Parity effects in the Casimir interaction

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In this talk

1. A novel gauge-invariant by construction method in the Casimir effect.
2. The Casimir-Polder potential of an anisotropic atom between two dielectric half spaces with Chern-Simons boundary layers.
3. The Casimir-Polder potential of an anisotropic atom between two Chern-Simons layers in vacuum expressed through special functions.
4. P-odd three-body vacuum effects.
5. Casimir energy of two Chern-Simons layers in vacuum.
6. Casimir energy of two dielectric half spaces with Chern-Simons boundary layers.
7. Appearance of a minimum in the Casimir energy due to presence of Chern-Simons layers at the boundaries of dielectrics.

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Chern-Simons layer on a dielectric half space

Chern-Simons layer on a dielectric half space

The action with Chern-Simons layer at $z = 0$ has the form:

$$S = \frac{a}{2} \int \varepsilon^{z\nu\rho\sigma} A_\nu F_{\rho\sigma} dtdxdy. \tag{1}$$

Equations of electromagnetic field in the presence of Chern-Simons action (1) can be written as follows:

$$\partial_\mu F^{\mu\nu} + a\, \varepsilon^{z\nu\rho\sigma} F_{\rho\sigma} \delta(z) = 0. \tag{2}$$

Consider a flat Chern-Simons layer put at $z = 0$ on a dielectric half space $z < 0$ characterized by a frequency dependent dielectric permittivity $\varepsilon(\omega)$, the magnetic permeability $\mu = 1$. Boundary conditions on the components of the electromagnetic field follow:

$$E_z|_{z=0^+} - \varepsilon(\omega) E_z|_{z=0^-} = -2aH_z|_{z=0}, \tag{3}$$

$$H_x|_{z=0^+} - H_x|_{z=0^-} = 2aE_x|_{z=0}, \tag{4}$$

$$H_y|_{z=0^+} - H_y|_{z=0^-} = 2aE_y|_{z=0}. \tag{5}$$

A special case: plane Chern-Simons layer in vacuum

TE or s-polarization (the factor $\exp(i\omega t + ik_y y)$ is omitted):

$$E_x = \exp(-ik_z z) + r_s \exp(ik_z z), z > 0 \tag{6}$$
$$E_x = t_s \exp(-ik_z z), z < 0 \tag{7}$$
$$H_x = r_{s\to p} \exp(ik_z z), z > 0 \tag{8}$$
$$H_x = t_{s\to p} \exp(-ik_z z), z < 0. \tag{9}$$

TM or p-polarization:

$$H_x = \exp(-ik_z z) + r_p \exp(ik_z z), z > 0 \tag{10}$$
$$H_x = t_p \exp(-ik_z z), z < 0 \tag{11}$$
$$E_x = r_{p\to s} \exp(ik_z z), z > 0 \tag{12}$$
$$E_x = t_{p\to s} \exp(-ik_z z), z < 0. \tag{13}$$

A special case: plane Chern-Simons layer in vacuum

In vacuum the reflection coefficients for TE mode from a Chern-Simons layer have the form:

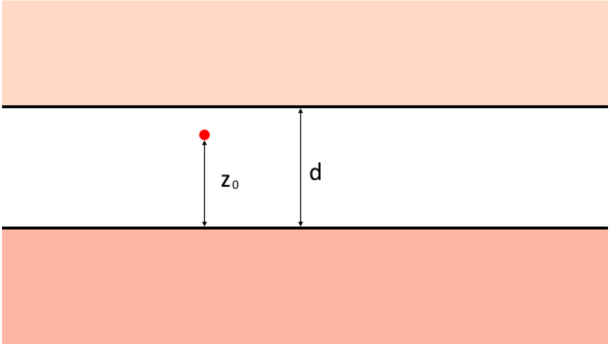
$$r_s = -\frac{a^2}{1+a^2}, \qquad t_s = \frac{1}{1+a^2},$$
$$r_{s\to p} = \frac{a}{1+a^2}, \qquad t_{s\to p} = -\frac{a}{1+a^2}, \tag{14}$$

for TM mode:

$$r_p = \frac{a^2}{1+a^2}, \qquad t_p = \frac{1}{1+a^2},$$
$$r_{p\to s} = \frac{a}{1+a^2}, \qquad t_{p\to s} = \frac{a}{1+a^2}. \tag{15}$$

[V.N.Marachevsky, Theor.Math.Phys., 2017]

The Casimir-Polder potential of an anisotropic atom between two Chern-Simons boundary layers



Anisotropic neutral atom between two dielectric half spaces with plane Chern-Simons boundary layers, z_0 is a distance of the atom from the layer and the dielectric medium characterized by the index 2, d is a width of the vacuum slit.
[V.N.Marachevsky and A.A.Sidelnikov, Phys.Rev.D, 2023].

Consider a dipole source at the point $\mathbf{r}' = (0, 0, z_0)$ characterized by electric dipole moment $d^l(t)$ with components of the four-current density [V.N.Marachevsky and Yu.M.Pis'mak, Phys.Rev.D, 2010]

$$\rho(t, \mathbf{r}) = -d^l(t)\partial_l \delta^3(\mathbf{r} - \mathbf{r}') \,, \tag{16}$$

$$j^l(t, \mathbf{r}) = \partial_t d^l(t) \delta^3(\mathbf{r} - \mathbf{r}') \,. \tag{17}$$

The Casimir-Polder potential is defined in terms of the scattered electric Green function $D_{ij}^{E,sc}(t_1 - t_2, \mathbf{r}, \mathbf{r}') = D_{ij}^{E}(t_1 - t_2, \mathbf{r}, \mathbf{r}') - D_{ij}^{E,vac}(t_1 - t_2, \mathbf{r}, \mathbf{r}')$ from the source (16),(17) and the atomic polarizability $\alpha_{ij}(t_1 - t_2) = i\langle T(\hat{d}_i(t_1), \hat{d}_j(t_2))\rangle$ as follows:

$$U(z_0) = -\int_0^\infty \frac{d\omega}{2\pi} \alpha^{ij}(i\omega) D_{ij}^{E,sc}(i\omega, \mathbf{r}', \mathbf{r}'). \tag{18}$$

From Weyl formula

$$\frac{e^{i\omega|\mathbf{r}'-\mathbf{r}|}}{4\pi|\mathbf{r}'-\mathbf{r}|} = i\iint \frac{e^{i(k_x(x'-x)+k_y(y'-y)+\sqrt{\omega^2-k_x^2-k_y^2}(z'-z))}}{2\sqrt{\omega^2-k_x^2-k_y^2}}\frac{dk_x dk_y}{(2\pi)^2}, \tag{19}$$

valid for $z'-z>0$, one can write electric and magnetic fields propagating downwards from the dipole source (16),(17) in the form [V.N.Marachevsky and A.A.Sidelnikov, Universe, 2021]

$$\mathbf{E^0}(\omega,\mathbf{r}) = \int \widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|})e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}}e^{-ik_z(z-z_0)}d^2\mathbf{k}_{\|}, \tag{20}$$

$$\mathbf{H^0}(\omega,\mathbf{r}) = \frac{1}{\omega}\int [\widetilde{\mathbf{k}}\times\widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|})]e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}}e^{-ik_z(z-z_0)}d^2\mathbf{k}_{\|}, \tag{21}$$

$$\widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|}) = \frac{i}{8\pi^2 k_z}\left(-(\mathbf{d}\cdot\widetilde{\mathbf{k}})\widetilde{\mathbf{k}}+\omega^2\mathbf{d}\right), \tag{22}$$

where $\mathbf{k}_{\|}=(k_x,k_y)$, $k_z=\sqrt{\omega^2-k_{\|}^2}$, $\widetilde{\mathbf{k}}=(\mathbf{k}_{\|},-k_z)$.

To solve a diffraction problem we write electric and magnetic fields for $z > 0$ in the form

$$\mathbf{E}^{\mathbf{1}}(\omega, \mathbf{r}) = \int \widetilde{\mathbf{N}}(\omega, \mathbf{k}_{\|}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-ik_z(z-z_0)} d^2\mathbf{k}_{\|} + \int \mathbf{v}(\omega, \mathbf{k}_{\|}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{ik_z z} d^2\mathbf{k}_{\|} , \tag{23}$$

$$\mathbf{H}^{\mathbf{1}}(\omega, \mathbf{r}) = \frac{1}{\omega} \int [\widetilde{\mathbf{k}} \times \widetilde{\mathbf{N}}(\omega, \mathbf{k}_{\|})] e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-ik_z(z-z_0)} d^2\mathbf{k}_{\|} + \frac{1}{\omega} \int [\mathbf{k} \times \mathbf{v}(\omega, \mathbf{k}_{\|})] e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{ik_z z} d^2\mathbf{k}_{\|} , \tag{24}$$

and for $z < 0$ in the form

$$\mathbf{E}^{\mathbf{2}}(\omega, \mathbf{r}) = \int \mathbf{u}(\omega, \mathbf{k}_{\|}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-iK_z z} d^2\mathbf{k}_{\|} , \tag{25}$$

$$\mathbf{H}^{\mathbf{2}}(\omega, \mathbf{r}) = \frac{1}{\omega} \int \left([\mathbf{k}_{\|} \times \mathbf{u}(\omega, \mathbf{k}_{\|})] - K_z [\mathbf{n} \times \mathbf{u}(\omega, \mathbf{k}_{\|})] \right) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-iK_z z} d^2\mathbf{k}_{\|} \tag{26}$$

with $K_z = \sqrt{\varepsilon(\omega)\omega^2 - k_x^2 - k_y^2}$ and $\mathbf{n} = (0, 0, 1)$.

Unknown vector functions $\mathbf{v}(\omega, \mathbf{k}_{\|})$ and $\mathbf{u}(\omega, \mathbf{k}_{\|})$ can be found from the system of boundary conditions imposed on electric and magnetic fields:

$$\mathrm{div}(\mathbf{E^1} - \mathbf{E^0}) = 0, \tag{27}$$

$$\mathrm{div}\mathbf{E^2} = 0, \tag{28}$$

$$E_x^1|_{z=0} = E_x^2|_{z=0}, \tag{29}$$

$$E_y^1|_{z=0} = E_y^2|_{z=0}, \tag{30}$$

$$H_x^1|_{z=0+} - H_x^2|_{z=0-} = 2aE_x^1|_{z=0}, \tag{31}$$

$$H_y^1|_{z=0+} - H_y^2|_{z=0-} = 2aE_y^1|_{z=0}. \tag{32}$$

We get in polar coordinates:

$$v_r = \left[-\frac{r_{TM} + a^2 T}{1 + a^2 T} \widetilde{N}_r + \frac{k_z}{\omega} \frac{aT}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{33}$$

$$v_\theta = \left[-\frac{\omega}{k_z} \frac{aT}{1 + a^2 T} \widetilde{N}_r + \frac{r_{TE} - a^2 T}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{34}$$

$$v_z = \frac{k_r}{k_z} \left[\frac{r_{TM} + a^2 T}{1 + a^2 T} \widetilde{N}_r - \frac{k_z}{\omega} \frac{aT}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{35}$$

where r_{TM}, r_{TE} are Fresnel reflection coefficients

$$r_{TM}(\omega, k_r) = \frac{\varepsilon(\omega) k_z - K_z}{\varepsilon(\omega) k_z + K_z}, \qquad r_{TE}(\omega, k_r) = \frac{k_z - K_z}{k_z + K_z} \tag{36}$$

and

$$T(\omega, k_r) = \frac{4 k_z K_z}{(k_z + K_z)(\varepsilon(\omega) k_z + K_z)}. \tag{37}$$

At this point it is convenient to define the local matrix R resulting from equations (33), (34):

$$R(a, \varepsilon(\omega), \omega, k_r) \equiv \frac{1}{1 + a^2 T} \begin{pmatrix} -r_{TM} - a^2 T & \frac{k_z}{\omega} a T \\ -\frac{\omega}{k_z} a T & r_{TE} - a^2 T \end{pmatrix}. \quad (38)$$

The tangential local components of the electric field in the interval $0 < z < d$ from the point dipole (16),(17) located at $(0, 0, z_0)$ are expressed in terms of matrices $R_1(\omega)$, $R_2(\omega)$ as follows:

$$\begin{pmatrix} E_r \\ E_\theta \end{pmatrix} = \frac{e^{ik_z z}}{I - R_2 R_1 e^{2ik_z d}} \left[R_2 R_1 \begin{pmatrix} N_r \\ N_\theta \end{pmatrix} e^{ik_z(2d - z_0)} + R_2 \begin{pmatrix} \widetilde{N_r} \\ \widetilde{N_\theta} \end{pmatrix} e^{ik_z z_0} \right]$$
$$+ \frac{e^{ik_z(2d - z)}}{I - R_1 R_2 e^{2ik_z d}} \left[R_1 R_2 \begin{pmatrix} \widetilde{N_r} \\ \widetilde{N_\theta} \end{pmatrix} e^{ik_z z_0} + R_1 \begin{pmatrix} N_r \\ N_\theta \end{pmatrix} e^{-ik_z z_0} \right], \quad (39)$$

in (39) the local components of the electric field are obtained by a summation of multiple reflections from media with indices 1 and 2.

It is convenient to define four matrices entering (39) after Wick rotation:

$$M^1 = \left(I - R_2(i\omega)R_1(i\omega)e^{-2k_z d}\right)^{-1} R_2(i\omega)R_1(i\omega), \qquad (40)$$

$$M^2 = \left(I - R_2(i\omega)R_1(i\omega)e^{-2k_z d}\right)^{-1} R_2(i\omega), \qquad (41)$$

$$M^3 = \left(I - R_1(i\omega)R_2(i\omega)e^{-2k_z d}\right)^{-1} R_1(i\omega)R_2(i\omega), \qquad (42)$$

$$M^4 = \left(I - R_1(i\omega)R_2(i\omega)e^{-2k_z d}\right)^{-1} R_1(i\omega). \qquad (43)$$

After integration over polar coordinates we express scattered electric Green functions at imaginary frequencies for coinciding arguments $\mathbf{r} = \mathbf{r}'$ in terms of matrix elements of matrices M:

$$D^{E,sc}_{xx}(i\omega, \mathbf{r} = \mathbf{r}') = D^{E,sc}_{yy}(i\omega, \mathbf{r} = \mathbf{r}') = -\frac{1}{8\pi} \int_0^\infty dk_r k_r$$
$$\times \Big[k_z (e^{-2k_z d} M^1_{11} + e^{-2k_z z_0} M^2_{11} + e^{-2k_z d} M^3_{11} + e^{-2k_z (d - z_0)} M^4_{11})$$
$$+ \frac{\omega^2}{k_z} (e^{-2k_z d} M^1_{22} + e^{-2k_z z_0} M^2_{22} + e^{-2k_z d} M^3_{22} + e^{-2k_z (d - z_0)} M^4_{22}) \Big] \qquad (44)$$

$$D_{zz}^{E,sc}(i\omega, \mathbf{r} = \mathbf{r}') = -\frac{1}{4\pi} \int_0^\infty dk_r \frac{k_r^3}{k_z}$$
$$\times \left[-e^{-2k_z d} M_{11}^1 + e^{-2k_z z_0} M_{11}^2 - e^{-2k_z d} M_{11}^3 + e^{-2k_z (d - z_0)} M_{11}^4) \right] \tag{45}$$

The Casimir-Polder potential can be evaluated by substituting (44), (45) into the formula

$$U(z_0) = -\int_0^\infty \frac{d\omega}{2\pi} \alpha^{ij}(i\omega) D_{ij}^{E,sc}(i\omega, \mathbf{r}', \mathbf{r}'). \tag{46}$$

For Chern-Simons layers in vacuum $\varepsilon(\omega)=1$ for $z<0$ and $z>d$.

$$M^1 = M^3 = -\frac{1}{(1+a_1^2)(1+a_2^2)\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_1 a_2(1 - a_1 a_2(1 - e^{-2k_z d})) & a_1 a_2 (a_1 + a_2)\frac{k_z}{\omega} \\ -a_1 a_2 (a_1 + a_2)\frac{\omega}{k_z} & a_1 a_2(1 - a_1 a_2(1 - e^{-2k_z d})) \end{pmatrix}, \quad (47)$$

$$M^2 = -\frac{1}{(1+a_1^2)(1+a_2^2)\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_2^2(1 + a_1^2(1 - e^{-2k_z d})) & -a_2(1 + a_1^2 + a_1 a_2 e^{-2k_z d})\frac{k_z}{\omega} \\ a_2(1 + a_1^2 + a_1 a_2 e^{-2k_z d})\frac{\omega}{k_z} & a_2^2(1 + a_1^2(1 - e^{-2k_z d})) \end{pmatrix}, \quad (48)$$

$$M^4 = -\frac{1}{(1+a_1^2)(1+a_2^2)\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_1^2(1 + a_2^2(1 - e^{-2k_z d})) & -a_1(1 + a_2^2 + a_1 a_2 e^{-2k_z d})\frac{k_z}{\omega} \\ a_1(1 + a_2^2 + a_1 a_2 e^{-2k_z d})\frac{\omega}{k_z} & a_1^2(1 + a_2^2(1 - e^{-2k_z d})) \end{pmatrix}. \quad (49)$$

Note that

$$\frac{1}{(1+a_1^2)(1+a_2^2)\det[I-R_1R_2e^{-2k_zd}]}$$
$$=\frac{1}{1+a_1^2+a_2^2+2a_1a_2e^{-2k_zd}+a_1^2a_2^2(1-e^{-2k_zd})^2}$$
$$=\frac{\gamma_1}{1+\beta_1 y}+\frac{\gamma_2}{1+\beta_2 y} \quad (50)$$

with $y=\exp(-2k_zd)$, $A=a_1^2a_2^2$, $B=2(a_1a_2-a_1^2a_2^2)$, $C=(1+a_1^2)(1+a_2^2)$, $y_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=(a_1a_2-1\pm i(a_1+a_2))/(a_1a_2)$, $\beta_1=-1/y_1$, $\beta_2=-1/y_2$, $\gamma_1=1/(Ay_1(y_2-y_1))$, $\gamma_2=1/(Ay_2(y_1-y_2))$.

Decomposition of the denominator in (50) into two terms leads to an analytic result for the Casimir-Polder potential in terms of Lerch transcendent functions. We change variables

$$\int_0^\infty k_r dk_r f(k_z) = \int_\omega^\infty k_z dk_z f(k_z) \tag{51}$$

and use the integral

$$G_0(\chi,\beta,\omega) \equiv \int_\omega^\infty \frac{e^{-2k_z\chi}}{1+\beta e^{-2k_z d}} dk_z = \frac{1}{2d} \int_0^{e^{-2\omega d}} \frac{y^{\frac{\chi}{d}-1}}{1+\beta y} dy$$
$$= \frac{e^{-2\omega\chi}}{2d} \Phi\left(-\beta e^{-2\omega d}, 1, \frac{\chi}{d}\right), \tag{52}$$

where $\Phi(\alpha_1, \alpha_2, \alpha_3)$ is a Lerch transcendent function.

At large distances of the atom from half spaces the Casimir-Polder potential has the following form:

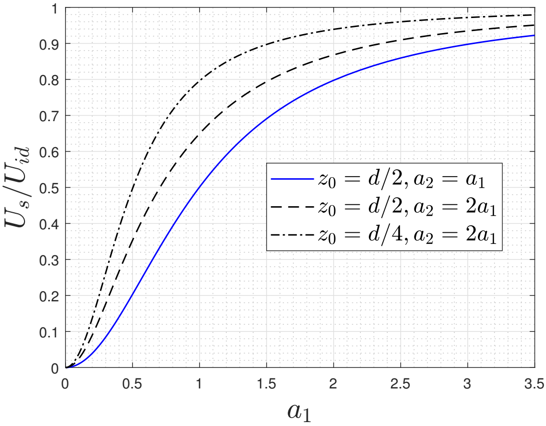
$$U_s(z_0, d) = U_{s1}(z_0, d) + U_{s2}(d) = \frac{\alpha_{xx}(0) + \alpha_{yy}(0) + \alpha_{zz}(0)}{32\pi^2 d^4}$$

$$\times \sum_{i=1,2} \gamma_i \Big[-a_2^2(1+a_1^2)\Phi\Big(y_i^{-1}, 4, \frac{z_0}{d}\Big) - a_1^2(1+a_2^2)\Phi\Big(y_i^{-1}, 4, \frac{d-z_0}{d}\Big)$$

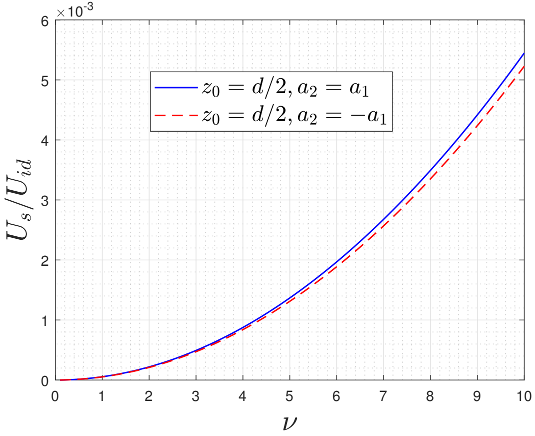
$$+ a_1^2 a_2^2 \Phi\Big(y_i^{-1}, 4, \frac{d+z_0}{d}\Big) + a_1^2 a_2^2 \Phi\Big(y_i^{-1}, 4, \frac{2d-z_0}{d}\Big)\Big] + U_{s2}(d), \quad (53)$$

$$U_{s2}(d) = \frac{\alpha_{xx}(0) + \alpha_{yy}(0) - \alpha_{zz}(0)}{32\pi^2 d^4} \Big(\mathrm{Li}_4\Big(\frac{a_1 a_2}{(a_1+i)(a_2+i)}\Big) + \mathrm{Li}_4\Big(\frac{a_1 a_2}{(a_1-i)(a_2-i)}\Big)\Big). \quad (54)$$

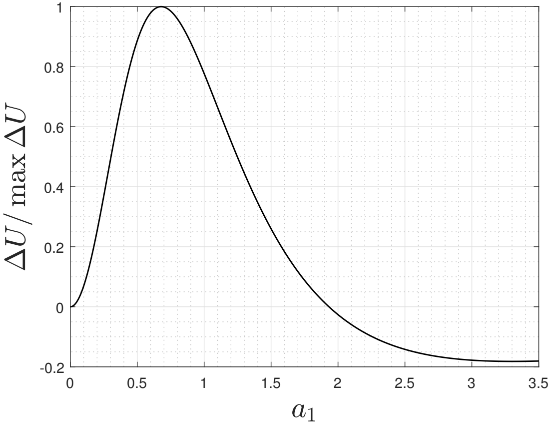
Here $\Phi(\alpha_1, \alpha_2, \alpha_3)$ - Lerch transcendent function, $\mathrm{Li}_4(z)$ is a polylogarithm function, $y_{1,2} = (a_1 a_2 - 1 \pm i(a_1 + a_2))/(a_1 a_2)$, $\gamma_1 = 1/(Ay_1(y_2 - y_1))$, $\gamma_2 = 1/(Ay_2(y_1 - y_2))$, $A = a_1^2 a_2^2$.



Ratios of the Casimir-Polder potential of a neutral polarizable isotropic atom located between two plane Chern-Simons layers in vacuum $U_s(z_0, d)$ to the potential of the same atom between two perfectly conducting planes $U_{id}(z_0, d)$, here z_0 is a distance of the atom from the layer characterized by a constant a_2, d is a distance between the layers.

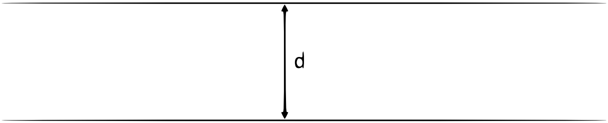


Ratios of the Casimir-Polder potentials $U_s(z_0, d)/U_{id}(z_0, d)$ differing by 180 degree rotation of the Chern-Simons layer characterized by a parameter a_2: $a_2 = a_1$ and $a_2 = -a_1$. Here z_0 is a distance of the atom from the layer characterized by a constant a_2, d is a distance between the layers, a dimensionless parameter $\nu = a_1/\alpha$ is quantized in quantum Hall layers and Chern insulators.



Ratio $\Delta U = U_s(z_0 = d/2, d, a_2 = -a_1) - U_s(z_0 = d/2, d, a_2 = a_1)$ to $\max \Delta U \approx 0.00587|U_{id}(z_0 = d/2, d)|$, $\max \Delta U$ holds at $a_1 \approx 0.678$.

The Casimir energy of two Chern-Simons layers in vacuum



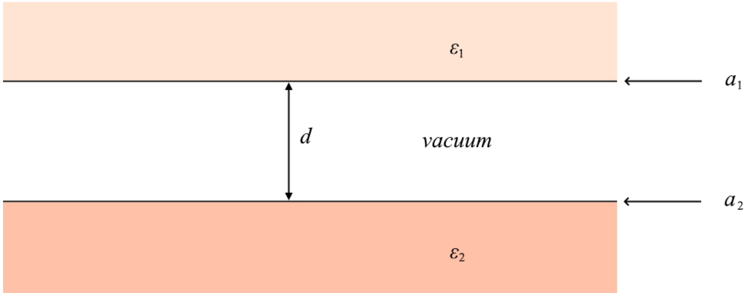
Two Chern–Simons layers in vacuum. The upper Chern–Simons layer is defined by a_1, the lower Chern–Simons layer is defined by a_2.

The Casimir energy of two Chern-Simons layers in vacuum is [V.N.Marachevsky, Theor.Math.Phys., 2017]:

$$E(a_1, a_2, d) = -\frac{1}{16\pi^2 d^3}\left(\mathrm{Li}_4\left(\frac{a_1 a_2}{(a_1+i)(a_2+i)}\right) + \mathrm{Li}_4\left(\frac{a_1 a_2}{(a_1-i)(a_2-i)}\right)\right), \qquad (55)$$

where $\mathrm{Li}_4(x) = \sum_{k=1}^{+\infty} x^k/k^4 = -\frac{1}{2}\int_0^{+\infty} drr^2 \ln(1 - xe^{-r})$.
Note that for $a_1 = -a_2$ the force is attractive for every a_1 (due to a theorem that the Casimir force between mirror objects is attractive). For $a_1 = a_2$ [V. N. Markov and Yu. M. Pis'mak, J. Phys. A: Math. Gen., 2006] one gets the Casimir energy of two Chern-Simons layers with identically selected directions of the layers in space. In this case the force is repulsive at all distances d for $a_1 \in [0, a_0]$, where $a_0 \approx 1.032502$, and attractive at all distances d for $a_1 > a_0$.

The Casimir effect for Chern-Simons layers at the boundaries of dielectric half spaces



Two dielectric half spaces with Chern–Simons boundary layers. The upper Chern–Simons boundary layer is defined by a_1; the lower Chern–Simons boundary layer is defined by a_2.

[V.N.Marachevsky, Phys.Rev.B, 2019]
[V.N.Marachevsky, Mod.Phys.Lett.A, 2020]
[V.N.Marachevsky and A.A.Sidelnikov, Physics, 2024]

Scattered magnetic Green's functions can be evaluated from reflected electric Green's functions:

$$D^H_{il}(\omega, \mathbf{r}, \mathbf{r}') = \frac{1}{\omega^2}\epsilon_{ijk}\epsilon_{lmn}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x'^m}D^E_{kn}(\omega, \mathbf{r}, \mathbf{r}'). \tag{56}$$

The Casimir pressure P equals the T_{zz} component of the fluctuation stress tensor in a slit between half spaces; it is expressed in terms of the scattered electric and magnetic Green's functions:

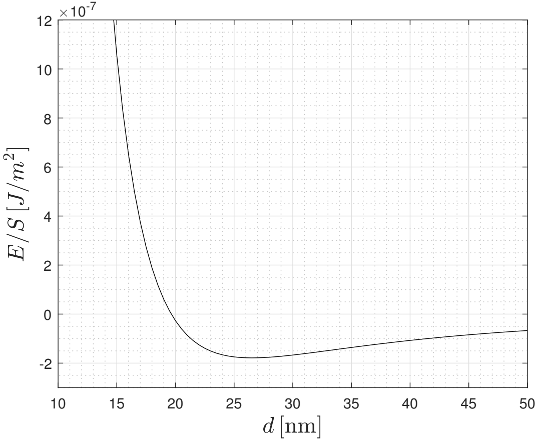
$$P = -\frac{i}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\Big[D^E_{xx}(\omega, \mathbf{r}, \mathbf{r}) + D^E_{yy}(\omega, \mathbf{r}, \mathbf{r}) - D^E_{zz}(\omega, \mathbf{r}, \mathbf{r}) + D^H_{xx}(\omega, \mathbf{r}, \mathbf{r}) + D^H_{yy}(\omega, \mathbf{r}, \mathbf{r}) - D^H_{zz}(\omega, \mathbf{r}, \mathbf{r})\Big]. \tag{57}$$

The Casimir pressure is expressed in terms of matrices $R_1(i\omega)$ and $R_2(i\omega)$ as follows:

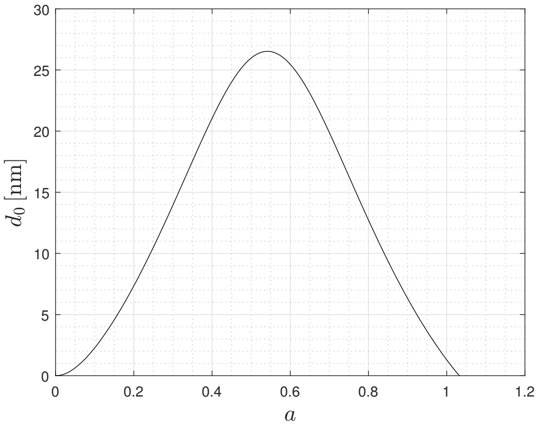
$$P = -\frac{1}{(2\pi)^2}\int_0^\infty d\omega \int_0^\infty dk_r k_r \widetilde{k}_z$$
$$\mathrm{Tr}\Big[\Big(I - R_2(i\omega)R_1(i\omega)e^{-2\widetilde{k}_z d}\Big)^{-1} R_2(i\omega)R_1(i\omega)e^{-2\widetilde{k}_z d} +$$
$$+\Big(I - R_1(i\omega)R_2(i\omega)e^{-2\widetilde{k}_z d}\Big)^{-1} R_1(i\omega)R_2(i\omega)e^{-2\widetilde{k}_z d}\Big], \quad (58)$$

where $\widetilde{k}_z \equiv \sqrt{\omega^2 + k_r^2}$.
The corresponding Casimir energy on a unit surface has the form

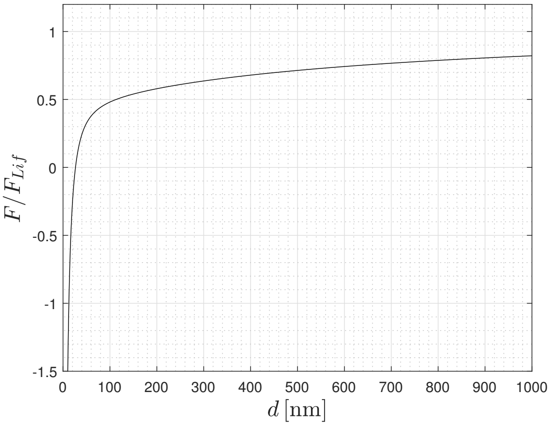
$$\frac{E}{S} = \frac{1}{(2\pi)^2}\int_0^\infty d\omega \int_0^\infty dk_r k_r \mathrm{Tr}\ln\Big(I - R_1(i\omega)R_2(i\omega)e^{-2\widetilde{k}_z d}\Big). \quad (59)$$



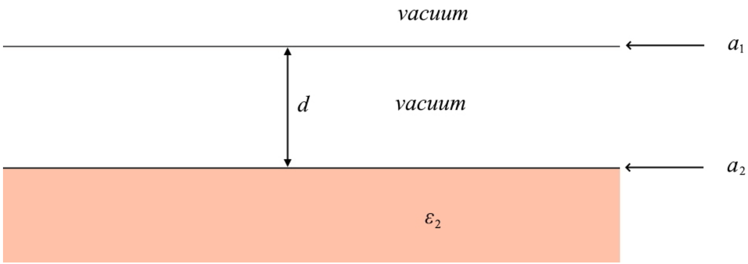
Energy on a unit surface for Chern-Simons layers with $a_1 = a_2 = 0.542$ at the boundaries of two $\mathrm{SiO_2}$ glass half spaces. The minimum of the energy is at $d_0 = 26.52$ nm.



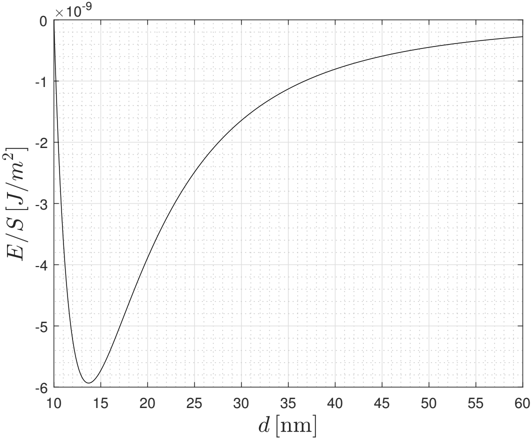
Position of the minimum of the energy d_0 for Chern-Simons layers at the boundaries of two $\mathrm{SiO_2}$ glass half spaces, $a \equiv a_1 = a_2$.



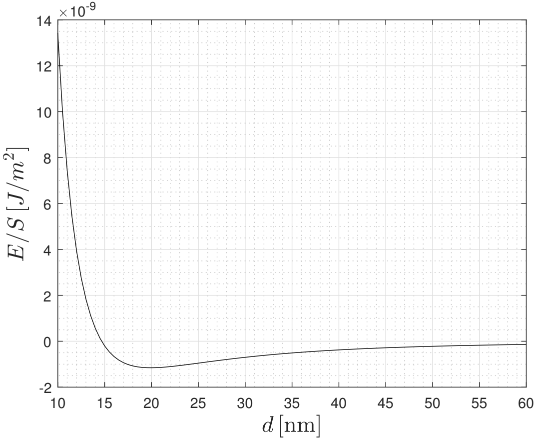
Ratio of the force F with Chern-Simons layers at the boundaries of two $\mathrm{SiO_2}$ glass half spaces to the force F_{Lif} between two $\mathrm{SiO_2}$ glass half spaces. Here $a_1 = a_2 = 0.542$.



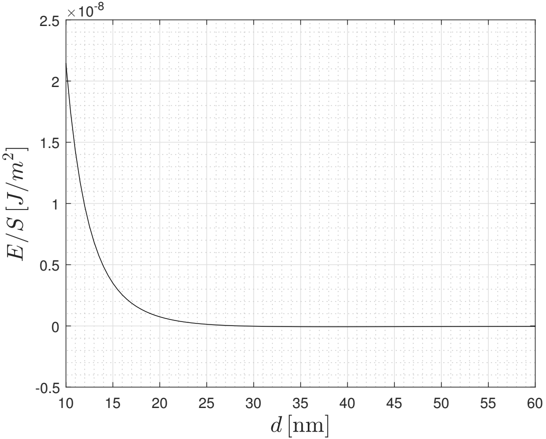
The Chern–Simons layer defined by a_1 is separated by a distance d from a dielectric half space, with the boundary Chern–Simons layer defined by a_2.



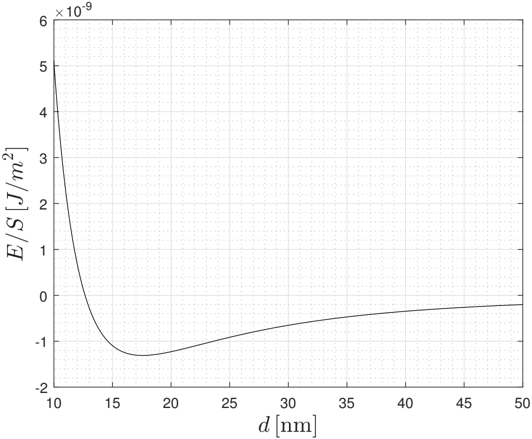
The Casimir energy for $\mathrm{SiO_2}$ glass half space substrate. Chern–Simons plane layers are defined by $a_1 = 6\alpha$, $a_2 = \alpha$.



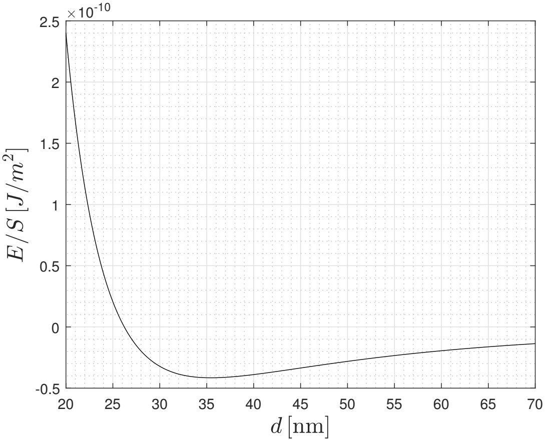
The Casimir energy for $\mathrm{SiO_2}$ glass half space substrate. Chern–Simons plane layers are defined by $a_1 = 5\alpha$, $a_2 = \alpha$.



The Casimir energy for $\mathrm{SiO_2}$ glass half space substrate. Chern–Simons plane layers are defined by $a_1 = 4\alpha$, $a_2 = \alpha$.



The Casimir energy for intrinsic Si half space substrate. Chern–Simons plane layers are defined by $a_1 = 2\alpha$, $a_2 = \alpha$.



The Casimir energy for intrinsic Si half space substrate. Chern–Simons plane layers are defined by $a_1 = \alpha$, $a_2 = \alpha$.

Explaining the minimum of the Casimir energy

Lifshitz force power law between two dielectrics/metals effectively changes from retarded d^{-4} to nonretarded d^{-3} behaviour at distances of the order $d \sim 10$ nm.

On the other hand, the force between two Chern-Simons layers in vacuum has d^{-4} behavior at all separations and thus dominates the total force at separations of the order $d \lesssim 10$ nm. For the condition $a \equiv a_1 = a_2$ the Casimir force between two Chern-Simons layers in vacuum is repulsive at all distances d for an interval $a \in [0, a_0]$, where $a_0 \approx 1.032502$.

As a result, the sum of the Lifshitz force and the force between two Chern-Simons layers in vacuum effectively leads to a repulsive force at short separations and to an attractive force at large separations.

Conclusions

1. A novel gauge-invariant formalism in the Casimir effect is presented.
2. Analytic results for the Casimir-Polder potential of a neutral anisotropic atom between two half-spaces with Chern-Simons boundary layers are derived and expressed through Lerch transcendent functions and polylogarithms.
3. P-odd three-body vacuum effects are predicted: there is a difference in values of the Casimir-Polder potential of a neutral atom after 180 degree rotation of one of the Chern-Simons layers. A neutral atom is described by QED dipole interaction.

Conclusions

4. Existence of a regime with the minimum of the Casimir energy due to presence of Chern-Simons layers at the surfaces of dielectrics, the Casimir force in this case is attractive at large distances and repulsive at short distances between the two dielectrics with Chern-Simons boundary layers.

Acknowledgments

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