Resonant entanglement of photon beams by magnetic field

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Introduction 1

Entanglement is a pure quantum property which is associated with quantum non-separability of parts of a composite system. Different views on what is actually happening in the process of quantum entanglement may be related to different interpretations of quantum mechanics and **QFT**. Entangled states are considered as key elements in quantum information theory in quantum computations and quantum cryptography technologies. See e.g. J.S. Bell, **Speakable and unspeakable in QM**, Quantum theory-Collected works (Cambridge 1987); Nielsen, Chuang, **Quantum computation and quantum information** (2000); Witten, **Notes on some entanglement properties of QFT**, RMP 90 (2018);
!!! C.I. Doronin, **Quantum Magic** (Ves. 2007).
We recall that maximally defined states in **QT** (**pure states, PS**) are described by vectors $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, $|\psi\rangle \in \mathcal{H}$, or by a statistical operator $\hat{\rho}$, which is in this case the projector $\hat{\rho} = \hat{P}_\psi = |\psi\rangle\langle\psi|$. In canonical interpretation of **QM** any pure state can be prepared as an ensemble of identical copies defined by certain external conditions.

Introduction 2

In general case external conditions do not define all the copials maximal exactly. In this case we speak of a mixed state (**MS**). A **MS** is described by statistical operator $\hat{\rho}$, which in a simplest case reads:

$$\hat{\rho} = \sum_n \lambda_n \hat{P}_{\psi_n}, \ \hat{P}_{\psi_n} = |\psi_n\rangle\langle\psi_n|, \ \langle A\rangle = \mathrm{tr}\hat{A}\hat{\rho} = \sum_n \lambda_n \langle\psi_n|\hat{A}|\psi_n\rangle.$$

MS are interpreted as a mixture with the weight λ_n of many maximally defined copies that correspond to pure states $\hat{P}_{\psi_n}$. On the other hand, in connection with discussions started by Einstein and others about the completeness of QM description of reality (the EPR paradox, hidden variables in QM, Bell's inequalities, and the possibility of considering QM as a global theory), it turned out to be useful to consider and study the so-called **entangled states** (**ES**) a definition of ES is given just below. Technically, such states often arise when considering so-called **composite systems**, which in their classical treatment consist of several parts.

Qubit systems

We recall that a **qubit** is a two-level quantum-mechanical system with state vectors (two columns)

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{H} = \mathbb{C}^2, \ \langle \psi' \,| \psi \rangle = \psi_1'^{*} \psi_1 + \psi_2'^{*} \psi_2 \ .$$

An orthogonal basis $|a\rangle$, $a = 0, 1$ in $\mathcal{H}$ can be chosen as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\langle a \,| a' \rangle = \delta_{aa'}, \ \sum_{a=0,1} |a\rangle \langle a| \ = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A photon with two possible linear polarizations is an example of qubit. Besides, two levels can be taken as spin up and spin down of an electron.

Two-qubit systems

Consider a composit system, a **two-qubit**, composed of two qubits A and B with the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_{A/B} = \mathbb{C}^2$. **Two-qubit** is a four level system. If $|a\rangle_A$ and $|b\rangle_B$, $a, b = 0, 1$, are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, then $|\alpha b\rangle = |a\rangle \otimes |b\rangle$ is a complete and orthonormalized basis in $\mathcal{H}_{AB}$,

$$|ab\rangle = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}.$$

The so-called computational basis $|\Theta\rangle_s$, $s = 1, 2, 3, 4$, reads:

$$|\Theta\rangle_1 = |00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \ |\Theta\rangle_2 = |01\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T,$$
$$|\Theta\rangle_3 = |10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \ |\Theta\rangle_4 = |11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Two photons moving in the same direction with different frequencies and any of two possible linear polarizations is an important example of a two-qubit system.

Entanglement in two-qubit systems 1

A **PS** $|\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ is called **separable** iff it can be represented as: $|\Psi\rangle_{AB} = |\Psi\rangle_{A} \otimes |\Psi\rangle_{B}$, $|\Psi\rangle_{A} \in \mathcal{H}_{A}$, $|\Psi\rangle_{B} \in \mathcal{H}_{B}$. Otherwise, it is **entangled**- **ES**. The first discovery within quantum information theory, which involves entanglement states, is due to Ekert (1991). He paid attention on the existence of a highly correlated state $|\psi\rangle_{\mathrm{Bell}} = \frac{1}{\sqrt{2}}\left(|0\rangle\,|1\rangle - |1\rangle\,|0\rangle\right)$-**Bell** state, and on the fact that Bell inequalities are violated by this state.
An **entanglement measure (EM)** $M\left(|\Psi\rangle_{AB}\right)$ of a state $|\Psi\rangle_{AB}$ is real and positive. The **EM** is zero for separable states, and is 1 for maximally **ES**. The information **EM** reads:

$$M\left(|\Psi\rangle_{AB}\right) = S\left(\hat{\rho}_{A}\right) = S\left(\hat{\rho}_{B}\right)\ ,\ S\left(\hat{\rho}_{A/B}\right) = -\mathrm{tr}\left(\hat{\rho}_{A/B}\log\hat{\rho}_{A/B}\right)\ ,$$
$$\hat{\rho}_{A} = \mathrm{tr}_{B}\hat{\rho}_{AB} = \sum_{b}\langle b\,|\hat{\rho}_{AB}|\,b\rangle\ ,\quad \hat{\rho}_{B} = \mathrm{tr}_{A}\hat{\rho}_{AB} = \sum_{a}\langle a\,|\hat{\rho}_{AB}|\,a\rangle\ ,$$

where $S\left(\hat{\rho}\right) = -\mathrm{tr}\left(\hat{\rho}\log\hat{\rho}\right)$ is von Neumann entropy of $\hat{\rho}$. One can see that $S\left(\rho_{A}\right) = S\left(\rho_{B}\right)$.

Entanglement in two-qubit systems 2

For **PS** $|\Psi\rangle_{AB} = \sum_{s=1}^{4} v_s |\Theta\rangle_s$, where $|\Theta\rangle_s$ is computational basis,

$$|\Theta\rangle_1 = |00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \; |\Theta\rangle_2 = |01\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T,$$
$$|\Theta\rangle_3 = |10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \; |\Theta\rangle_4 = |11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T,$$

we obtain the corresponding density operator:

$$\hat{\rho}_{AB} = {}_{AB}|\Psi\rangle\langle\Psi|_{AB} = [v_1 |00\rangle + v_2 |01\rangle + v_3 |10\rangle + v_4 |11\rangle]$$
$$\times [v_1^* \langle 00| + v_2^* \langle 01| + v_3^* \langle 10| + v_4^* \langle 11|] .$$

The reduced density operator of the subsystem A can be easily obtained:

$$\hat{\rho}_A = {}_B\langle 0|\hat{\rho}_{AB}|0\rangle_B + {}_B\langle 1|\hat{\rho}_{AB}|1\rangle_B$$
$$= |v_1|^2 |0\rangle\langle 0| + v_1 v_3^* |0\rangle\langle 1| + |v_2|^2 |0\rangle\langle 0| + v_2 v_4^* |0\rangle\langle 1|$$
$$+ v_3 v_1^* |1\rangle\langle 0| + |v_3|^2 |1\rangle\langle 1| + v_4 v_2^* |1\rangle\langle 0| + |v_4|^2 |1\rangle\langle 1| .$$

Entanglement in two-qubit systems 3

The corresponding density matrix reads:

$$\rho_{11}^{(A)} = |v_1|^2 + |v_2|^2, \ \rho_{12}^{(A)} = v_1 v_3^* + v_2 v_4^* \ ,$$
$$\rho_{21}^{(A)} = v_3 v_1^* + v_4 v_2^*, \ \rho_{22}^{(A)} = |v_3|^2 + |v_4|^2.$$

One can calculate its eigenvalues and eigenvectors,

$$\hat{\rho}^{(A)} P_a = \mu_a P_a \ , \ \mu_a = \frac{1}{2}\left(|v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + (-1)^a y\right),$$
$$y = \sqrt{\left(|v_1|^2 + |v_2|^2 - |v_4|^2 - |v_3|^2\right)^2 + 4\left|v_1 v_3^* + v_2 v_4^*\right|^2},$$
$$P_a = \left(\frac{|v_1|^2 + |v_2|^2 - |v_4|^2 - |v_3|^2 + (-1)^a y}{2\left(v_3 v_1^* + v_4 v_2^*\right)}, 1\right)^T .$$

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Thus,

$$M\left(|\Psi\rangle_{AB}\right) = -\sum_{a=1,2} \mu_a \log_2 \mu_a$$
$$= -\left[z \log_2 z + (1-z)\log_2(1-z)\right], \; z = \frac{1}{2}(1+y).$$

Let us consider **EM** of the Bell state $|\psi\rangle_{\mathrm{Bell}}$,

$$|\psi\rangle_{\mathrm{Bell}} = \frac{1}{\sqrt{2}}\left(|\Theta\rangle_2 - |\Theta\rangle_3\right), \; v_1 = v_4 = 0, \; v_2 = -v_3 = \frac{1}{\sqrt{2}},$$
$$y = \sqrt{|v_2|^2 - |v_3|^2} = 0, \; z = 1/2 \Longrightarrow M\left(|\psi\rangle_{\mathrm{Bell}}\right) = 1,$$

therefore, $|\psi\rangle_{\mathrm{Bell}}$ is really a maximally **ES**. If $|\psi\rangle_{\mathrm{Bell}}$ can be prepared experimentally, this is one of the way to proof of the violation of Bell's inequalities.

Setting of the problem

Consider photons with two different momenta $\mathbf{k}_s = \kappa_s \mathbf{n}$, $s = 1, 2$ (frequencies $\omega = kc$), moving in the same direction $\mathbf{n} = (0, 0, 1)$ and interacting with quantized charged scalar particles-electrons placed in a constant magnetic field $\mathbf{B} = B\mathbf{n}$, potentials of which in the Landau gauge are: $\mathbf{A}_{\text{ext}}(\mathbf{r}) = \left(-Bx^2, 0, 0\right)$. Photons with each moment may have two possible linear polarizations $\lambda = 1, 2$. In the beginning, we consider electron subsystem consisting of one charged particle. Both quantized fields (electromagnetic and the **KG** one) are placed in a box of the volume $V = L^3$ and periodic conditions are supposed. For such a model we find some exact solutions. We interpret $\rho = V^{-1}$ as the electron density, in supposition that the model describes the photon beam interacting with many free spinless charged particles, the totality of which we call electron medium. We show that in a certain approximation, solutions of the model correspond to two independent subsystems, one of which is a q-electron medium and another one is a set of some q-photons. We use these solutions for calculating entanglement of photon beam by electron medium and by constant magnetic field, see Breev, Gitman, EPJC 2024.

QED model 1

The operator potentials $\hat{A}^{\mu}(\mathbf{r})$, $\mu = 0, \ldots, 3$, $\mathbf{r} = \left(x^1, x^2, x^3 = z\right)$ of the photon beam are chosen in the Coulomb gauge, $\hat{A}^{\mu}(\mathbf{r}) = \left(0, \hat{\mathbf{A}}(\mathbf{r})\right)$, $\operatorname{div} \hat{\mathbf{A}}(\mathbf{r}) = 0$, in fact, they depend only on z,

$$\hat{\mathbf{A}}(\mathbf{r}) = \sum_{s;\lambda=1,2} \sqrt{\frac{1}{2\kappa_s V}} \left[\hat{a}_{s,\lambda} \exp\left(i\kappa_s z\right) + \hat{a}^{\dagger}_{s,\lambda} \exp\left(-i\kappa_s z\right)\right] \mathbf{e}_{\lambda} = \hat{\mathbf{A}}(z) \ ,$$

$$\left[\hat{a}_{s,\lambda}, \hat{a}_{s',\lambda'}\right] = 0, \ \left[\hat{a}_{s,\lambda}, \hat{a}^{\dagger}_{s',\lambda'}\right] = \delta_{s,s'}\delta_{\lambda,\lambda'}, \ s, s' = 1, 2; \quad \lambda, \lambda' = 1, 2 \ . \quad (1)$$

Here $\hat{a}_{s,\lambda}$ and $\hat{a}^{\dagger}_{s,\lambda}$ are creation and annihilation operators of the f-photons from the beam, $\mathbf{e}_{\lambda}$ are real polarization vectors, $(\mathbf{e}_{\lambda}\mathbf{e}_{\lambda'}) = \delta_{\lambda,\lambda'}$, $(\mathbf{n}\mathbf{e}_{\lambda}) = 0$. The photon Fock space $\mathfrak{H}_{\gamma}$ is constructed by the creation and annihilation operators and by the vacuum vector $|0\rangle_{\gamma}$, $\hat{a}_{s,\lambda}|0\rangle_{\gamma} = 0$, $\forall s, \lambda$. Photon vectors are denoted as $|\Psi\rangle_{\gamma}$, $|\Psi\rangle_{\gamma} \in \mathfrak{H}_{\gamma}$. The Hamiltonian of f-photon beam reads:

$$\hat{H}_{\gamma} = \sum_{s=1,2} \sum_{\lambda=1,2} \kappa_s \hat{a}^{\dagger}_{s,\lambda} \hat{a}_{s,\lambda}, \ \kappa_s = \kappa_0 d_s, \ \kappa_0 = 2\pi L^{-1}, \ d_s \in \mathbb{N} \ . \qquad (2)$$

QED model 2

Electrons are described by a scalar field $\varphi(x)$, $x = (x^\mu) = (t, \mathbf{r})$, interacting with the external constant magnetic field $A^0_{\text{ext}} = 0$, $\mathbf{A}_{\text{ext}}(\mathbf{r}) = \left(-Bx^2, 0, 0\right)$. After canonical quantization, the scalar field and its canonical momentum $\pi(\mathbf{r})$ become operators $\hat{\varphi}(\mathbf{r})$ and $\hat{\pi}(\mathbf{r})$. The corresponding Heisenberg operators $\hat{\varphi}(x)$ and $\hat{\pi}(x)$, satisfy the equal-time nonzero commutation relations $[\hat{\varphi}(x), \hat{\pi}(x')]_{t=t'} = i\delta(\mathbf{r} - \mathbf{r}')$. These operators act in the electron Fock space $\mathfrak{H}_{\text{e}}$ constructed by a set of creation and annihilation operators of the scalar particles and by a corresponding vacuum vector $|0\rangle_{\text{e}}$. Electron vectors are denoted as $|\Psi\rangle_{\text{e}}$, $|\Psi\rangle_{\text{e}} \subset \mathfrak{H}_{\text{e}}$. The Fock space $\mathfrak{H}$ of the complete system is $\mathfrak{H} = \mathfrak{H}_\gamma \otimes \mathfrak{H}_{\text{e}}$. Vectors from $\mathfrak{H}$ are denoted by $|\Psi\rangle$, $|\Psi\rangle \in \mathfrak{H}$. The Hamiltonian of the complete system has the following form:

$$\hat{H} = \int \left\{ \hat{\pi}^{+}(\mathbf{r})\hat{\pi}(\mathbf{r}) + \hat{\varphi}^{+}(\mathbf{r}) \left[\hat{\mathbf{P}}^2(\mathbf{r}) + m^2\right] \hat{\varphi}(\mathbf{r}) \right\} d\mathbf{r} + \hat{H}_\gamma \ ,$$

$$\hat{\mathbf{P}}(\mathbf{r}) = \hat{\mathbf{p}} + e\left[\hat{\mathbf{A}}(z) + \mathbf{A}_{\text{ext}}(\mathbf{r})\right], \ \hat{\mathbf{p}} = -i\nabla, \ \ e > 0 \ ,$$

see S.P.Gavrilov, D.M. Gitman, Sov. Phys. Journ. **23** (1980) 491.

QED model 3

The quantity $\varphi(x) = {}_{\mathrm{e}}\langle 0| \hat{\varphi}(\mathbf{r}) |\Psi(t)\rangle$ is on the one side the projection of a vector $|\Psi(t)\rangle$ onto a one-electron state, on the other side is a vector in $\mathfrak{H}_\gamma$; this amplitude-vector is denoted then by **AV**. Many-electron or -positron amplitudes describe photons interacting with many charged particles. We neglect such amplitudes in the accepted further approximation. In such an approximation, **AV** $\Phi(x) = U_\gamma(t)\,\varphi(x)$, $U_\gamma(t) = \exp\left(i\hat{H}_\gamma t\right)$, satisfies **KG**-like equation:

$$\left[\hat{P}_\mu \hat{P}^\mu - m^2\right]\Phi(x) = 0,\ \ \hat{P}^\mu = i\partial^\mu + e\left[\hat{A}^\mu(u) + A^\mu_{\mathrm{ext}}(\mathbf{r})\right],$$

$$\hat{A}^\mu(u) = \left(0, \hat{\mathbf{A}}(u)\right),\ \ u = t - z,\ \ \hat{\mathbf{A}}(u) = U_\gamma(t)\,\hat{\mathbf{A}}(z) U_\gamma^{-1}(t)$$

$$= \frac{1}{e}\sum_{s=1,2}\sum_{\lambda=1,2}\sqrt{\frac{\varepsilon}{2\kappa_s}}\left[\hat{a}_{s,\lambda}\exp\left(-i\kappa_s u\right) + \hat{a}^\dagger_{s,\lambda}\exp\left(i\kappa_s u\right)\right]\mathbf{e}_\lambda\ ,$$

where $\varepsilon = \alpha\rho$, $\alpha = e^2/\hbar c = 1/137$, and ρ is density of electron media. The quantity ε characterizes the strength of the interaction between charged particles and the photon beam. We suppose that both ε and α are small, this **supposition** defines the above mentioned approximation.

Solutions of the model 1

Here, we have three commuting integrals of motion (**IM**): $\hat{G}_\mu = i\partial_\nu + n_\nu \hat{H}_\gamma$, $\mu = 0,1,3$, $n^\mu = (1, \mathbf{n})$; the operator $\hat{G}_0$ can be interpreted as the total energy, $\hat{G}_{1,3}$ as momentum operators in x^1, z directions. Recall that $\hat{I}$ is **IM** if its mean value

$$(\varphi, \hat{I}\varphi) = \int \varphi^*(x) \left(i\overleftrightarrow{\partial_0} - 2eA_0 \right) \hat{I}\varphi(x)\, d\mathbf{r}\,, \quad \overleftrightarrow{\partial_0} = \overrightarrow{\partial_0} - \overleftarrow{\partial_0}\,,$$

with respect to any φ satisfying KGE is time-independent. If $\hat{I}$ is **IM**, then $\left[\hat{I}, \hat{P}_\mu \hat{P}^\mu - m^2\right] = 0$. If $\hat{I}$ is **IM**, then, apart from satisfying the KGE, the wave function could be choose as an eigenfunction of $\hat{I}$. Then we look for $\Phi(x)$ that are also eigenvectors for **IM**s,

$$\left[\hat{P}_\mu \hat{P}^\mu - m^2\right] \Phi(x) = 0, \;\; \hat{G}_\mu \Phi(x) = g_\mu \Phi(x), \;\; \mu = 0, 1, 3. \qquad (3)$$

Solutions of the model 2

It follows from Eqs. (3) that

$$\left[\hat{P}_\mu \hat{P}^\mu - m^2\right]\Phi(x) = \frac{1}{2(ng)}[\hat{H}_\chi(u) - (g_0 - g_3)/2]\Phi(x),$$

$$\hat{H}_\chi(u) = \hat{H}_\gamma + \frac{1}{2(ng)}\left\{\left[eBx^2 - g^1 - e\hat{A}^1(u)\right]^2 + \left[i\partial_2 - eA^2(u)\right]^2 + m^2\right\}.$$

Thus, $\hat{H}_\chi(u)$ is **IM**. A solutions of Eqs. (3) has the form:

$$\Phi(x) = \exp\left[-i(g_0 t + g_1 x^1 + g_3 z)\right]\hat{U}_\gamma(u)\,\chi(x^2),$$

$$\hat{H}_\chi(0)\chi(x^2) = \frac{g_0 - g_3}{2}\chi(x^2),\ \hat{H}_\chi(0) = \hat{U}_\gamma(u)^{-1}\hat{H}_\chi(u)\hat{U}_\gamma(u). \qquad (4)$$

Solutions of the model 3

In order to solve the latter equation, we pass to a description of electron motion in magnetic field in an adequate Fock space (Malkin, Man'ko 1968),

$$\hat{a}_0 = (2)^{-1/2}\left(\eta + \partial_\eta\right)\,,\ [\hat{a}_0, \hat{a}_0^\dagger] = 1,$$
$$\hat{a}_0^\dagger = (2)^{-1/2}\left(\eta - \partial_\eta\right),\ (eB)^{1/2}\eta = \left(eBx^2 - g^1\right)\,. \quad (5)$$

These operators commute with all the photon operators $a_{s,\lambda}^\dagger$ and $\hat{a}_{s,\lambda}$, $s = 1, 2$, $\lambda = 1, 2$. We denote the totality of f-photon and q-electron creation and annihilation operators as $a_{s,\lambda}^\dagger$ and $\hat{a}_{s,\lambda}$, $s = 0, 1, 2$, where $\hat{a}_{0,\lambda}^\dagger = \hat{a}_0^\dagger\delta_{\lambda,1}$ and $\hat{a}_{0,\lambda} = \hat{a}_0\delta_{\lambda,1}$.
$\hat{H}_\chi(0)$ is quadratic in terms of $a_{s,\lambda}^\dagger$ and $\hat{a}_{s,\lambda}$ and can be diagonalized with the help of a linear canonical transformation

$$\hat{a} = u\hat{c} - v\hat{c}^\dagger,\ \hat{a}^\dagger = \hat{c}^\dagger u^\dagger - \hat{c}v^\dagger,\ [\hat{c}_{s,\lambda}, \hat{c}_{s',\lambda'}^\dagger] = \delta_{s,s'}\delta_{\lambda,\lambda'}\,,$$

where $\lambda, \lambda' = 1, 2$, $s, s' = 0, 1, 2$.

Solutions of the model 4

$$\hat{H}_{\chi}(0) = \hat{H}_{\rm qph}(0) + \hat{H}_{\rm qe}(0), \quad \hat{H}_{\rm qph}(0) = \sum_{s=1,2}\sum_{\lambda=1,2} \tau_{s,\lambda} c^{\dagger}_{s,\lambda} c_{s,\lambda} + f_1,$$

$$\hat{H}_{\rm qe}(0) = \tau_0 \hat{c}^{\dagger}_0 \hat{c}_0 + f_2, \quad c_{s,\lambda}|0\rangle_{\rm qph} = 0, \ \hat{c}_0|0\rangle_{\rm qe} = 0, \qquad (6)$$

where functions $f_{1.2}(\tau, \kappa, \nu, \epsilon)$, matrices u,v are found analytically, and $\tau_{k,\lambda}$ are positive roots of equation

$$\sum_{s=1,2} \frac{\epsilon}{\tau^2_{k,\lambda} - \kappa^2_s} = 1 + \frac{(-1)^{\lambda-1}\omega}{\tau_{k,\lambda}}, \ \omega = eB(ng)^{-1}, \ k = 0, 1, 2, \qquad (7)$$

satisfying conditions $\tau_{0,\lambda} = \tau_0 \delta_{\lambda,1}$, $\tau_0(\epsilon = 0) = \omega$, $\tau_{k,\lambda}(\epsilon = 0) = \kappa_k$.

IM $\hat{H}_\chi(u) = \hat{U}_\gamma(u)\,\hat{H}_\chi(0)\,\hat{U}_\gamma(u)^{-1}$ is the sum of two commuting **IM**s $\hat{H}_{\mathrm{qph}}(u)$ and $\hat{H}_{\mathrm{qe}}(u)$,

$$\hat{H}_\chi(u) = \hat{H}_{\mathrm{qph}}(u) + \hat{H}_{\mathrm{qe}}(u),\ \left[\hat{H}_{\mathrm{qph}}(u), \hat{H}_{\mathrm{qe}}(u)\right] = 0,$$
$$\left\{\hat{H}_{\mathrm{qph}}(u), \hat{H}_{\mathrm{qe}}(u)\right\} = \hat{U}_\gamma(u)\left\{\hat{H}_{\mathrm{qph}}(0), \hat{H}_{\mathrm{qe}}(0)\right\}\hat{U}_\gamma(u)^{-1}.$$

In a sence, $\hat{H}_{\mathrm{qph}}(u)$ corresponds to q-photons, while $\hat{H}_{\mathrm{qe}}(u)$ to q-electrons. Commuting operators $\hat{\mathcal{P}}_\mu = i\partial_\mu - n_\mu\left[\hat{H}_{\mathrm{qph}}(u) - \hat{H}_\gamma\right]$, $\mu = 0, 1, 2$, are also **IM**s. We choose **AV** $\Phi(x)$ to be eigenvectors for all these **IM**,

$$\hat{H}_{\mathrm{qph}}(u)\Phi(x) = E_{\mathrm{qph}}\Phi(x),\ \hat{H}_{\mathrm{qe}}(u)\Phi(x) = E_{\mathrm{qe}}\Phi(x),\ \hat{\mathcal{P}}_\mu\Phi(x) = p_\mu\Phi(x)$$
$$\hat{H}_\chi(u)\Phi(x) = E\Phi(x),\ E = E_{\mathrm{qph}} + E_{\mathrm{qe}} \Longrightarrow \hat{H}_\chi(0)\chi(x^2) = E\chi(x^2).$$

Equations for $\Phi(x)$ are consistent if

$g_0 = p_0 + E_{\mathrm{qph}},\ g_1 = p_1,\ g_3 = p_3 - E_{\mathrm{qph}}$, which implies $ng = np$.

Solutions of the model 6

Finally we have

$$\Phi(x) = \left|\Phi_{\mathrm{qph}}\right\rangle \otimes \left|\Phi_{\mathrm{qe}}\right\rangle ,$$

where

$$\left|\Phi_{\mathrm{qph}}\right\rangle = \prod_{s,\lambda=1,2} \frac{\left(\hat{c}^{\dagger}_{s,\lambda}\right)^{N_{s,\lambda}}}{\sqrt{N_{s,\lambda}!}} \left|0\right\rangle_{\mathrm{qph}} ,$$

$$\left|\Phi_{\mathrm{qe}}\right\rangle = \exp\left\{-i\left(p_0 t + p_1 x^1 + p_3 z\right)\right\} \frac{\left(\hat{c}^{\dagger}_{0}\right)^{N_0}}{\sqrt{N_0!}} \left|0\right\rangle_{\mathrm{qe}} ,$$

$$E_{\mathrm{qph}} = \sum_{s=1,2} \sum_{\lambda=1,2} \tau_{s,\lambda} N_{s,\lambda} + f_1 ,\ E_{\mathrm{qe}} = \tau_0 N_0 + f_2 ,\ N \in \mathbb{N} .$$

Entanglement of photons

Consider $\left|\Phi_{\mathrm{qph}}(\lambda_1,\lambda_2)\right\rangle$ describing two q-photons, with different frequencies and with polarizations λ_1,λ_2,

$$\left|\Phi_{\mathrm{qph}}(\lambda_1,\lambda_2)\right\rangle = \hat{c}^{\dagger}_{1,\lambda_1}\hat{c}^{\dagger}_{2,\lambda_2}\left|0\right\rangle_{\mathrm{qph}} \ . \tag{8}$$

For small ε and $\Delta\kappa = |\kappa_2 - \kappa_1| \gg 1$, we have $|0\rangle_{\mathrm{qph}} = |0\rangle_{\gamma} + O(\sqrt{\varepsilon})$. We believe that corresponding f-photon nonentangled beam after passing through the macro region, is deformed to this form, and there exists an analyzer detecting a two-photon state for measuring its entanglement. In terms of computational basis $|\vartheta_1\rangle = a^{+}_{1,1}a^{+}_{2,1}|0\rangle$, $|\vartheta_2\rangle = a^{+}_{1,1}a^{+}_{2,2}|0\rangle$, $|\vartheta_3\rangle = a^{+}_{1,2}a^{+}_{2,1}|0\rangle$, $|\vartheta_4\rangle = a^{+}_{1,1}a^{+}_{2,1}|0\rangle$, two-photon state $\left|\Phi_{\mathrm{ph}}(\lambda_1,\lambda_2)\right\rangle$, contained in (8), reads:

$$\left|\Phi_{\mathrm{ph}}(\lambda_1,\lambda_2)\right\rangle = D\sum_{j=1}^{4} v_j\left|\vartheta_j\right\rangle, \ D = \left(\sum_{i=1}^{4}|v_i|^2\right)^{-1/2},$$

$$v_1 = u_{1,1}\tilde{u}_{2,1} + u_{2,1}\tilde{u}_{1,1}, \quad v_4 = v_1 \ ,$$

$$v_2 = u_{1,1}\tilde{u}_{2,2} + u_{2,2}\tilde{u}_{1,1}, \quad v_3 = -v_2 \ .$$

EM of photons with anti-parallel polarizations 1

Further, we calculate **EM** $M(|\Phi_{\rm ph}(\lambda_1,\lambda_2)\rangle) = M(\lambda_1,\lambda_2)$ for $\lambda_2 \neq \lambda_1$ in ε^4-order (in this order for $\lambda_2 = \lambda_1$ it is equal to zero),

$$M(\lambda_1,\lambda_2) = -\left[z\log_2 z + (1-z)\log_2(1-z)\right],\ z = (1+y)/2\ ,$$
$$y = \sqrt{\left(|v_1|^2 + |v_2|^2 - |v_4|^2 - |v_3|^2\right)^2 + 4\left|v_1 v_3^* + v_2 v_4^*\right|^2}.$$

For $p_3 = 0$ and up to ε^4-order, $y = 1 - \beta\varepsilon^4$,

$$M(\lambda_1,\lambda_2) = -2\beta\varepsilon^4\log_2\varepsilon + \frac{\beta}{2\ln 2}\left(1 - \ln\frac{\beta}{2}\right)\varepsilon^4,$$
$$\beta = \frac{(\bar{p}_0/\Delta\kappa)^4}{8\left[\omega_0 + (-1)^{\lambda_1}\kappa_1\right]^2\left[\omega_0 + (-1)^{\lambda_2}\kappa_2\right]^2}, \qquad (9)$$

where $\omega_0 = eB\left[2eB\left(N_0 + 1/2\right) + m^2\right]^{-1/2}$.

EM of photons with anti-parallel polarizations 2

The quantity y is singular, if

$$\omega_0 = \begin{cases} \kappa_1, & \text{if} \quad \lambda_1 = 1, \quad \lambda_2 = 2 \\ \kappa_2, & \text{if} \quad \lambda_1 = 2, \quad \lambda_2 = 1 \end{cases} .$$

The corresponding to such ω_0 strengths of magnetic field B, are called **resonant** ones. There exist two resonant values, $B = B_1$ at $\omega_0 = \kappa_1$ for $\lambda_1 = 1$ and $B = B_2$ at $\omega_0 = \kappa_2$ for $\lambda_1 = 2$:

$$\begin{aligned} B_1 &= \frac{\kappa_1}{e}\sqrt{(N_0 + 1/2)^2\kappa_1^2 + m^2} + (N_0 + 1/2)\,\kappa_1^2 \,, \\ B_2 &= \frac{\kappa_2}{e}\sqrt{(N_0 + 1/2)^2\kappa_2^2 + m^2} + (N_0 + 1/2)\,\kappa_2^2 \,. \end{aligned}$$

EM of photons with anti-parallel polarizations 3

One can find that when resonant values are reached, entanglement manifests itself already in ε^3 order. For $B = B_1$:

$$y = 1 - \delta_1 \varepsilon^3 + O(\varepsilon^4), \ \delta_1 = \frac{1}{4(\Delta\kappa)^4(\kappa_1 + \kappa_2)^2} \left(\frac{\kappa_1}{eB_1} \right)^3 ,$$

$$M(\lambda_1, \lambda_2) = -\frac{3}{2}\delta_1 \varepsilon^3 \log_2 \varepsilon + \frac{\delta_1}{2 \ln 2} \left(1 - \ln \frac{\delta_1}{2} \right) \varepsilon^3 + O(\varepsilon^4) ,$$

whereas for $B = B_2$, we obtain:

$$y = 1 - \delta_2 \varepsilon^3 + O(\varepsilon^4), \quad \delta_2 = \frac{1}{4(\Delta\kappa)^4(\kappa_1 + \kappa_2)^2} \left(\frac{\kappa_2}{eB_2} \right)^3 ,$$

$$M(\lambda_1, \lambda_2) = -\frac{3}{2}\delta_2 \varepsilon^3 \log_2 \varepsilon + \frac{\delta_2}{2 \ln 2} \left(1 - \ln \frac{\delta_2}{2} \right) \varepsilon^3 + O(\varepsilon^4) .$$

Numerical calculations 1

We consider all electrons located on zero Landau level $N_0 = 0$ and photons with polarization $\lambda_1 = 2$ and $\lambda_2 = 1$. It follows from Eqs. (9) that

$$\beta = \frac{\left(\bar{p}_0 / \Delta\kappa\right)^4}{8\left(\omega_0 + \kappa_1\right)^2 \left(\omega_0 - \kappa_2\right)^2}, \quad \omega_0 = \frac{eB}{\sqrt{|eB| + m^2}} .$$

One can see that at $\omega_0 > 0$ resonant entanglement is related to frequency of the second photon. At $\omega_0 < 0$ the resonant entanglement will be related to frequency of first photon.
Furter, we consider the case $\omega_0 > 0$, where the resonant value of the magnetic field is B_2.

Numerical calculations 2

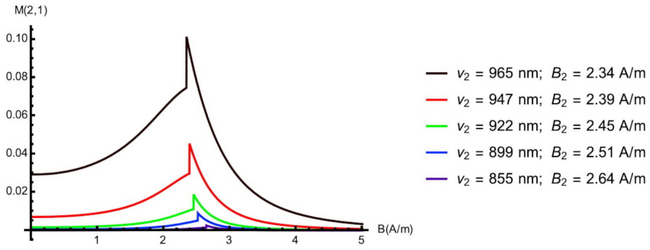

Figure: Fig. 1. The entanglement measure as a function of the magnetic field.

Numerical calculations 3

Fig. 1, shows the $M(2,1)$ as a function of B at fixed $\nu_1 = 10^3$ nm, and different second photon frequencies ν_2. The electron density is $\rho = 10^{14}\mathrm{el\ m}^{-3}$. The $M(2,1)$ increases with increasing $B < B_2$. At $B = B_2$ has a jump. At $B > B_2$ the is a smooth decrease in the $M(2,1)$. The $M(2,1)$ decreases as the difference in photon frequencies increases. Considering entanglement at $B = 0$, one can see that it is the same for $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_1 = 2$, $\lambda_2 = 1$. The presence of the magnetic field removes this degeneracy. Increasing B increases entanglement. The resonant value of B increases with increasing frequency of the second photon. But resonant values are not large, for example, for photons with frequencies ν_2 corresponding to ultraviolet range 380 nm – 10 nm, resonant values range from 6 A/m to 225 A/m.

Numerical calculations 4

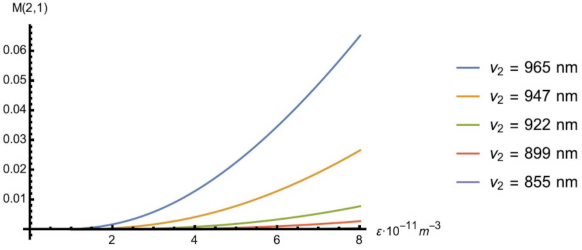

Figure: Fig 2. The entanglement measure $M\ (2,1)$ as a function of the electron medium density.

Numerical calculations 5

On Fig. 2, The $M\ (2,1)$ is calculated as a function of ε (in fact of ρ) for $\nu_1 = 10^3$ nm, different ν_2 and for $B = 2$ A/m which is less than corresponding resonant values. The $M\ (2,1)$ increases with increasing ε.
$\varepsilon = \alpha\rho$, $\rho = 137 \cdot 10^{11} x$. $x = 6 \Longrightarrow \rho \approx 8 \cdot 10^{13} m^{-3}$.
Magnetic field in CGS in Oersteds. Amperes/meter (A/m) in CI.
Oersted=1000/(4π)(A/m).

Some final remarks

In our calculations the entanglement measure does not exceed $0,1$. However, such a magnitude of the entanglement is usual in laboratory experiments, for example, similar magnitudes appear when an entangled biphoton state is scattered inside an optical cavity, see Refs. H. Piryatinski et al, J.Chem.Phys.**150**(18) (2019); R. Malatesta et al. (2023). arXiv:2309.04751. We stress that performed numerical calculations are intended to illustrate the existence of a possible resonant entanglement within the framework of chosen model and approximations made. On the other hand, if our consideration motivates possible experiments to detect the effect of the resonant entanglement then there may be an incentive to refine the corresponding model, under weaker restrictions on the density of electron medium and photon frequencies.

The end