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ФИАН

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A RELATIVISTIC EQUATIONS FOR BOUND-STATES WITH SOLUTIONS

Never do anything against conscience
even if the state demands it

Albert Einstein

Introduction and Summary

Ever since of success of the Feynman formalism in QED, corresponding field-theoretic formulations have been in the forefront of strong interaction dynamics since the early fifties of past century, the main strategy being to device various 'closed' form of approaches which are represented as appropriate 'integral' equations. One of the earliest efforts in this direction was the Tamm-Dancoff formalism,

I.E. Tamm:*J. Phys. 9:449, 1945;*
S.M. Dancoff:Phys. Rev. 78:382, 1950

which showed a great intuitive appeal. See also

V.P. Silin, I.Y. Tamm, V.Ya. Fainberg: *ZhETF 29:6, 1955.*

Introduction and Summary

The 3D Tamm-Dankoff equation and the 4D Schwinger-Dyson equation (SDE) have been the source of much wisdom underlying the formulation of many approaches to strong interaction dynamics. To these one should add the Bethe-Salpeter equation (BSE),

E.E. Salpeter and H.A. Bethe: *Phys. Rev. 84:1232, 1951*

which is an approximation to SDE for the dynamics of a 4D 2-particle amplitude, characterized by an effective interaction, for the N-N interaction, but now adapted to the quark level.

Introduction and Summary

and to Logunov-Tavkhelidze Quasi-potential equation

A.A. Logunov and A.N. Tavkhelidze: *Nuovo Cimento 29:380, 1963*

and Kadyshevsky formalism:

V.G. Kadyshevsky: *Sov Phys. JETF 19:443;597, 1964,;* Nucl. Phys. B6:125, 1968,;
C. Itzikson, V.G. Kadyshevsky, I.T. Todorov: *Phys. Rev. D1:2823, 1970.*

Introduction and Summary

The report is organized as follows:
In sections 2-3 we results the solutions of ladder perturbative BSE. And discuss about multi-fermion equations in QED.
In Section 4 we describe the method of construction of the MFE with the fermion bilocal source for the NJL model with the $SU(2)$–symmetric 4-quark interaction and, for the sake of completeness, consider the well-known leading approximation results of this model. Also in this section we investigate the first-after-leading step of the iteration scheme, which gives us the equations for the leading order 2-particle Green function and NLO correction to the propagator of quarks.
In Section 5 we describe the second step of the iteration scheme. As a result we obtain the equations for 4-quark Green function and for the 3-quark Green function. We also obtain in this step the equations for NLO 2-quark function and NNLO correction to quark propagator. We discuss the structure of second step equations and obtain the solutions of 4-quark and 3-quark equations.

Introduction and Summary

In Section 6 we describe the 3th step of iteration scheme. As a result we obtain the equations for 6-quark Green function and for the 5-quark Green function, and, the NLO equations for 4-quark and 3-quark Green functions. We also obtain in this step the equations for NNLO 2-particle function and NNNLO correction to quark propagator.
In Section 7 the modification of the MFE for the NJL model in the formalism with the multilocal diquark and triple-quark sources is briefly discussed.

Section 2. The ladder Bethe-Salpeter equations and their possible solutions

The ladder approximation in method of BSE for the scattering amplitude in field theory models was originally used justify Regge behavior at high energies

B.A. Arbuzov, A.A. Logunov, A.N. Tavkhelidze, R.N. Faustov: *Phys. Lett. 2:150, 1962;*
J.C. Polkinghorne: J. Math. Phys.4:503, 1963;
D. Amati, S. Fubini, A. Stanghellini: *Nuovo Cim.26:896, 1962;*
L. Bertocchi, S. Fubini, M. Tonin: Nuovo Cim.25:626, 1962

and was the point of departure in the construction of the multi-peripheral model.

Section 2.

Different methods have been used to obtain exact solutions of ladder BSE for forward scattering amplitude in a number of models and other works by B.A. Arbuzov and Co

B.A. Arbuzov, V.E. Rochev: *Yad.Fiz. 21:883, 1975;*
B.A. Arbuzov, V.Yu. Diakonov, V.E. Rochev: Yad.Fiz. 23:904, 1976;
K.G. Klimenko, V.E. Rochev: *Yad.Fiz. 31:448, 1980;*
V.Yu. Diakonov: TMF 43:218, 1980

Section 2.

In particular for ladder BSE for imaginary part of scattering amplitude $p + p' = k + k'$

$$F(s,t) = \pi\lambda^2\delta(s-\mu^2) + \frac{\pi\lambda^2}{(2\pi)^4}\int d^4q \frac{\theta(s'-q_0)\delta(q^2-\mu^2)}{[(p-q)^2-m^2][(k-q)^2-m^2]}F(s',t) \quad (1)$$

Here $s = (p+p')^2$, $s' = (p+p'-q)^2$, $t = (p-k)^2$ and μ is exchange mass, and m_0 is the mass in other propagators, $d^4q = dq_0|\overrightarrow{q}|d|\overrightarrow{q}|dcos\theta d\varphi$.

Section 2. Forward scattering:

In these works to find via different mathematical way (the inverse Mellin transformation,a and/or via diagonalized way by means of an expansion in Gegenbauer polinomial and at $t = 0$ - only forward scattering, in common approximately in form

$$F(s') = C(g^2, \alpha)(\frac{s}{m^2})^{\alpha},$$

where $g^2 = \frac{\lambda^2}{32\pi^2 m^2}$, and Regge parameter α has the form

$$\alpha = -\frac{1}{2} + \sqrt{\frac{1}{4} + g^2}.$$

Such result lead us to idea, which consist in finding the solution in starting as Regge form of behavior of scattering amplitude.

Subsection 2.1. Easy way for solution of ladder BSE for imaginary part of forward scattering amplitude

Let us to introduce in kernel of integral (1) a one as integral $1 = \int \delta((p + p' - q)^2 - s')ds'$. In case of forward scattering $p = k$, $p' = k'$ and $p^2 = m^2$, the integration with respect to φ, $dcos\theta$, dq_0 and $d|\overrightarrow{q}|$ in c.m.s. $|\overrightarrow{p}| + |\overrightarrow{p'}| = 0$ is trivial. The result is

$$F(s) = \frac{\pi^2\lambda^2}{2(2\pi)^4 m^2} \int d(\frac{s'}{s}) \frac{(1 - \frac{s'}{s})F(s')}{(1 - \frac{s'}{s})^2 + \frac{\mu^2}{m^2}}$$

Let us to find the solution as

$F(s) = s^{\alpha}$.

The result of integration is the sum of two hypergeometric equations

$$64\pi^2\mu^2 \frac{(\alpha + 1)(\alpha + 2)}{\lambda^2} = F(1, 2; \alpha + 3; -i\frac{m}{\mu}) + F(1, 2; \alpha + 3; i\frac{m}{\mu}). \quad (2)$$

Subsection 2.1. Forward scattering

1) In case $m << \mu$,

$$\alpha = -\frac{3}{2} \pm \frac{1}{2}\sqrt{1 + \frac{\lambda^2}{8\pi^2\mu^2}}.$$

2)At $\mu << m$

$$\alpha = -n \pm (-\frac{32\pi^2 m^2}{\lambda^2} + \frac{1}{2} ln \frac{\mu^2}{m^2}), \quad n = 1, 2, 3, ...$$

S.A. Gadjiev, R.G. Jafarov: *Dokl.AN Azerb. , v.XLII: No11: 20, 1986;*
S.A. Gadjiev, R.G. Jafarov: Dokl.AN Azerb. , v.XLIII: No1:34, 1987.

The Eq.(1) after the integration with respect to φ, $dcos\theta$, dq_0 and $d|\overrightarrow{q}|$ in c.m.s. $|\overrightarrow{p}| + |\overrightarrow{p'}| = 0$, in case $p = k$, $p' = k'$ and $p^2 = m^2$ and $k^2 = m^2$ receive the form

$$F(s,t) = \frac{\pi^2\lambda^2}{8(2\pi)^4|\overrightarrow{p}|^2\sqrt{s}\sqrt{\frac{(s-s'+\mu^2)^2}{4s} - \mu^2}} \cdot$$

$$\cdot \int ds' dz \frac{F(s',t)}{(\beta + z)(z^2 + 2\beta z_0 z + \beta^2 + {z_0}^2 - 1)^{1/2}},$$

where $z = cos\theta$, $z_0 = cos\theta_0$, θ_0-the scattering angle and $\beta = \frac{\mu^2 - s + s'}{4|\overrightarrow{p}|\sqrt{\frac{(s-s'+\mu^2)^2}{4s} - \mu^2}}$.

Let us take in place $z_0 = 1 + \epsilon$, where $\epsilon << 1$ and to expand to series. We find the solution as $F(s,t) = s^{\alpha,t}$. The result of integrations is

$$64\pi^2\mu^2 \frac{(\alpha+1)(\alpha+2)}{\lambda^2(1 + \frac{t}{6m^2})} = F(1,2;\alpha+3; -i\frac{m}{\mu}) + F(1,2;\alpha+3; i\frac{m}{\mu}). \quad (3)$$

Subsection 2.2.

1) In case $m << \mu$,

$$\alpha = -\frac{3}{2} \pm \frac{1}{2}\sqrt{1 + \frac{\lambda^2}{8\pi^2\mu^2}(1 + \frac{t}{6m^2})}.$$

2)At $\mu << m$

$$\alpha = -n \pm (-\frac{32\pi^2 m^2}{\lambda^2(1 + \frac{t}{6m^2})} + \frac{1}{2} ln \frac{\mu^2}{m^2}), \quad n = 1, 2, 3, ...$$

In case $t = 0$ all results have exact co-ordinate with results of forward scattering.

S.A. Gadjiev, R.G. Jafarov: *Krat. Soobsh. po Fizike FIAN, No11:25, 1986.*
S.A. Gadjiev, R.G. Jafarov, A.I. Livashvili: Izvest. Vuzov. Fizika No5:49, 1989.

Section 3. Multi-particle equations

The multi-particle (three or more particle) generalizations of the 4D BSE have been studied in detail. A straightforward generalization of two-particle BSE has bee intensively studied in sixties-seventies of last century. A best exposition of these studied can be found in the work of Huang and Weldon

K. Huang and H.A. Weldon: *Phys. Rev.D11:257, 1975.*

These generalizations are based on the analysis of Feynman diagrams, and all statements have a perturbative sense only. A form of the equations was chosen arbitrary. An additional disadvantage of the diagrammatic method is the fact almost all propositions can be formulated in words and cannot be formalized. The above-mentioned difficulties cannot be resolved in the framework of the diagramma. However, the natural language exists for the description multi-particle equations in the framework of the Lagrangian field theory. There are Legendre transformations of the generating functional for the Green's functions.

Section 3.

Functional Legendre transformations were firstly introduced in quantum statistics and applied to the quantum field theory

Quantum Field Theory and Quantum Statistics (Essays in Honour of the sixtieth birthday of E.S. Fradkin) (Eds. I.A. Batalin, C. Isham, and G.A. Vilkovisky) Vols 1 and 11 (Bristol: Adam Hilger, 1987);
Rochev V.E.: *Teor. Mat. Fiz. 51:22, 1982.*

With these transformations one can obtain multi-particle equations as a consequence of Schwinger ones. These multi-particle equations are model-independent, and they do not depend on perturbation theory.
A number of perspective physical applications of the effective models are connected with multi-particle functions, which are, in the main, the subject of present report.

Section 3.1. New non-perturbative method in QED and the multi-fermion equations

The problem of nonperturbative calculations in QED arose practically simultaneously with the principal solution of the problem of perturbative calculations with based on renormalized coupling constant perturbation theory. It is necessary to recognize, however, that the progress in the nonperturbative calculations during last decades is not to large. A new approach to nonperturbative calculations in quantum electrodynamics is proposed in work

Rochev V.E.:*J.Phys. A33:7379, 2000.*

Section 3.1.

This approach is based on a regular iteration scheme for the solution of Schwinger-Dyson equations for generating the functional of Green functions of QED by an exactly soluble equation. Its solution generates a linear iteration scheme each step of which is described by a closed system of integro-differential equation.

Note that equations of Green function at leading approximation and at the first step of iteration scheme in two versions. First of them on the language of Feynman diagrams of perturbative theory is analog of summation of chain diagrams with fermion loop.

The second version of the iteration scheme can be compared on the diagram language a ladder summation. The generating functional has the form

$$G(J,\eta) = \int D(\psi,\bar{\psi},A) exp i\{\int (L + J_\mu(x)A_\mu(x)) - \int dxdy\bar{\psi}^\beta \eta^{\beta\alpha}(y,x)\psi^\alpha(x)\}.$$

Functional derivatives of G with respect to sources are vacuum expectation values. SDEs for the generating functional of Green functions of QED has the forms:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu)\frac{1}{i}\frac{\delta G}{\delta J_\nu} + ietr\left[\gamma_\mu \frac{\delta G}{\delta \eta}\right] + J_\mu G = 0, \quad (4)$$

$$G + (i\widehat{\partial} - m)\frac{\delta G}{\delta \eta} + \frac{e}{i}\gamma_\mu \frac{\delta^2 G}{\delta J_\mu \eta} - \eta \star \frac{\delta G}{\delta \eta} = 0. \quad (5)$$

In correspondence with the choice of the leading approximation i-th term of the iteration expansion of the generating functional

$$G = G^{(0)} + G^{(1)} + G^{(2)} + \cdots, \quad (6)$$

which is solution of iteration scheme equations. A solution of equations (4), (5) is looked in the form:

$$G^{(i)} = P^{(i)} G^{(0)}.$$

Since $P^{(0)} \equiv 1$, it is evident that for any "i" the functional $P^{(i)}$ is a polynomial in functional variables J and η. This circumstance is very important since it means the system of equations for coefficient functions of this functional take closed in any order of the iteration scheme.
This iteration scheme has no explicit small parameter. In some sense, the sources J and η play the role of such a parameter. Expansion (6) of the generating functional should be treated as an approximation of $G(J, \eta)$ near the point $J_\mu = 0$, $\eta = 0$.
The iteration equation for the generating functional of Green functions of chain approximation in switching off photon sources $J_\mu = 0$ has the form:

$$G^{(i)} + (i\widehat{\partial} - m_0)\frac{\delta G^{(i)}}{\delta \eta} - ie^2\{D_{\mu\nu} \star \gamma_\mu \frac{\delta}{\delta \eta} tr[\gamma_\nu \frac{\delta G^{(i)}}{\delta \eta}]\} = \eta \star \frac{\delta G^{(i-1)}}{\delta \eta}. \quad (7)$$

The solution of first step equation is

$$G^{(1)} = \left\{ \frac{1}{2} S_2 \star \eta^2 + S^{(1)} \right\}$$

Subsection 3.1.1.

Let us use the following Feynman graphical rules

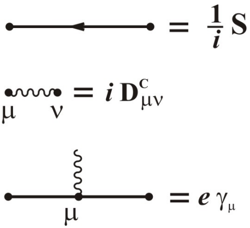


Figure 1.

we obtain the BSE in chain approximation

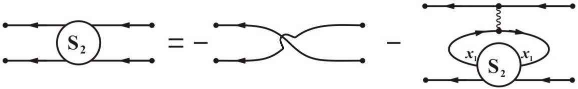


Figure 2.

as means as series (see Fig.2)

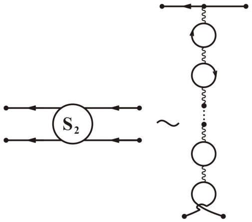


Figure 3.

The second step generation functional is

$$G^{(2)}(\eta) = P^{(2)}(\eta)G^{(0)}, \tag{8}$$

where

$$P^{(2)} = \frac{1}{4!}S_4 \star \eta^4 + \frac{1}{3!}S_3 \star \eta^3 + \frac{1}{2}S_2^{(1)} \star \eta^2 + S^{(2)} \star \eta.$$

The second iteration step contains the equations for the 4 S_4- and 3 S_3 functions and also the equations for the first order correction to 2-fermion function $S_2^{(1)}$ and 2-order correction equation to electron propagator $S^{(2)}$. For these 4 functions we have a system of four integral equations, which , and all equations, (also for next, ladder approximation equations) posses the similar structure.

$$S_n = S_n^0 - ie^2 \left\{ (D_{\mu\nu}^c \star S \cdot \gamma_\mu S) \star tr[\gamma_\nu S_n] \right\}$$

and differ from each other by the structure of inhomogeneous terms S_n^0.

Subsection 3.1.1.

The inhomogeneous term S_4^0 for 4-electron function is

$$S_4^0 = -3 \cdot \{S \cdot S \cdot S_2\},$$

where S_2 is very well known form.

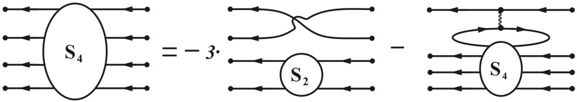


Figure 4.

The inhomogeneous term of 3-electron equation is

$$S_3^0 = -2 \cdot \{S \cdot S \cdot S^{(1)}\} - 2 \cdot \{S \cdot S_2\} - ie^2 \left\{ (D^c_{\mu\nu} \star S\gamma_\mu) \star tr[\gamma_\nu S_4] \right\}.$$

Here $S^{(1)}$ is first step correction electron function, which is defined in preceding step.

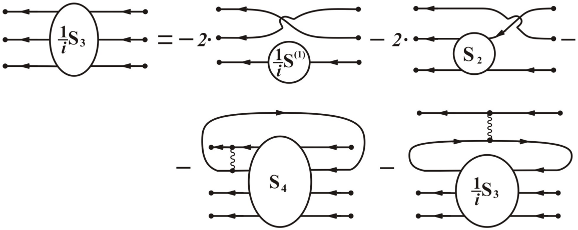


Figure 5.

Subsection 3.1.1.

The inhomogeneous term of the first order correction for two-electron function has the following form

$$(S_2^{(1)})^0 = -\{S \cdot S^{(1)}\} - ie^2 \left\{(D^c_{\mu\nu} \star S\gamma_\mu) \star tr[\gamma_\nu S_3]\right\},$$

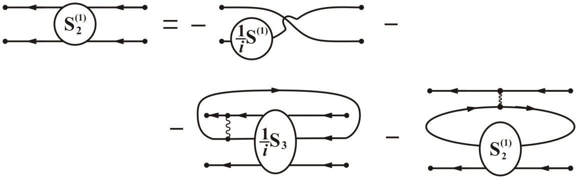


Figure 6.

and the inhomogeneous term of second-order correction for single electron function absence

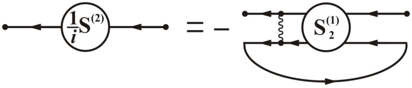


Figure 7.

Subsection 3.1.2. Ladder BSE

As we note the leading order and first step equations are very well known

Rochev V.E.:*J.Phys. A33:7379, 2000.*

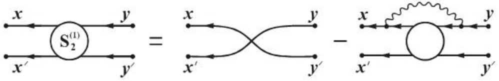
Here we would like to demonstrate the solution of ladder BSE for two-electron bound state and the constructing of second order equations.

Jafarov R.G.:*Izv. Akad. nauk Azerb. 25, No5:19, 2005;*
Gadjiev S.A., Jafarov R.G.:Izv. Akad. nauk Azerb. 26, No5:20, 2006.

This step leads us very to well known two-electron function equation is

$$S_2 = -S \cdot S + K \star S_2$$

where $K = ie^2\{tr[D_{\mu\nu} \star SS\gamma_\mu S_2 \gamma_\nu S],\}$ is the kernel of equation.

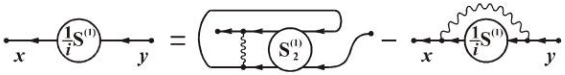


Subsection 3.1.2. Ladder BSE

The equation for first step electron propagator is

$$S^{(1)} = ie^2 D^c_{\mu\nu} \star S\gamma_\mu S_2 \gamma_\nu + ie^2 D^c_{\mu\nu} \star S\gamma_\mu S^{(1)} \gamma_\nu S$$

which have a following graphical form



Subsection 3.1.2. Ladder BSE

BSE in momentum space is

$$S^{-1} \cdot S_2 \cdot S^{-1} = 1 \cdot 1 + ie^2 D^c_{\mu\nu} \star \gamma_\mu S_2 \gamma_\nu$$

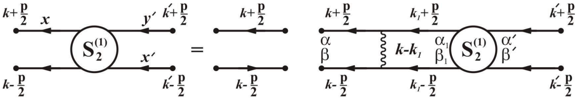
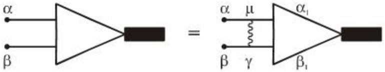


Figure 10.

The BSE for bound states is

$$S^{-1} \chi^{\alpha\beta} S^{-1} = ie^2 D^c_{\mu\nu} \star \gamma_\mu \chi^{\alpha\beta} \gamma_\nu$$



Section 4. Mean-field expansion for Nambu–Jona-Lasinio model and the multi-quark functions

A number of perspective physical applications of the effective models are connected with multi-quark functions, which are the subject of present report. The basic method of calculations is a formalism of multi-local (double, triple, etc.) sources

Aydan A. Garibli, Rauf G. Jafarov, and Vladimir E. Rochev Mean-Field Expansion, Regularization Issue, and Multi-Quark Functions in Nambu–Jona-Lasinio Model :*Symmetry 11 (2019) 5, 668.*

As an object of application of the method we choose Nambu - Jona-Lasinio (NJL) model
This model is one of the most successful effective models of quantum chromodynamics for the light hadrons. For review see

Klevansky S.P.:*Rev. Mod. Phys. 64:649, 1992;*
Hatsuda T. and Kunihiro T.:Phys. Reports 247:221, 1994;
Volkov M.K., Radjabov A.E.:*Uspekhi Fiz. Nauk 176:569, 2006.*

Section 4.

It is necessary to note, that this method has been successfully applied for the other field-theoretic models and can be applied also for analogous calculations in other similar effective models.

The multi-quark functions arise in higher orders of the MFE for the NJL model. To formulate the MFE we have used an iteration scheme of solution of the Schwinger-Dyson equation with the fermion bilocal source, which has been developed in works by Rochev. We have considered the equations for Green functions of the NJL model up to the third order of the MFE. The leading approximation and first order of the MFE maintains equations for the quark propagator and the 2-quark function and also the NLO correction to the quark propagator. The second order of MFE includes the equations for the 4-quark and the 3-quark functions and also the equations for the NLO 2-quark function and NNLO quark propagator.

Furthermore we have considered the generalization of the method in the framework of the NJL-type models, which includes the other multi-local sources (specifically, the diquark and 3-quark sources).
We have found a solution of the 4-quark and 3-quark equations. The solution of the 3-quark function is a disconnected combination of the leading-order functions and, consequently, the corresponding physical effects (i.e., pion-pion scattering) are suppressed in this order of the MFE. Therefore, we also investigate the third step of iterations, which gives us the equations for the 6-quark and 5-quark functions and the equations for the NLO 4-quark and 3-quark functions. The solution of the 6-quark functions equation has the disconnected form, which is similar to the solution for the 4-quark function of the preceding step. The solution of the second-step four-quark equation gives us a possibility to close the equation for the 3-quark function.

Subsection 4.1. The method. Leading order and first step equations

The Lagrangian of the 2-flavor NJL model may be written in the well-known form

$$L = \bar{\psi} i \hat{\partial} \psi + \frac{g}{2}\Big[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \tau^a \psi)^2\Big]. \qquad (9)$$

To construct the MFE we use an iteration scheme of the solution of functional-differential SDE

$$G + i\hat{\partial}\frac{\delta G}{\delta \eta} + ig\Big\{\frac{\delta}{\delta \eta} tr\Big[\frac{\delta G}{\delta \eta}\Big] + i\gamma_5\tau^a \frac{\delta}{\delta \eta} tr\Big[i\gamma_5\tau^a \frac{\delta G}{\delta \eta}\Big]\Big\} = \eta \star \frac{\delta G}{\delta \eta} \qquad (10)$$

for the generating functional G of Green functions.

The generating functional G can be represented as the functional integral with bilocal fermion source η:

$$G(\eta) = \int D(\psi, \bar{\psi}) \exp i\Big\{ \int dxL - \int dxdy\bar{\psi}(y)\eta(y,x)\psi(x) \Big\}. \quad (11)$$

We shall solve Eq. (10) employing the method which proposed in work by

Rochev V.E.Jafarov R.G.:*Central Eur. J. Phys. 2:367, 2004 (arXiv:hep-ph/0311339).*

The solution of the equation of leading approximation,i.e., the functional-differential SDE (10) with zero r.h.s., is the following functional $G^{(0)} = \exp\left\{\text{Tr}\left(S \star \eta\right)\right\}$, where S is solution of the equation

$$1 + i\hat{\partial}S + igS \cdot tr[S(0)] = 0. \quad (12)$$

The leading approximation generates the linear iteration scheme

$$G = G^{(0)} + G^{(1)} + \cdots + G^{(n)} + \cdots ,$$

consists in the step-by-step solutions of the equations

$$G^{(n)} + i\hat{\partial}\frac{\delta G^{(n)}}{\delta\eta} + ig\left\{\frac{\delta}{\delta\eta} tr\left[\frac{\delta G^{(n)}}{\delta\eta}\right] - \gamma_5\tau^a\frac{\delta}{\delta\eta} tr\left[\gamma_5\tau^{\mathbf{a}}\frac{\delta G^{(n)}}{\delta\eta}\right]\right\} = \eta\star\frac{\delta G^{(n-1)}}{\delta\eta}. \quad (13)$$

Functional $G^{(n)}$ is $G^{(n)} = P^{(n)}G^{(0)}$, where $P^{(n)}$ is a polynomial of $2n$ -th degree on the bilocal source η.

The unique connected Green function of the leading approximation S is the quark propagator. A solution of Eq. (12) is

$$S(p) = (m - \hat{p})^{-1},$$

where m is the dynamical quark mass, which is a solution of the gap equation of the NJL model in the chiral limit.

The other connected Green's functions appear in the subsequent steps of the iterative scheme.
The first iteration step contains the leading-order equation for the 2-particle S_2 quark function

$$S_2 = -S \cdot S + K \star S_2 \tag{14}$$

$K = ig\Big\{(S \cdot S) \star tr[S_2] - (S\gamma_5\tau^a S) \star tr[\gamma_5\tau^a S_2]\Big\}$ is the kernel of equation

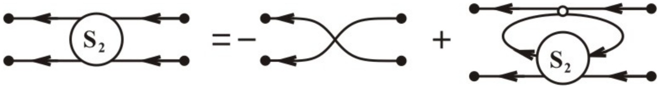


Figure 12.

and first order quark function equation

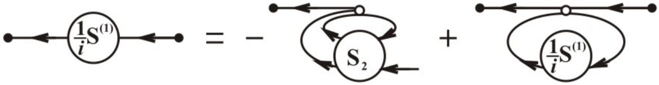


Figure 13.

Section 5. Second step equations

The second step contains the equations for the four S_4- and 3-particle S_3 functions and also the equations for the two-particle function $S_2^{(1)}$ and the second-order corrections to the quark propagator $S^{(2)}$. For these 4 functions we have a system of four integral equations. All these equations (and all equations of following steps of the iteration scheme) possess the structure, which is similar to the structure of Eq. (14):

$$S_n = S_n^0 + ig\Big\{(S \cdot S) \star tr[S_n] - (S\gamma_5\tau^a \cdot S) \star tr[\gamma_5\tau^a S_n]\Big\}, \qquad (15)$$

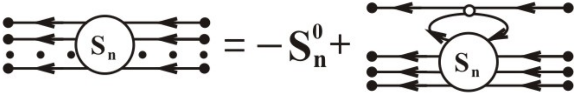


Figure 14.

and differ from each other by the structure of inhomogeneous terms S_n^0.

The inhomogeneous term in the equation for 4-quark function S_4 is

$$S_4^0 = -3 \cdot \Big\{ S \cdot S \cdot S_2 \Big\}, \qquad (16)$$

where S_2 is defined in preceding section by Eq. (14).

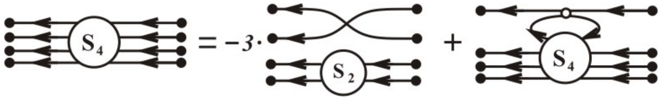


Figure 15.

Section 5. Second step equations

The inhomogeneous term in the equation for 3-quark function S_3

$$S_3^0 = -2\left\{S\cdot S\cdot S^{(1)}\right\} - 2\cdot\left[S\cdot S_2\right] + ig\cdot S\star\left\{tr[S_4] - \gamma_5\tau tr[\gamma_5\tau S_4]\right\}. \quad (17)$$

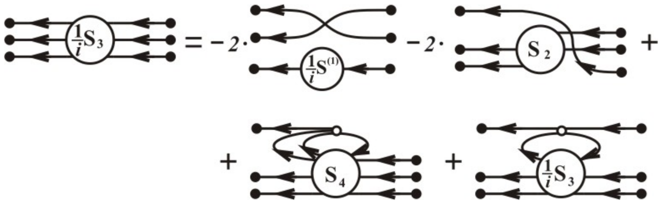


Figure 16.

The solution of 4-quark equation is the sum of products of 2-quark functions S_2:

$$S_4 = 3 \cdot \left\{ S_2 \cdot S_2 \right\} \qquad (18)$$

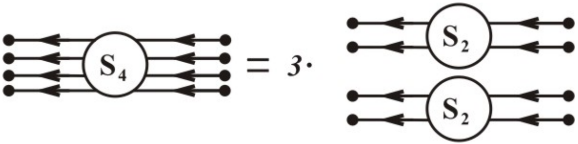


Figure 17.

R.G. Jafarov and V.E. Rochev: *Proceedings of the XXVIII International Workshop on the FPHEP and Field Theory(2005), New Physics at Colliders and Cosmic Rays, Moscow Region, Protvino, p.27-33, 2005 and in Proceedings of Workshop LHP06, Tehran, Iran, 2006 (arXiv: hep-ph/0609183).*

Section 6. Structure of third step of iteration step equations

As we have showed above the equation for the 4-quark function S_4 has a simple exact solution which is the product of first-order 2-quark functions (see Eq. (18)). As it seen from this solution, the pion-pion scattering in NJL model is suppressed, i.e. in the second order of MFE this scattering is absent. This process arises in the third order of our iterative scheme, i.e. in NLO 4-quark function $S_4^{(1)}$.

The third-step generating functional is

$$G^{(3)}[\eta] = \left\{ \frac{1}{6!} \mathrm{Tr}\left(S_6 * \eta^6 \right) + \frac{1}{5!} \mathrm{Tr}\left(S_5 * \eta^5 \right) + \frac{1}{4!} \mathrm{Tr}\left(S_4^{(1)} * \eta^4 \right) + \frac{1}{3!} \mathrm{Tr}\left(S_3^{(1)} * \eta^3 \right) + \frac{1}{2} \mathrm{Tr}\left(S_2^{(2)} * \eta^2 \right) + \mathrm{Tr}\left(S^{(3)} * \eta \right) \right\} G^{(0)}.$$

R.G. Jafarov: *Fizika Azerb NAS, XI, No 3:27,2005.*

Section 6. Structure of third step of iteration step equations

After standard operations we obtain the equations for six-quark function S_6 and for 5-quark function S_5. Inhomogeneous terms are following:

$$S_6^0 = 5 \cdot \left\{ -S \cdot S \cdot S_4 \right\} \tag{19}$$

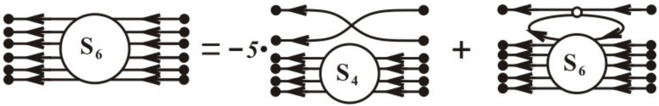


Figure 18.

and

$$S_5^0 = -4\cdot\left\{S\cdot S\cdot S_3\cdot\right\} - 4\cdot\left[S\cdot S_4\right] + ig\left\{tr\left[S\star S_6\right] - tr\left[S\gamma_5\tau^a\star S_6\gamma_5\tau^a\right]\right\}, \qquad (20)$$

accordingly. The equations for 6-quark function and for the 5-quark function with inhomogeneous term (19) and (20) in our iteration scheme are new. The third step of iterative scheme gives us the equation for 4-quark function ($S_4^{(1)}$).

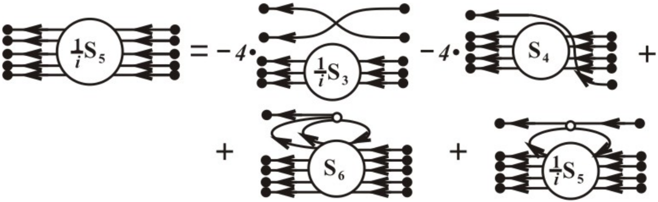


Figure 19.

Section 6. Structure of third step of iteration step equations

As we note above the structure of this equation have are the form (15) with following inhomogeneous term

$$(S_4^{(1)})^0 = -3\cdot\left\{S\cdot S\cdot S_2^{(1)}\right\} - 3\cdot\left[S\cdot S_3\right] + ig\left\{tr\left[S\star S_5\right] - tr\left[S\gamma_5\tau^a\star S_5\gamma_5\tau^a\right]\right\}. \tag{21}$$

The equation for NLO 4-quark function $S_4^{(1)}$ gives us possibility to describe the pion-pion scattering in quark fields context. The inhomogeneous term (21) of equations for 4-quark function $S_4^{(1)}$ contains 5-quark function S_5, three-quark function S_3 and 2-quark function $S_2^{(1)}$. The inhomogeneous term (20) for 5-quark equation include the 6-quark function S_6, 4-quark function S_4 and 3-quark function S_3. Before the investigation of four-quark function $S_4^{(1)}$ it is necessary to find the solution of equation for 6-quark function S_6, because the inhomogeneous part (20) includes function S_6. Also it is necessary to find a solution of equation for NLO 2-quark function $S_2^{(1)}$.

Section 6. Structure of third step of iteration step equations

The solution of six-quark equation is the sum of products of 2-quark functions S_2 and 4-quark functions S_4:

$$S_6 = 5 \cdot \left\{ S_2 \cdot S_4 \right\} \tag{22}$$

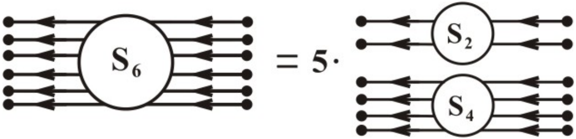


Figure 20.

In this step we obtain also the equations for NLO 3-quark function $S_3^{(1)}$, NNLO 2-quark function $S_2^{(2)}$ and the equation for NNNLO correction to the quark propagator $S^{(3)}$, which matter the forms (15), at $n = 3$, $n = 2$, $n = 1$, accordingly.

Jafarov R.G.:*Izv. Akad. nauk Azerb. v.XXVI, No2:3, 2006.*

Section 7. The formalism of other type sources

In this last Section we consider the generalization of MFE of Section 2, which includes other types of multi-quark sources except of bilocal source η. Such generalization can be useful for the description of baryons in the framework of MFE.

R.G. Jafarov and V.E. Rochev: *Talk given in QUARKS-2010 16th International Seminar on High Energy Physics Kolomna, Russia, 6-12 June, 2010;*

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Aydan A. Garibli, Rauf G. Jafarov, and Vladimir E. Rochev Mean-Field Expansion, Regularization Issue, and Multi-Quark Functions in Nambu–Jona-Lasinio Model :Symmetry 11 (2019) 5, 668.

Firstly, consider the formalism with diquark sources. For this purpose, we add two diquark-source terms ξ and $\bar{\xi}$ in the exponent of Eq. (11) for generating functional G:

$$G(\eta,\xi,\bar{\xi}) = \int D(\psi,\bar{\psi}) \exp i\Big\{ \int dx L - \int dxdy \bar{\psi}(y)\eta(y,x)\psi(x) + \\ + \int dx_1 dx_2 \bar{\psi}(x_1)\bar{\psi}(x_2)\xi(x_1,x_2) + \int dx_1 dx_2 \bar{\xi}(x_1,x_2)\psi(x_1)\psi(x_2)\Big\}. \quad (23)$$

With these sources SDE (10) is modified as follows:

$$G + i\hat{\partial}\frac{\delta G}{\delta \eta} + ig\Big\{\frac{\delta}{\delta\eta} tr\left[\frac{\delta G}{\delta\eta}\right] - \gamma_5\tau^a \frac{\delta}{\delta\eta} tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta}\right]\Big\} = \\ = \eta\star\frac{\delta G}{\delta\eta} + 2\cdot\frac{\delta G}{\delta\xi}\star\xi. \quad (24)$$

We have, apart from SDE (24), the additional SDE, which generates by new sources:

$$i\hat{\partial}\frac{\delta G}{\delta\bar{\xi}} + ig\left\{\frac{\delta}{\delta\bar{\xi}}tr\left[\frac{\delta G}{\delta\eta}\right] - \gamma_5\tau^a\frac{\delta}{\delta\bar{\xi}}tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta}\right]\right\} =$$
$$= \eta\star\frac{\delta G}{\delta\bar{\xi}} - 2\cdot\bar{\xi}\star\frac{\delta G}{\delta\eta}. \qquad (25)$$

It should be noted, that the presence of the new diquark source leads to the connection condition for derivatives of generating functional:

$$\frac{\delta^2 G}{\delta\bar{\xi}(x_2,x_1)\delta\eta(y,x)} = -\frac{\delta^2 G}{\delta\bar{\xi}(x_1,x)\delta\eta(y,x_2)}. \qquad (26)$$

Due to this connection condition SDE (25) can be rewritten in the alternative forms. These alternative forms, being fully equivalent from the point of view of an exact solution of SDE's, can lead to different approximations in the MFE. The choice of the suitable forms for the construction of MFE in the case should be made with an assistance of corresponding physical reasons.

In the very similar manner one can introduce 3-quark, or baryon sources. These sources can be used for the direct description of nucleons and other baryons omitting the intermediate diquark modelling. The generating functional with anti-commutative three-quark sources ζ and $\bar{\zeta}$ is

$$G(\eta,\zeta,\bar{\zeta}) = \int D(\psi,\bar{\psi}) \exp i\Big\{ \int dx L - \int dx dy \bar{\psi}(y)\eta(y,x)\psi(x) +$$
$$+ \int dx_1 dx_2 dx_3 \bar{\psi}(x_1)\bar{\psi}(x_2)\bar{\psi}(x_3)\zeta(x_1,x_2,x_3) +$$
$$+ \int dx_1 dx_2 dx_3 \bar{\zeta}(x_1,x_2,x_3)\psi(x_1)\psi(x_2)\psi(x_3)\Big\}. \tag{27}$$

The master-equations for generating of SDEs are follows

$$0=\int D(\psi,\bar{\psi})\frac{\delta}{\delta\bar{\psi}^{\alpha}(x)^{c}_{j}}\bar{\psi}^{\beta}(y)^{d}_{k}\times$$

$$\times\exp i\Big[\int dxL-\int dxdy\bar{\psi}(y)\eta(y,x)\psi(x)+$$

$$+\int dxdydz\bar{\xi}(xyz)\psi(x)\psi(y)\psi(z)+\int dxdydz\bar{\psi}(x)\bar{\psi}(y)\bar{\psi}(z)\xi(xyz)\Big],$$

$$0=\int D(\psi,\bar{\psi})\frac{\delta}{\delta\bar{\psi}^{\alpha_1}(x_1)^{c_1}_{j_1}}\psi^{\alpha_3}(x_3)^{c_3}_{j_3}\psi^{\alpha_2}(x_2)^{c_2}_{j_2}\times$$

$$\times\exp i\Big[\int dxL-\int dxdy\bar{\psi}(y)\eta(y,x)\psi(x)+$$

$$+\int dxdydz\bar{\xi}(xyz)\psi(x)\psi(y)\psi(z)+\int dxdydz\bar{\psi}(x)\bar{\psi}(y)\bar{\psi}(z)\xi(xyz)\Big].$$

i.e.

$$\bar{\psi}^{\beta}(y)_k^d\,\psi^{\alpha}(x)_j^c \to i\frac{\delta}{\delta\eta^{\beta\alpha}(y,x)_{kj}^{dc}},$$

$$\psi^{\gamma}(z)_l^e\psi^{\beta}(y)_k^d\,\psi^{\alpha}(x)_j^c \to -i\frac{\delta}{\delta\bar{\xi}^{\gamma\beta\alpha}(zyx)_{lkj}^{edc}},$$

$$\bar{\psi}^{\gamma}(z)_l^e\bar{\psi}^{\beta}(y)_k^d\,\bar{\psi}^{\alpha}(x)_j^c \to i\frac{\delta}{\delta\xi^{\gamma\beta\alpha}(zyx)_{lkj}^{edc}}.$$

Subsection 7.2. The formalism with triple-sources

SDE (10) with 3-quark sources is modified as follows:

$$G + i\hat{\partial}\frac{\delta G}{\delta\eta} + ig\Big\{\frac{\delta}{\delta\eta}tr\Big[\frac{\delta G}{\delta\eta}\Big] - \gamma_5\tau^a\frac{\delta}{\delta\eta}tr\Big[\gamma_5\tau^a\frac{\delta G}{\delta\eta}\Big]\Big\} =$$

$$= \eta \star \frac{\delta G}{\delta\eta} - 3\cdot\frac{\delta G}{\delta\xi}\star\zeta. \qquad (28)$$

As above, apart from SDE (28), the additional SDE exists, which generates by the 3-quark sources:

$$i\hat{\partial}\frac{\delta G}{\delta\bar{\zeta}} + ig\Big\{\frac{\delta}{\delta\bar{\zeta}}tr\Big[\frac{\delta G}{\delta\eta}\Big] - \gamma_5\tau^a\frac{\delta}{\delta\bar{\zeta}}tr\Big[\gamma_5\tau^a\frac{\delta G}{\delta\eta}\Big]\Big\} =$$

$$= \eta \star \frac{\delta G}{\delta\bar{\zeta}} + 3i\cdot\frac{\delta^2 G}{\delta\eta\delta\eta}\bar{\zeta}. \qquad (29)$$

The connection condition for the derivatives of the generating functional, which is very similar to the condition(26), also exists in the three-quark-source formalism, and also leads to alternative forms of SDE(29).

Section 7

The method of the construction of MFE for these system of equations is similar to that of Section 2.
An analysis of this construction is the object of future investigations!.

Acknowledgments

Thanks for patience and attention!

Thanks for questions!

Much to my regret!

to encounter