k}_1}\big) - \frac{1}{2} N_c \big(W^l_{\mathbf{k}} N^l_{\mathbf{k}_1} - N^l_{\mathbf{k}} W^l_{\mathbf{k}_1}\big) \Big\}$$

$$\times (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k} - \mathbf{k}_1)).$$

Here we have introduced the notation for a colorless quadratic combination of the averaged color charge $\mathfrak{q}_2(t) \equiv \langle \mathcal{Q}^e \rangle \langle \mathcal{Q}^e \rangle$. The coefficient $d_A \equiv N_c^2 - 1$ on the left-hand side of (29) is an invariant for the group $SU(N_c)$.

Let us return to the matrix kinetic equation (27). We now contract the left- and right-hand sides of this equation with the color matrix $(T^s)^{a'a}$. After somewhat cumbersome calculations of the traces of the product of generators in the adjoint representation up to fifth order the kinetic equation for $W^l_{\mathbf{k}}$ takes the form

$$N_c \frac{\partial \big(\langle \mathcal{Q}^s \rangle W^l_{\mathbf{k}}\big)}{\partial t} = 2N \big(\mathrm{Im}\, \mathcal{T}_{\mathbf{k},\mathbf{k}}\big) N^l_{\mathbf{k}} \langle \mathcal{Q}^s \rangle \tag{30}$$

$$- \int d\mathbf{k}_1 \left|\mathcal{T}_{\mathbf{k},\mathbf{k}_1}\right|^2 \Big\{ 2\delta^{sd}\delta^{ce} + \frac{1}{4} N_c\, d^{sd\lambda} d^{ce\lambda} \Big\} \big(W^l_{\mathbf{k}} - W^l_{\mathbf{k}_1}\big) \langle \mathcal{Q}^c \rangle \langle \mathcal{Q}^d \rangle \langle \mathcal{Q}^e \rangle$$

$$\times (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k} - \mathbf{k}_1))$$

$$+ \frac{1}{2} N_c^2 \int d\mathbf{k}_1 \left|\mathcal{T}_{\mathbf{k},\mathbf{k}_1}\right|^2 N^l_{\mathbf{k}} N^l_{\mathbf{k}_1} \langle \mathcal{Q}^s \rangle (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k} - \mathbf{k}_1)),$$

where d^{abc} is totally symmetric structure constants of the Lie algebra $su(N_c)$.

Equation for the expected value of color charge $\langle \mathcal{Q}^a \rangle$

The kinetic equations obtained (29) and (30) contain the averaged color charge $\langle \mathcal{Q}^a \rangle$ of a hard particle, which itself is an unknown function of time. To determine the equation to which a given charge obeys, the first step is to average the original equation (26) and then to use approximate expression for the fourth-order correlation function. As a result we obtain

$$\frac{d\langle \mathcal{Q}^d \rangle}{dt} = \frac{1}{2}\, A(t)\langle \mathcal{Q}^d \rangle, \quad \langle \mathcal{Q}^d \rangle|_{t=t_0} = \mathcal{Q}_0^d, \tag{31}$$

where

$$A(t) \equiv N_c^2 \int d\mathbf{k}_1\, d\mathbf{k}_2 \left|\mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}\right|^2 N^l_{\mathbf{k}_1} N^l_{\mathbf{k}_2}\, (2\pi)\,\delta(\omega^l_{\mathbf{k}_1} - \omega^l_{\mathbf{k}_2} - \mathbf{v}\cdot(\mathbf{k}_1 - \mathbf{k}_2)) \tag{32}$$

and $\mathcal{Q}_0^d$ is some fixed (nonrandom) vector in the internal color space. We are interested in the time dependence of the colorless quadratic combination

$$\mathfrak{q}_2(t) = \langle \mathcal{Q}^e \rangle \langle \mathcal{Q}^e \rangle$$

as well as the colorless combination of the fourth order

$$\mathfrak{q}_4(t) = \mathfrak{q}_2^a(t)\, \mathfrak{q}_2^a(t), \tag{33}$$

where

$$\mathfrak{q}_2^a(t) \equiv d^{abc} \langle \mathcal{Q}^b \rangle \langle \mathcal{Q}^c \rangle.$$

Equation for the expected value of color charge $\langle \mathcal{Q}^a \rangle$

From (31), we immediately find the desired time dependence of these combinations as nonlinear functionals of the colorless part $N^l_{\mathbf{k}}$ of the plasmon number density

$$\mathfrak{q}_2(t) = \mathfrak{q}_2(t_0) \exp\left\{ \int_{t_0}^{t} A(\tau)\, d\tau \right\}, \quad \mathfrak{q}_4(t) = \mathfrak{q}_4(t_0) \exp\left\{ 2 \int_{t_0}^{t} A(\tau)\, d\tau \right\}. \quad (34)$$

Thus, the square of the averaged color charge $\mathfrak{q}_2(t)$ of a hard particle is not conserved in the interaction with the random soft bosonic excitations of a hot gluon plasma.

The solutions (34) allow us to close the kinetic equations (29) and (30) and thereby, within the approximation employed in this work, to obtain a complete self-consistent description of the dynamics of soft gluon excitations in the presence of an external high-energy color-charged particle in the medium.

For the special case $N_c = 2$, when $d^{abc} \equiv 0$ and, as a consequence $\mathfrak{q}_4(t_0) \equiv 0$, we have

$$\mathfrak{q}_4(t) \equiv 0.$$

In another, more nontrivial special case $N_c = 3$, by virtue of the definition (33) and the property

$$d^{abe} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce} = \frac{1}{3} \left(\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right),$$

we find

$$\mathfrak{q}_4(t_0) = d^{abc} d^{ab'c'} \langle \mathcal{Q}_0^b \rangle \langle \mathcal{Q}_0^c \rangle \langle \mathcal{Q}_0^{b'} \rangle \langle \mathcal{Q}_0^{c'} \rangle = \frac{1}{3} \left(\langle \mathcal{Q}_0^a \rangle \langle \mathcal{Q}_0^a \rangle \right)^2 \equiv \frac{1}{3} \left(\mathfrak{q}_2(t_0) \right)^2.$$

5. System of kinetic equations for soft gluon excitations

We write out once more the kinetic equations for the scalar plasmon number densities $N^l_{\mathbf{k}}$ and $W^l_{\mathbf{k}}$ obtained above

$$d_A \frac{\partial N^l_{\mathbf{k}}}{\partial t} = 2\mathfrak{q}_2(t) N_c \left(\operatorname{Im}\mathcal{T}_{\mathbf{k},\mathbf{k}}\right) W^l_{\mathbf{k}} - \mathfrak{q}_2(t) N_c \int d\mathbf{k}_1 \left|\mathcal{T}_{\mathbf{k},\mathbf{k}_1}\right|^2 \times \tag{35}$$

$$\times \left\{ \left(N^l_{\mathbf{k}} - N^l_{\mathbf{k}_1}\right) - \frac{1}{2} N_c \left(W^l_{\mathbf{k}} N^l_{\mathbf{k}_1} - N^l_{\mathbf{k}} W^l_{\mathbf{k}_1}\right)\right\} (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k}-\mathbf{k}_1)),$$

$$\frac{\partial W^l_{\mathbf{k}}}{\partial t} = -\frac{1}{2} A(t) W^l_{\mathbf{k}} + 2\left(\operatorname{Im}\mathcal{T}_{\mathbf{k},\mathbf{k}}\right) N^l_{\mathbf{k}} \tag{36}$$

$$-\int d\mathbf{k}_1 \left|\mathcal{T}_{\mathbf{k},\mathbf{k}_1}\right|^2 \left\{ \rho\, \mathfrak{q}_2(t) \left(W^l_{\mathbf{k}} - W^l_{\mathbf{k}_1}\right) - \frac{1}{2} N_c N^l_{\mathbf{k}} N^l_{\mathbf{k}_1}\right\} (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k}-\mathbf{k}_1)),$$

where $\mathcal{T}_{\mathbf{k},\mathbf{k}_1}$ is amplitude of elastic scattering of plasmon off a hard color-charged particle; the functions $\mathfrak{q}_2(t)$ and $A(t)$ are defined, respectively, by the expressions

$$\mathfrak{q}_2(t) \equiv \left\langle \mathcal{Q}^a \right\rangle \left\langle \mathcal{Q}^a \right\rangle,$$

$$A(t) \equiv N_c^2 \int d\mathbf{k}_1 d\mathbf{k}_2 \left|\mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}\right|^2 N^l_{\mathbf{k}_1} N^l_{\mathbf{k}_2} (2\pi)\, \delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}_1} - \mathbf{v}\cdot(\mathbf{k}-\mathbf{k}_1)), \tag{37}$$

$$\rho \equiv \frac{2}{N_c} + \frac{1}{4}\frac{\mathfrak{q}_4(t)}{(\mathfrak{q}_2(t))^2} = \frac{2}{N_c} + \frac{1}{4}\frac{\mathfrak{q}_4(t_0)}{(\mathfrak{q}_2(t_0))^2}.$$

The coefficient ρ in the last expression is equal to 1 for the color $SU(2_c)$ group and 3/4 for the color $SU(3_c)$ group.

To get some understanding of the behavior of the solution of the system of kinetic equations (35) and (36), we consider the model problem of the interaction of two infinitely narrow packets with typical wavevectors $\mathbf{k}_0$ и $\mathbf{k}'_0$. Let us introduce the scalar plasmon number densities $N^l_{\mathbf{k}}$ and $W^l_{\mathbf{k}}$ as follows

$$
\begin{aligned}
N^l_{\mathbf{k}}(t) &= N_1(t)\,\delta(\mathbf{k}-\mathbf{k}_0) + N_2(t)\,\delta(\mathbf{k}-\mathbf{k}'_0),\\
W^l_{\mathbf{k}}(t) &= W_1(t)\,\delta(\mathbf{k}-\mathbf{k}_0) + W_2(t)\,\delta(\mathbf{k}-\mathbf{k}'_0),
\end{aligned}
\tag{38}
$$

at that $\mathbf{k}_0 \neq \mathbf{k}'_0$. Let us substitute (38) into the left- and right-hand sides of equations (35) and (36). As a result, we obtain a system of four nonlinear ordinary differential equations of the first order

$$
\begin{aligned}
\frac{dN_1(t)}{dt} &= A_{13}W_1 - \frac{N_c}{d_A}\,\mathfrak{q}_2(t)B\left(N_1W_2 - N_2W_1\right),\\
\frac{dN_2(t)}{dt} &= A_{24}W_2 + \frac{N_c}{d_A}\,\mathfrak{q}_2(t)B\left(N_1W_2 - N_2W_1\right),\\
\frac{dW_1(t)}{dt} &= A_{31}N_1 - \frac{1}{2}\,A(t)W_1 + B\,N_1N_2,\\
\frac{dW_2(t)}{dt} &= A_{42}N_2 - \frac{1}{2}\,A(t)W_2 + B\,N_1N_2.
\end{aligned}
\tag{39}
$$

Interaction of infinitely narrow packets

Nonzero "matrix elements" A_{ij}, $i, j = 1, \ldots, 4$ are defined by the following expressions:

$$A_{31} = 2\left(\operatorname{Im}\mathcal{T}_{\mathbf{k}_0,\mathbf{k}_0}\right),\; A_{13} = \frac{N_c}{d_A}\,\mathfrak{q}_2(t)\,A_{31},$$

$$A_{42} = 2\left(\operatorname{Im}\mathcal{T}_{\mathbf{k}'_0,\mathbf{k}'_0}\right),\; A_{24} = \frac{N_c}{d_A}\,\mathfrak{q}_2(t)\,A_{42},$$

and the coefficient B has the form

$$B = (1/2)\,N_c\,\mathfrak{q}_2(t)\left|\mathcal{T}_{\mathbf{k}_0,\mathbf{k}'_0}\right|^2 (2\pi)\,\delta(\omega^l_{\mathbf{k}_0} - \omega^l_{\mathbf{k}'_0} - \mathbf{v}\cdot(\mathbf{k}_0 - \mathbf{k}'_\mathbf{0})).$$

This coefficient is, generally speaking, a *generalized* function.

Let us simplify the resulting system as much as possible. The matrix elements A_{ij} in front of the linear terms on the right-hand sides of (39) are proportional to the imaginary parts $\operatorname{Im}\mathcal{T}_{\mathbf{k}_0,\mathbf{k}_0}(\mathbf{v})$ and $\operatorname{Im}\mathcal{T}_{\mathbf{k}'_0,\mathbf{k}'_0}(\mathbf{v})$ These factors are actually related to the collisionless (Landau) damping of soft gluon oscillations and thus must contain the Dirac delta function which reflects the corresponding conservation laws for energy and momentum:

$$\operatorname{Im}\mathcal{T}_{\mathbf{k}_0,\mathbf{k}_0}(\mathbf{v}) \sim \int \frac{d\Omega_{\mathbf{v}'}}{4\pi}\,w_{\mathbf{v}'}(\mathbf{v};\mathbf{k}_0)\,(2\pi)\,\delta(\omega^l_{\mathbf{k}_0} - \mathbf{v}'\cdot\mathbf{k}_0),$$

where $w_{\mathbf{v}'}(\mathbf{v};\mathbf{k}_0)$ is the probability for the Landau damping process and $d\Omega_{\mathbf{v}'}$ is a differential solid angle. An explicit form of this probability can be obtained by using the expression for the scattering amplitude found in the work Yu. Markov *et al.* (2024).

However, as is well known, the *linear Landau damping* is kinematically forbidden in hot gluon plasma and therefore, these matrix elements can be set to zero, i.e.,

$$A_{13} = A_{24} = A_{31} = A_{42} = 0.$$

Next, we consider the terms in the last two equations in (39) containing the function $A(t)$. By virtue of the definition (37) this function is quadratic in the colorless part of plasmon number density, and thus, the terms $A(t)W_1$ and $A(t)W_2$ in (39) are of the third order. In constructing the kinetic equations (35) and (36), we limited ourselves to linear and quadratic contributions of the plasmon number density. For this reason, within the accepted accuracy, in the last two equations in (39) one should drop the contributions with the function $A(t)$, and in the first two equations, due to the fact that the function $\mathfrak{q}_2(t)$ depends exponentially on the $A(t)$, the function $\mathfrak{q}_2(t)$ should be assumed equal to its initial value, i.e.

$$\mathfrak{q}_2(t) \simeq \mathfrak{q}_2(t_0) \equiv \mathfrak{q}_2^0.$$

Taking all the above into account, the system of four equations (39) can be reduced to a system of two equations

$$\begin{cases} \dfrac{dN_1(t)}{dt} = \beta B\big[N_1(\mathcal{C}_2 - W_1) + W_1(\mathcal{C}_1 - N_1)\big], \\ \dfrac{dW_1(t)}{dt} = BN_1(\mathcal{C}_1 - N_1), \end{cases} \tag{40}$$

where for the sake of brevity, we have designated $\beta \equiv N_c\,\mathfrak{q}_2^0/d_A$.

The functions $N_2(t)$ and $W_2(t)$ are defined from relations of the form

$$N_1(t) + N_2(t) = \mathcal{C}_1, \qquad W_1(t) - W_2(t) = \mathcal{C}_2,$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are some constants. Obviously, the system (40) has two stationary points, one of which is trivial: $N_1 = W_1 = 0$, and the second one is defined as $N_1 = \mathcal{C}_1$, $W_1 = \mathcal{C}_2$.

It will be shown below, that at a certain relation between the constants $\mathcal{C}_1$ and $\mathcal{C}_2$ we can obtain the exact solution of the system (40). For this purpose, the first step, due to the autonomy of the right-hand sides, is to reduce this system to a single equation

$$\frac{dN_1}{dW_1} = \beta\left(\frac{\mathcal{C}_2 - W_1}{\mathcal{C}_1 - N_1} + \frac{W_1}{N_1}\right),$$

or, in a slightly different form, which defines W_1 as a function of N_1

$$\left[(2N_1 - \mathcal{C}_1)W_1 - \mathcal{C}_2 N_1\right]\frac{dW_1}{dN_1} = \frac{1}{\beta}\,(N_1^2 - \mathcal{C}_1 N_1). \qquad (41)$$

7. Constructing the exact solution of the equation (41) and the system (40)

Let us rewrite the system (40) and equation (41) in a slightly different form, introducing the notations generally accepted in the theory of differential equations. We set $y \equiv W_1$, $x \equiv N_1$, then instead of the original system (40), we have

$$\begin{cases} \dfrac{dy(t)}{dt} = x\,(\mathcal{C}_1 - x), \\ \dfrac{dx(t)}{dt} = \beta\left[x(\mathcal{C}_2 - y) + y(\mathcal{C}_1 - x)\right], \end{cases} \tag{42}$$

and instead of (41), in turn, we can write down

$$\left[(2x - \mathcal{C}_1)\,y - \mathcal{C}_2\,x\right]\frac{dy}{dx} = \frac{1}{\beta}\,x\,(x - \mathcal{C}_1), \tag{43}$$

or in more standard notations (A.D. Polyanin and V.F. Zaitsev (1995)):

$$\left[g_1(x)\,y + g_0(x)\right]\frac{dy}{dx} = f_0(x), \tag{44}$$

where

$$g_0(x) \equiv -\mathcal{C}_2\,x, \qquad g_1(x) \equiv 2x - \mathcal{C}_1, \qquad f_0(x) \equiv \frac{1}{\beta}\,x\,(x - \mathcal{C}_1).$$

In the system (42) we eliminated the parameter B, formally redefining the time $t \to t/B$.

Equation (44) belongs to the class of the Abel equations of the second kind. The first step is to reduce it to the "normal" form. We perform replacement of the unknown function

$$w = y + \frac{g_0(x)}{g_1(x)}.$$

This transformation reduces the original equation (43) to the form:

$$w\,\frac{dw}{dx} = F_1(x)\,w + F_0(x), \qquad (45)$$

where

$$F_1(x) = \frac{d}{dx}\left(\frac{g_0(x)}{g_1(x)}\right) = \frac{\mathcal{C}_1\mathcal{C}_2}{(2x-\mathcal{C}_1)^2}, \qquad F_0(x) = \frac{f_0(x)}{g_1(x)} = \frac{1}{\beta}\,\frac{x(x-\mathcal{C}_1)}{2x-\mathcal{C}_1}.$$

Next, the replacement of the argument of the function

$$\xi = \int F_1(x)\,dx = -\frac{1}{2}\,\frac{\mathcal{C}_1\mathcal{C}_2}{2x-\mathcal{C}_1}, \quad \text{or} \quad x = \frac{1}{2}\,\mathcal{C}_1 - \frac{1}{4\xi}\,\mathcal{C}_1\mathcal{C}_2$$

allows us to bring equation (45) to the canonical form

$$w\,\frac{dw}{d\xi} = w + F(\xi), \quad \text{where} \;\; F(\xi) = \frac{F_0(x)}{F_1(x)} = \frac{\mathcal{C}_1^2}{8\beta}\left(\frac{1}{\xi} - \frac{\mathcal{C}_2^2}{4\xi^3}\right). \qquad (46)$$

For the equation (46), at a certain ratio between the parameters $\mathcal{C}_1$ and $\mathcal{C}_2$, namely, for

$$\mathcal{C}_2^2 = \mathcal{C}_1^2 / 2\beta$$

there is an exact solution in the parametric form (Eq. 7 in subsection 1.3.1. of A.D. Polyanin and V.F. Zaitsev (1995)):

$$\xi = \frac{a}{\tau}\left(\tau - \ln|1+\tau| - C\right)^{1/2},$$

$$w = a\left[\frac{1+\tau}{\tau}\left(\tau - \ln|1+\tau| - C\right)^{1/2} - \frac{1}{2}\tau\left(\tau - \ln|1+\tau| - C\right)^{-1/2}\right],$$

where τ is a parameter, C is an arbitrary constant and $a^2 = \mathcal{C}_1^2/4\beta$. Returning to the original function y and to its argument x, we determine the exact solution for the equation (43):

$$\begin{cases} x = x(\tau, C) = \dfrac{1}{2}\mathcal{C}_1 - \dfrac{1}{4a}\mathcal{C}_1\mathcal{C}_2\dfrac{\tau}{f(\tau)}, \\ y = y(\tau, C) = a\left[\dfrac{1+\tau}{\tau}f(\tau) - \dfrac{1}{2}\dfrac{\tau}{f(\tau)}\right] + \dfrac{\mathcal{C}_2 x}{2x - \mathcal{C}_1}. \end{cases} \tag{47}$$

Here, we introduce the notation $f(\tau) \equiv \left(\tau - \ln|1+\tau| - C\right)^{1/2}$.

The solution of the initial dynamical system (42) is determined by the formulae

$$x = x(\tau, C), \quad y = y(\tau, C),$$
$$t = \frac{1}{\beta} \int \frac{\dot{x}_\tau \, d\tau}{[\mathcal{C}_1 - 2x(\tau, C)]\, y(\tau, C) + \mathcal{C}_2 x(\tau, C)} + \widetilde{C}. \qquad (48)$$

Here $\dot{x}_\tau \equiv dx(\tau, C)/d\tau$, and $\widetilde{C}$ is another arbitrary constant. The latter relation defines the implicit dependence of parameter τ on time t: $\tau = \tau(t, C, \widetilde{C})$. Using the formulae (47), we can find the dependence of x and y and, in this way, the original functions N_1 and W_1 on time t.

Substituting the exact solutions (47) in the expression (48) after some cumbersome algebraic transformations, we finally find the desired time parameterization

$$t = t(\tau, C, \widetilde{C}) = -\frac{2a}{\mathcal{C}_1^2} \int \frac{d\tau}{(1+\tau)\, f(\tau)} + \widetilde{C}. \qquad (49)$$

Recall that $f(\tau) = \big(\tau - \ln|1+\tau| - C\big)^{1/2}$. Unfortunately, this indefinite integral is not calculated explicitly.

The only thing that can be done here is to reduce the integral of transcendental function in (49) to the integral of the so-called Lambert W-function (R.M. Corless *et al.*, 1996) that has been well studied. For this purpose, let us replace the integration variable

$$\ln|1+\tau| = \zeta,$$

which gives

$$\int \frac{d\tau}{(1+\tau)\big(\tau - \ln|1+\tau| - C\big)^{1/2}} = \int \frac{d\zeta}{\big[\pm \mathrm{e}^{\zeta} - \zeta - \big(1+C\big)\big]^{1/2}}, \qquad (50)$$

where on the right-hand side in the integrand we have

$$\begin{cases} +\mathrm{e}^{\zeta}, & \text{for } \tau > -1, \\ -\mathrm{e}^{\zeta}, & \text{for } \tau < -1. \end{cases}$$

Let us again perform the replacement of the integration variable

$$\pm \mathrm{e}^{\zeta} - \zeta - \big(1+C\big) = \lambda. \qquad (51)$$

The solution of this expression with respect to the new variable λ can be represented in the following form:

$$\zeta = \zeta(\lambda) = -\lambda - \big(1+C\big) - W\big(\mp \mathrm{e}^{-(1+C)}\mathrm{e}^{-\lambda}\big),$$

where $W(x)$ is the Lambert W-function (solution to the equation $W\mathrm{e}^{W} = x$).

Further, by using the rule of differentiation for this function (R.M. Corless *et al.*, 1996, 1997), we find

$$d\zeta = -d\lambda + \frac{W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\lambda}\right)}{1 + W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\lambda}\right)}\,d\lambda \equiv -\frac{1}{1 + W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\lambda}\right)}\,d\lambda.$$

Substituting the replacement of the variable (51) and the differential $d\zeta$ into (50), we find, instead of the last integral in (50),

$$-\int \frac{d\lambda}{\lambda^{1/2}\left[1 + W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\lambda}\right)\right]}$$

or, eventually, after a trivial replacement $\xi \equiv \lambda^{1/2}$

$$t = t(\tau, C, \widetilde{C}) = \frac{4a}{\mathcal{C}_1^2}\int \frac{d\xi}{1 + W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\xi^2}\right)} + \widetilde{C}, \quad \xi = f(\tau), \qquad (52)$$

where we choose plus sign if $\tau > -1$ and minus sign, if $\tau < -1$.

However, such an integral cannot be calculated directly either. The difficulty is that the Lambert W-function depends on the variable ξ as the function $\mathrm{e}^{-\xi^2}$.

Here, we can use the particular integral representation for the Lambert W-function given in G.A. Kalugin *et al.* (2011). In our case it will look as follows

$$\int \frac{d\xi}{1+W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\xi^2}\right)} = \frac{1}{\pi}\int\limits_0^{\pi} dv \int \frac{d\xi}{1 \mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\xi^2}\mathrm{e}^{v\cot v}\sin v/v}. \qquad (53)$$

Further we present the integrand as a series expansion

$$\frac{1}{1 \mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\xi^2}\mathrm{e}^{v\cot v}\sin v/v} = 1 + \sum_{\nu=1}^{\infty} (\pm 1)^{\nu}\,\mathrm{e}^{-\nu(1+C)}\,\mathrm{e}^{-\nu\xi^2}\left\{\mathrm{e}^{v\cot v}\,\frac{\sin v}{v}\right\}^{\nu}.$$

Let us write the integral over ξ through the Gauss error function

$$\int\limits_0^{\xi} \mathrm{e}^{-\nu\xi^2}d\xi = \frac{1}{2}\sqrt{\frac{\pi}{\nu}}\,\mathrm{erf}(\sqrt{\nu}\,\xi),$$

and for the integration over v we use Corollary 3.4. from G.A. Kalugin *et al.* (2011)

$$\int\limits_0^{\pi}\left\{\mathrm{e}^{v\cot v}\,\frac{\sin v}{v}\right\}^{\nu} dv = \frac{\pi\,\nu^{\nu}}{\nu!}.$$

As a result, for the integral (53) we find a representation in the form of a series

$$\int \frac{d\xi}{1 + W\left(\mp \mathrm{e}^{-(1+C)}\,\mathrm{e}^{-\xi^2}\right)} = \xi + \frac{\sqrt{\pi}}{2} \sum_{\nu=1}^{\infty} \frac{(\pm 1)^{\nu}}{\nu!\sqrt{\nu}} \left(\mathrm{e}^{-\,(1+C)}\nu\right)^{\nu} \operatorname{erf}(\sqrt{\nu}\,\xi).$$

Substituting this representation into (52) and returning to the original variable τ (we simply replace ξ with $f(\tau)$), we find the following representation for the time parameterization (49):

$$t(\tau, C, \widetilde{C}) = \frac{4a}{\mathcal{C}_1^2} \left\{ f(\tau) + \frac{\sqrt{\pi}}{2} \sum_{\nu=1}^{\infty} \frac{(\pm 1)^{\nu}}{\nu!\sqrt{\nu}} \left(\mathrm{e}^{-\,(1+C)}\nu\right)^{\nu} \operatorname{erf}\left(\sqrt{\nu}\, f(\tau)\right) \right\} + \widetilde{C}.$$

Here, we recall that under the sum sign we choose $(+1)^{\nu} \equiv 1$, if $\tau > -1$, and $(-1)^{\nu}$, if $\tau < -1$.

Perhaps this representation is more convenient for approximate expressions of time t as a function of the parameter τ, using, for example, several first terms of the series or, vice versa, using asymptotic approximation at $\nu \to \infty$ for the terms of this series.

Conclusion

In this work, a generalization of the Lie-Poisson bracket for the case of a composite system – a continuous medium described by a bosonic normal field variable $a^a_{\mathbf{k}}$ and hard test particle with a non-Abelian charge Q^a is performed and the corresponding Hamilton equations are presented. The canonical transformations for the bosonic normal variable and for the color charge of a hard test particle are constructed in an explicit form.

A complete system of the canonicity conditions for these transformations is derived and the important notion of the plasmon number density $\mathcal{N}^{a\,a'}_{\mathbf{k}}$, which is a nontrivial matrix in the color space, is introduced. An explicit form of the effective fourth-order Hamiltonian describing the elastic scattering of plasmon off a hard color particle, is found. The matrix kinetic equation for the function $\mathcal{N}^{a\,a'}_{\mathbf{k}}$ is obtained.

A color decomposition of the matrix function $\mathcal{N}^{a\,a'}_{\mathbf{k}}$ is proposed and the first moment with respect to color of the matrix kinetic equation is calculated that defines the scalar equation for the colorless part $N^l_{\mathbf{k}}$ of this decomposition.

The second color moment from the matrix equation defining the scalar kinetic equation for the color part $W^l_{\mathbf{k}}$ of the decomposition of the matrix function $\mathcal{N}^{a\,a'}_{\mathbf{k}}$, is determined. The equation of motion for the mean value of color charge $\big\langle \mathcal{Q}^a(t) \big\rangle$ is obtained. This allowed the description of the system to be completely closed.

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Thank you for attention!