Vacuum instability effects in strong field QED with asymmetric electric field of analytic form

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Outline

- Introduction (SFQED)
- Known exactly solvable cases
- New non-stationary analytical asymmetric electric field
- General solution of Dirac equation
- Differential and total mean numbers
- New regularization of Klein step

Introduction

1. In QED with strong electric-like external fields (strong-field QED in what follows) there exists the so-called vacuum instability due to the effect of real particle creation from the vacuum caused by the external fields (the so-called Schwinger effect [1]).
2. A number of publications, reviews and books are devoted to this effect itself and to developing different calculation methods in theories with unstable vacuum, see Refs. [1–4] for a review.

In strong-field QED, nonperturbative (with respect to strong external fields) methods are well-developed for two classes of external backgrounds, namely for the so-called t-electric potential steps (t-steps) and x-electric potential steps (x-steps). The latter fields can also create particles from the vacuum, the Klein paradox is closely related to this process.

[1] J. Schwinger, *On Gauge Invariance and Vacuum Polarization*, Phys. Rev. **82**, 664 (1951).

[2] A. I. Nikishov, in *Quantum Electrodynamics of Phenomena in Intense Fields*, Proc. P.N. Lebedev Phys. Inst. **111**, 153 (Nauka, Moscow 1979).

[3] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982); A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory, St. Petersburg, 1994).

[4] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, Berlin, 1991).

Introduction

- t-steps represent uniform time-dependent external electric fields that are switched on and off at the initial and the final time instants. A general nonperturbative formulation of strong-field QED with t-steps was developed many years ago in Refs.:
  - D. M. Gitman, *Quantum processes in an intense electromagnetic field. I and II*, Sov. Phys. Journ. **19**, 1314 (1976); S. P. Gavrilov and D. M. Gitman, *Quantum processes in an intense electromagnetic field producing pairs. III*, Sov. Phys. Journ. **20**, 75 (1977);
  - D. M. Gitman, *Processes of arbitrary order in quantum electrodynamics with a pair-creating external field*, Journ. Phys. A **10**, 2007 (1977);
  - E. S. Fradkin and D. M. Gitman, *Furry picture for quantum electrodynamics with pair-creating external field*, Fortschr. Phys. **29**, 381 (1981); E. S. Fradkin, D.M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, Berlin, 1991).
- x-steps represent time-independent inhomogeneous electriclike external fields of a constant direction. A nonperturbative approach in QED with the x-steps, was developed in Refs.:
  - S. P. Gavrilov and D. M. Gitman, *Quantization of Charged Fields in the Presence of Critical Potential Steps*, Phys. Rev. D **93**, 045002 (2016); *Regularization, Renormalization and Consistency Conditions in QED with x-Electric Potential Steps*, EPJ C **80**, 820 (2020);
  - A. I. Breev, S. P. Gavrilov and D. M. Gitman, *Calculations of vacuum mean values of spinor field current and energy-momentum tensor in a constant electric background*, EPJ C **83**, 108 (2023);

The t-case step

The homogeneous fields with constant direction are considered in $d = D + 1$ - dimensions, parametrized by coordinates $X = (t, \mathbf{r})$, $\mathbf{r} = \left(x^1 = x, x^2, \ldots, x^D\right)$. The electromagnetic potentials can be chosen as time-like steps,

$$A^0 = 0,\ \mathbf{A} = \left(A^1(t) = A_x(t) = A(t), 0, \ldots, 0\right),\ A(-\infty) > A(+\infty), \quad (1)$$

$$\mathbf{E}(t) = \left(E^1(t), 0, \ldots, 0\right),\ E^1(t) = E_x(t) = E(t) = -A'(t) \geq 0. \quad (2)$$

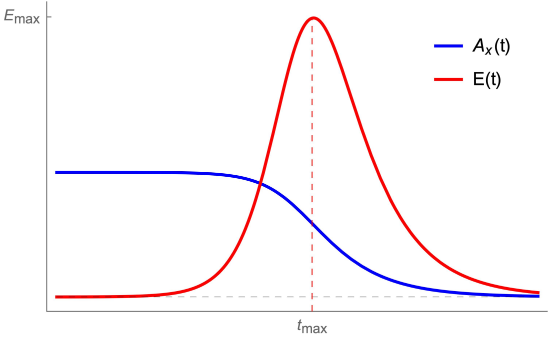


Figure: General view of electric field (red line) and its vector potential (blue line) corresponding to a t-step.

Example 1: T-constant electric field

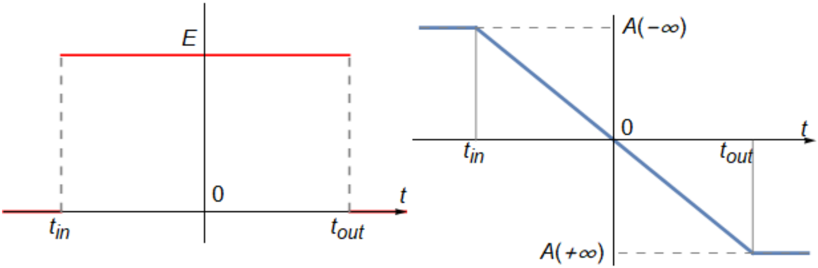


Figure: T-constant electric field and corresponding vector potential.

The vacuum instability in the T-constant electric field was studied in

- V. G. Bagrov, D. M. Gitman and Sh. M. Shvartsman, JETP **41** (1975);
- S. P. Gavrilov and D. M. Gitman, Phys. Rev. D **53** (1996);

The field corresponds to a regularized version of the constant field. The vacuum instability in the latter field was studied e.g. in

- J. Schwinger, Phys. Rev. **82** (1951);
- A. I. Nikishov, JETP **30** (1970); Proc. P.N. Lebedev Phys. Inst. **111** (1979);

Example 2: Sauter-like electric field

$$E(t) = E_{\max} \cosh^{-2}(t/T_S), \; A_x(t) = -T_S \tanh(t/T_S), \; E_{\max} > 0 \,. \qquad (3)$$

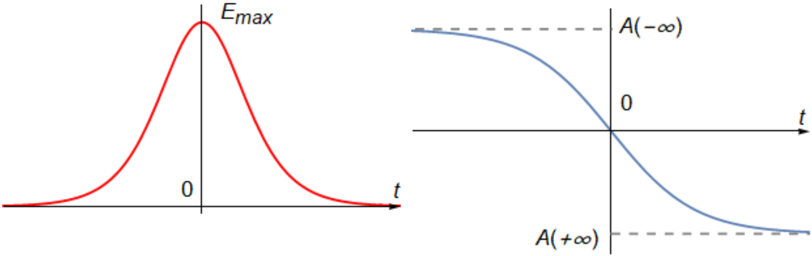


Figure: Sauter-like electric field and corresponding vector potential

The vacuum instability in the Sauter-like electric field was first studied by N. B. Narozhny and A. I. Nikishov, Sov. J. Nucl. Phys. **11** (1970) and then many researchers returned to this problem, since in the case under consideration it was convenient to test various approaches.

Example 3: Exponential peak electric field

The field is exponentially growing and decaying electric field. This configuration is parametrized by three arbitrary parameters $E_{\max} > 0$, $k_1 > 0$ and $k_2 > 0$:

$$E(t) = E_{\max} \begin{cases} \exp(k_1 t), & t \in (-\infty, 0] \\ \exp(-k_2 t), & t \in (0, +\infty) \end{cases}, \tag{4}$$

$$A_x(t) = E_{\max} \begin{cases} k_1^{-1} [-\exp(k_1 t) + 1], & t \in (-\infty, 0] \\ k_2^{-1} [\exp(-k_2 t) - 1], & t \in (0, +\infty) \end{cases}. \tag{5}$$

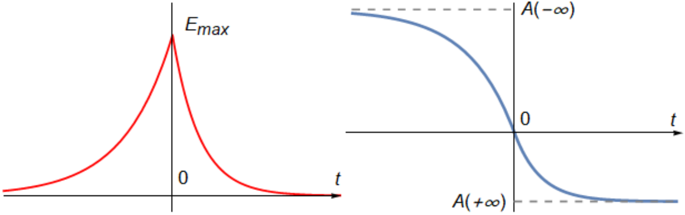


Figure: Expotential peak field and its vector potential

The vacuum instability in the exponential peak field was studied in Refs. T. C. Adorno, S. P. Gavrilov, and D. M. Gitman, Phys. Scr. **90** (2015); EPJ C **76** (2016); S. P. Gavrilov, D. M. Gitman, and A. A. Shishmarev, Phys. Rev. D **96** (2017); T. C. Adorno et. al., Int. JMPA **33** (2018).

Example 4: Inverse square peak electric field

The inverse square peak field is a combination of two parts one increasing and another one decreasing, both of them inversely proportional to square of the time,

$$E(t) = E_{\max} \begin{cases} (1 - t/\tau_1)^{-2}, & t \in (-\infty, 0] \\ (1 + t/\tau_2)^{-2}, & t \in (0, +\infty) \end{cases}, \quad (6)$$

$$A_x(t) = E_{\max} \begin{cases} \tau_1 \left[1 - (1 - t/\tau_1)^{-1}\right], & t \in (-\infty, 0] \\ \tau_2 \left[(1 + t/\tau_2)^{-1} - 1\right], & t \in (0, +\infty) \end{cases}. \quad (7)$$

This peak configuration is parametrized by three arbitrary parameters $E_{\max} > 0$, $\tau_1 > 0$ and $\tau_2 > 0$.

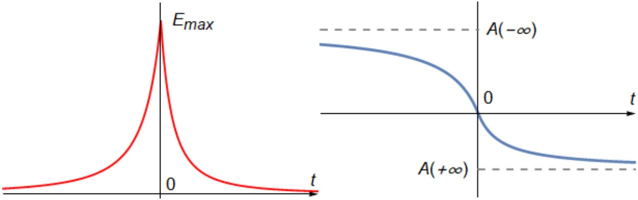


Figure: Inverse square peak field and its vector potential

The vacuum instability in inverse square peak field was studied in T. C. Adorno, S. P. Gavrilov, D. M. Gitman, EPJ C **78** (2018).

New Example: Analytic asymmetric electric field

Here we present a new example of exactly solvable case:

$$E(t) = \frac{E_0}{8}\sqrt{1+\exp(t/\sigma)}\cosh^{-2}(t/2\sigma), \;\; E_0 > 0, \;\; \sigma > 0 \; , \tag{8}$$

$$A_x(t) = \frac{\sigma E_0}{\sqrt{1+\exp(t/\sigma)}} \; . \tag{9}$$

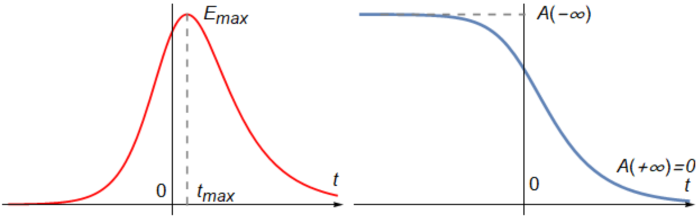


Figure: Analytic asymmetric field and its potential.

In contrast to the Sauter-like electric field this field is asymmetrical with respect to the time instant $t_{\max} = \sigma \ln 2$, where it reaches its maximum value $E_{\max} = 3^{-3/2} E_0$. The vacuum instability in inverse square peak field was studied in A. I. Breev, S. P. Gavrilov, D. M. Gitman, and A. A. Shishmarev, Phys. Rev. D **104** (2021).

The analytic asymmetric electric field and Sauter-like electric field

It is useful to compare analytic asymmetric electric field and Sauter-like electric field. We present graphs of both fields, analytic asymmetric field with $\sigma = T_S/2$ (by green line) and Sauter-like field shifted to the right in time by $(T_S/2)\log 2$ (by red line). In this case, both fields reach the same maximum value $E_{\max}$ at the time instant $t_{\max} = (T_S/2)\log 2$,

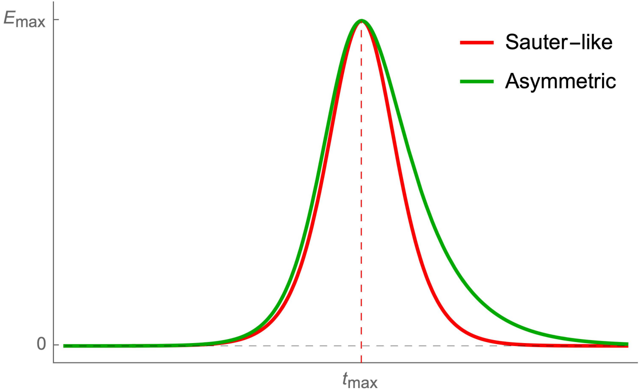


Figure: Comparison of Sauter-like and asymmetric electric fields

The Dirac equation, variable separation

The Dirac equation in has the form

$$i\partial_t\psi(X) = H(t)\psi(X), \ H(t) = \gamma^0\left\{\gamma^1\left[-i\partial_x + eA_x(t)\right] - i\boldsymbol{\nabla}_\perp\gamma_\perp + m\right\}, \quad (10)$$

where $H(t)$ is one-particle Dirac Hamiltonian, $\psi(X)$ is $2^{[d/2]}$-component spinor. We seek solutions of Dirac equation in the following form:

$$\psi_n(X) = \exp(i\mathbf{pr})\psi_n(t), \ n = (\mathbf{p}, s) \ , \quad (11)$$

$$\psi_n(t) = \left\{\gamma^0 i\partial_t - \gamma^1\left[p_x + eA_x(t)\right] - \boldsymbol{\gamma}_\perp\mathbf{p}_\perp + m\right\}\phi_n(t) \ , \quad (12)$$

$$\phi_n(t) = \varphi_n(t)v_s, \ \ \gamma^0\gamma^1 v_s = v_s. \quad (13)$$

where v_s is a set of constant orthonormalized spinors. The scalar functions $\varphi_n(t)$ satisfy the following second-order differential equation:

$$\left\{\frac{d^2}{dt^2} + \left[p_x + eA_x(t)\right]^2 + \mathbf{p}_\perp^2 + m^2 + ie\dot{A}(t)\right\}\varphi_n(t) = 0 \ . \quad (14)$$

General solution of Dirac equation

To find general solution of the Dirac equation, we use the following ansatz:

$$\varphi_n(t) = (1+z)^{\alpha_1}(1-z)^{\alpha_2} u_n(z), \quad z = \sqrt{1+\exp(t/\sigma)}\,, \tag{15}$$

$$\alpha_1 = i\sigma\sqrt{(p_x - eE_0\sigma)^2 + \mathbf{p}_\perp^2 + m^2}, \quad \alpha_2 = i\sigma\sqrt{(p_x + eE_0\sigma)^2 + \mathbf{p}_\perp^2 + m^2}\,. \tag{16}$$

For functions $u_n(z)$ the Heun equation holds,

$$\boxed{\hat{H}_n u_n(z) = 0}, \quad \hat{H}_n = \frac{d^2}{dz^2} + \left(-\frac{1}{z} + \frac{1+2\alpha_2}{z-1} + \frac{1+2\alpha_1}{z+1}\right)\frac{d}{dz} +$$
$$+ \frac{z\left[\alpha_3^2 - (\alpha_1-\alpha_2)^2\right] + (\alpha_1 - \alpha_2 + \alpha_3)}{z(z-1)(z+1)}, \quad \alpha_3 = -2ie\sigma^2 E_0. \tag{17}$$

The differential operator $\hat{H}_n$ satisfies the identity

$$\boxed{\hat{H}_n \hat{M}_n \equiv \hat{B}_n \hat{R}_n\,,} \tag{18}$$

where

$$\hat{M}_n = \frac{bz - \alpha_1 + \alpha_2 - \alpha_3}{(a-1)\,b}\frac{d}{dz} + 1, \tag{19}$$

$$\hat{R}_n(z) = \frac{d^2}{dz^2} + \left(\frac{2\alpha_1}{z+1} + \frac{2\alpha_2}{z-1}\right)\frac{d}{dz} + \frac{(a-1)\,b}{z^2-1}, \tag{20}$$

$$a = \alpha_1 + \alpha_2 - \sqrt{2(\alpha_1^2+\alpha_2^2) - \alpha_3^2}, \quad b = \alpha_1 + \alpha_2 + \sqrt{2(\alpha_1^2+\alpha_2^2) - \alpha_3^2}, \tag{21}$$

Generalized Darboux Transformation Method

Taking into account that the differential operator $\hat{H}_n$ satisfies the identity

$$\boxed{\hat{H}_n \hat{M}_n \equiv \hat{B}_n \hat{R}_n\,,} \tag{22}$$

we represent the functions $u_n(z)$ as:

$$u_n(z) = U_n \hat{M}_n w_n\left(\frac{z+1}{2}\right), \tag{23}$$

$$\hat{H}_n u_n(z) = U_n \hat{H}_n \hat{M}_n w_n\left(\frac{z+1}{2}\right) = U_n \hat{B}_n \underbrace{\hat{R}_n w_n\left(\frac{z+1}{2}\right)} = 0\,, \tag{24}$$

where the function $w_n\left(\frac{z+1}{2}\right)$ is determined from Eq. $\hat{R}_n w_n = 0$.
One sees that Wronskians of the functions

$$\varphi_n(t) = (1+z)^{\alpha_1}(1-z)^{\alpha_2}\, U_n \hat{M}_n w_n\left(\frac{z+1}{2}\right), \tag{25}$$

$$\tilde{\varphi}_n(t) = (1+z)^{\alpha_1}(1-z)^{\alpha_2}\, \tilde{U}_n \hat{M}_n \tilde{w}_n\left(\frac{z+1}{2}\right), \tag{26}$$

are proportional to Wronskians of the functions w and $\tilde{w}$:

$$W(\varphi_n, \tilde{\varphi}_n) = U_n \tilde{U}_n \Omega_n(z)\, W(w_n, \tilde{w}_n)\,, \tag{27}$$

$$\Omega_n(z) = 2^{2(\alpha_1+\alpha_2)}(1+z)^{2\alpha_1}(1-z)^{2\alpha_2}\,\frac{(a-b)\,\alpha_3 + 2\left(\alpha_1^2-\alpha_2^2\right)}{4\,(a-1)\,b}\,. \tag{28}$$

In- and out-solutions: Definition

In- and out-solutions $\psi(t)$ of the Dirac equation have special asymptotics as $t \to \pm\infty$ and correspond to initial $(-)$ and final $(+)$ particles and antiparticles. The functions $\varphi(t)$ that correspond to spinors $\psi(t)$, that are in-solutions, are denoted as ${}_{\zeta}\varphi_n(t)$, while functions $\varphi(t)$ that correspond to spinors $\psi(t)$, that are out-solutions, are denoted as ${}^{\zeta}\varphi_n(t)$. Both sets are classified by a quantum number $\zeta = \pm$, which labels particles $(\zeta = +)$ and antiparticles $(\zeta = -)$. Solutions ${}_{\zeta}\varphi_n(t)$ and ${}^{\zeta}\varphi_n(t)$ have the following asymptotic behavior:

$$
{}^{\zeta}\varphi_n(t) = {}^{\zeta}\mathcal{N} \exp\left(-i\,{}^{\zeta}\varepsilon_n t\right), \quad {}^{\zeta}\varepsilon_n = \zeta\omega_1, \quad t \to +\infty, \tag{29}
$$

$$
{}_{\zeta}\varphi_n(t) = {}_{\zeta}\mathcal{N} \exp\left(-i\,{}_{\zeta}\varepsilon_n t\right), \quad {}_{\zeta}\varepsilon_n = \zeta\omega_2, \quad t \to -\infty, \tag{30}
$$

$$
\omega_1 = \sqrt{p_x^2 + \mathbf{p}_\perp^2 + m^2}, \quad \omega_2 = \sqrt{\left(p_x + eE_0\sigma\right)^2 + \mathbf{p}_\perp^2 + m^2}\,. \tag{31}
$$

Using the equal-time inner product

$$
(\psi, \psi) = \int d\mathbf{r}\, \psi^\dagger(X)\,\psi(X), \quad d\mathbf{r} = dx^1 dx^2 \ldots dx^D \tag{32}
$$

of Dirac bispinors, we easily calculate the normalization constants ${}_{\zeta}\mathcal{N}$ and ${}^{\zeta}\mathcal{N}$, using explicit forms of their asymptotics,

$$
{}^{\zeta}\mathcal{N} = {}^{\zeta}CY, \quad {}^{\zeta}C = \left[2\omega_1\left(\omega_1 - \zeta p_x\right)\right]^{-1/2}, \quad Y = V_{(d-1)}^{-1/2}, \tag{33}
$$

$$
{}_{\zeta}\mathcal{N} = {}_{\zeta}CY, \quad {}_{\zeta}C = \left\{2\omega_2\left[\omega_2 - \zeta\left(p_x + eE_0\sigma\right)\right]\right\}^{-1/2}, \tag{34}
$$

In- and Out-solutions: Explicit form

In this case in- and out-solutions have the form:

$$_{\pm}\varphi_n(t) = {}_{\pm}\mathcal{N} U_{n,1/2} (1+z)^{\alpha_1} (1-z)^{\alpha_2} \hat{M}_n w_{n,1/2}\left(\frac{z+1}{2}\right), \quad (35)$$

$$^{\pm}\varphi_n(t) = {}^{\pm}\mathcal{N} U_{n,3/4} (1+z)^{\alpha_1} (1-z)^{\alpha_2} \hat{M}_n w_{n,3/4}\left(\frac{z+1}{2}\right), \quad (36)$$

where

$$U_{n,1} = \frac{2^{1-\alpha_1-3\alpha_2} e^{i\pi\alpha_2} (a-1) b}{(2\alpha_2 - 1)(b - \alpha_1 + \alpha_2 - \alpha_3)}, \quad U_{n,2} = \frac{2^{\alpha_2-\alpha_1+2} e^{-i\pi\alpha_2}\alpha_2}{a - \alpha_1 + \alpha_2 + \alpha_3}, \quad (37)$$

$$U_{n,3} = \frac{2^{-b} e^{-i\pi(\alpha_2 - b)} a - 1}{a - b - 1}, \quad (38)$$

$$U_{n,4} = \frac{2^{1-a} e^{-i\pi(\alpha_2 - a)} b(a-b)}{a(b - \alpha_1 + \alpha_2 - \alpha_3) - b(a + \alpha_1 - \alpha_2 - \alpha_3)}; \quad (39)$$

$$\begin{aligned} w_{n,1}(\xi) &= \xi^{a-2\alpha_1-1}(1-\xi)^{2\alpha_1-a-b+1} \times \\ &\times F\left(2\alpha_1 - a + 1, 2 - a; 2\alpha_1 - a - b + 2; 2 - \alpha_1; 1 - \xi^{-1}\right), \quad (40) \\ w_{n,2}(\xi) &= \xi^{1-a} F\left(a-1, a-2\alpha_1; a+b-2\alpha_1; 1-\xi^{-1}\right), \quad (41) \\ w_{n,3}(\xi) &= (-\xi)^{-b} F\left(b, b-2\alpha_1+1; b-a+2; \xi^{-1}\right), \quad (42) \\ w_{n,4}(\xi) &= (-\xi)^{1-a} F\left(a-1, a-2\alpha_1; a-b; \xi^{-1}\right), \xi = \frac{z+1}{2}. \quad (43) \end{aligned}$$

In- and Out-solutions: g-coefficients

One can also see that in-solutions with quantum numbers n are expressed via out-solutions with the same quantum numbers n. Thus,

$$
{}^{\zeta}\psi_n(t) = \sum_{\zeta'} g_n\left({}_{\zeta'}|^{\zeta}\right)\ {}_{\zeta'}\psi_n(t)\,, \quad {}_{\zeta}\psi_n(t) = \sum_{\zeta'} g_n\left({}^{\zeta'}|_{\zeta}\right)\ {}^{\zeta'}\psi_n\,, \tag{44}
$$

where

$$
\left({}_{\zeta}\psi_n,\ {}^{\zeta'}\psi_{n'}\right) = g_n\left({}_{\zeta'}|^{\zeta}\right)\delta_{nn'}\,, \quad g_n\left({}_{\zeta'}|^{\zeta}\right) = g_n\left({}^{\zeta}|_{\zeta'}\right)^{*}\,, \tag{45}
$$

$$
\sum_{\zeta'} g_n\left({}^{\zeta}|_{\zeta'}\right) g_n\left({}_{\zeta'}|^{\zeta''}\right) = \delta_{\zeta\zeta''}\,. \tag{46}
$$

Using the Kummer relations for the hypergeometric equation, we find:

$$
\begin{aligned}
g_n\left({}^{+}|_{+}\right) &= \frac{{}_{+}\mathcal{N}}{{}^{+}\mathcal{N}} \frac{2^{b-\alpha_1-3\alpha_2+1}\sin(\pi b)\,\Gamma(a-b)\,\Gamma(b+1)}{(b-\alpha_1+\alpha_2-\alpha_3)\sin(2\pi\alpha_2)\,\Gamma(2\alpha_1-b)\,\Gamma(2\alpha_2)}, \\
g_n\left({}^{-}|_{+}\right) &= -\frac{{}_{+}\mathcal{N}}{{}^{-}\mathcal{N}} \frac{2^{a-\alpha_1-3\alpha_2}(a-\alpha_1+\alpha_2+\alpha_3)\sin(\pi a)\,\Gamma(b-a)\,\Gamma(a)}{\sin(2\pi\alpha_2)\,\Gamma(b-2\alpha_2+1)\,\Gamma(2\alpha_2)}, \\
g_n\left({}^{+}|_{-}\right) &= -\frac{{}_{-}\mathcal{N}}{{}^{+}\mathcal{N}} \frac{2^{b-\alpha_1+\alpha_2+1}\pi\Gamma(a-b)}{(a-\alpha_1+\alpha_2+\alpha_3)\sin(2\pi\alpha_2)\,\Gamma(a)\,\Gamma(-2\alpha_2)\,\Gamma(a-2\alpha_1)}, \\
g_n\left({}^{-}|_{-}\right) &= -\frac{{}_{-}\mathcal{N}}{{}^{-}\mathcal{N}} \frac{2^{a-\alpha_1+\alpha_2}\pi(b-\alpha_1+\alpha_2-\alpha_3)\,\Gamma(b-a)}{\sin(2\pi\alpha_2)\,\Gamma(b+1)\,\Gamma(-2\alpha_2)\,\Gamma(1-a+2\alpha_2)}.
\end{aligned} \tag{47}
$$

Generalized Furry representation

Decomposing the Dirac operator $\hat{\Psi}(x)$ (Heisenberg representation) in the complete sets of in- and out-solutions

$$\hat{\Psi}(X) = \sum_n \left[\hat{a}_n(\text{in})\ {}_+\psi_n(X) + \hat{b}_n^\dagger(\text{in})\ {}_-\psi_n(X)\right] =$$

$$= \sum_n \left[\hat{a}_n(\text{out})\ {}^+\psi_n(X) + \hat{b}_n(\text{out})^\dagger\ {}^-\psi_n(X)\right] . \quad (48)$$

we introduce in- and out-creation and annihilation Fermi operators,

$$\left[\hat{a}_n(\text{in/out}), \hat{a}_m^\dagger(\text{in/out})\right]_+ = \left[\hat{b}_n(\text{in/out}), \hat{b}_m^\dagger(\text{in/out})\right]_+ = \delta_{nm} . \quad (49)$$

The initial $|0,\text{in}\rangle$ and final $|0,\text{out}\rangle$ vacuum vectors, as well as many-particle in- and out-states, are defined by

$$\hat{a}_n(\text{in})|0,\text{in}\rangle = \hat{b}_n(\text{in})|0,\text{in}\rangle = 0\,,\ \hat{a}_n(\text{out})|0,\text{out}\rangle = \hat{b}_n(\text{out})|0,\text{out}\rangle = 0\,, \quad (50)$$

$$|\text{in}\rangle = \hat{b}_n^\dagger(\text{in})\cdots\hat{a}_n^\dagger(\text{in})\cdots|0,\text{in}\rangle\,, \quad |\text{out}\rangle = \hat{b}_n^\dagger(\text{out})\cdots\hat{a}_n^\dagger(\text{out})\cdots|0,\text{out}\rangle\,. \quad (51)$$

The in- and out-operators are related by linear canonical transformations,

$$\hat{a}_n(\text{out}) = g_n\left({}^+|_+\right)\hat{a}_n(\text{in}) + g_n\left({}^+|_-\right)\hat{b}_n^\dagger(\text{in})\,, \quad (52)$$

$$\hat{b}_n^\dagger(\text{out}) = g_n\left({}^-|_+\right)\hat{a}_n(\text{in}) + g_n\left({}^-|_-\right)\hat{b}_n^\dagger(\text{in})\,. \quad (53)$$

These relations allow one to calculate the differential mean numbers of created pairs

$$N_n = \langle 0,\text{in}|\hat{a}_n^\dagger(\text{out})\hat{a}_n(\text{out})|0,\text{in}\rangle = \left|g_n\left({}_-|^+\right)\right|^2 . \quad (54)$$

Vacuum-to-vacuum probability, differential and total mean numbers

Here, using exact solutions that were found above, we already can calculate characteristics of the vacuum instability in the analytic asymmetric electric field, namely the vacuum-to-vacuum transition probability P_{v}, differential N_n and total N mean numbers of created pairs.
As it follows from the general formulation of strong–field *QED* with *t*-electric potential steps, all these characteristics are expressed via coefficients

$$P_{\mathrm{v}} = \exp\left[\sum_n \ln\left(1 - N_n\right)\right], \quad N_n = \left|g_n\left({}_-\middle|{}^+\right)\right|^2, \quad N = \sum_n N_n. \tag{55}$$

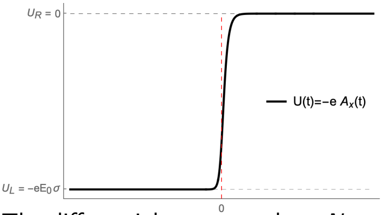
Using the above calculated coefficients $g_n\left({}_-\middle|{}^+\right)$, we can find:

$$\boxed{N_n = \frac{\sinh 2\sigma\pi\left(\omega_0 + \omega_1 - \omega_2/2\right)\sinh 2\sigma\pi\left(\omega_0 - \omega_1 + \omega_2/2\right)}{\sinh 4\sigma\pi\omega_1 \sinh 2\sigma\pi\omega_2}}, \tag{56}$$

$$\omega_0 = \frac{1}{2}\sqrt{\left(p_x - eE_0\sigma\right)^2 + \pi_\perp^2}\ , \tag{57}$$

$$\omega_1 = \sqrt{p_x^2 + \mathbf{p}_\perp^2 + m^2}, \quad \omega_2 = \sqrt{\left(p_x + eE_0\sigma\right)^2 + \mathbf{p}_\perp^2 + m^2}\ . \tag{58}$$

Small values of the parameter $\sigma \ll (eE_0)^{-1}\sqrt{p_x^2 + \mathbf{p}_\perp^2 + m^2}$.



In this case, the analytic asymmetric electric field and its potential change rapidly, and the electric field is a short pulse corresponding to a small increment ΔW,

$$\Delta W = P_x(t \to -\infty) - P_x(t \to +\infty) = \\ = eE_0\sigma\,, \quad (59)$$

$$P_x(t) = p_x + eA_x(t)\;. \quad (60)$$

The differential mean numbers N_n are also small enough for any p_x and $\pi_\perp$,

$$N_n = \frac{(eE_0\sigma)^2\,\pi_\perp^2}{4\left(p_x^2 + \pi_\perp^2\right)^2}\left[1 + O\left(\frac{eE_0\sigma}{p_x^2 + \pi_\perp^2}\right)\right]\;. \quad (61)$$

It is the case of a weak external field such a result can be derived in the frame of perturbation theory with respect to the external field. At small longitudinal momenta, $p_x^2 \ll \pi_\perp^2$,

$$N_n \approx \frac{(\Delta W)^2}{4\pi_\perp^2}, \quad (62)$$

which coincides with the result obtained, for example, for a weak pulse of T-constant electric field with the height $\Delta W = eET$ of a corresponding step in the same range of longitudinal momenta. In the case of a small ΔW the leading term of the distribution N_n is given by Eq. (62) does not depend on the field configuration.

Big values of the parameter sigma – a slowly varying electric field

$$\sigma \gg (eE_0)^{-1/2} \max\left\{1, m^2/eE_0\right\} . \quad (63)$$

In this case the differential mean numbers can be approximately presented as:

$$N_n \approx \exp\left[-\pi\tau\right], \quad \tau = 2\sigma\left(2\omega_1 + \omega_2 - 2\omega_0\right) . \quad (64)$$

The main contribution to the total number $N = \sum_n N_n$ is given by the expression

$$\boxed{N \approx V_{(d-1)}\rho, \; \rho = k\frac{\Delta W}{eE_{\max}}\beta, \quad \beta = \frac{J_{(d)}}{(2\pi)^{d-1}}\left[eE_{\max}\right]^{d/2}\exp\left[-\frac{\pi m^2}{eE_{\max}}\right]}, \quad (65)$$

$$k = \frac{2}{3}\int_0^{+\infty}\frac{dq}{(q^2+2q)^{1/2}(q+1)^{d/2}}\cos\frac{\arccos\left[(q+1)^{-1}\right]}{3}\exp\left[-\frac{\pi m^2}{eE_{\max}}q\right], \quad (66)$$

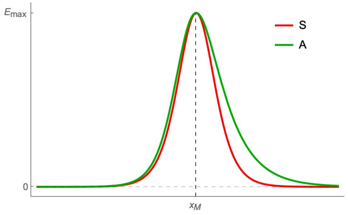
We see that the number density ρ of created pairs is proportional to the increment ΔW of kinetic momentum. This latter quantity defines the total number of states $\Delta WL/2\pi$ with the longitudinal momenta p_x, in which particles can be created (here L is the length of the system along the axis x). Note that this is typical to any slowly varying field. The density of created pairs obtained with the help of the slowly varying field approximation (see S. P. Gavrilov and D. M. Gitman, Phys. Rev. D. **95** (2017)) coincide with (65).

x-potential of analytic asymmetric electric field

We note that among the above exactly solvable cases only the Sauter electric field is given by an analytic function,

$$A_0^{(\mathrm{Sauter})}(x) = -LE_S \tanh(x/L),$$
$$E_{(\mathrm{Sauter})}(x) = E_S \cosh^{-2}(x/L), \; E_S > 0, \; L > 0. \tag{67}$$

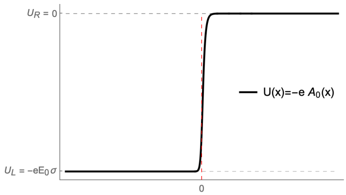
This field reaches its maximum value at $x = 0$ and is symmetric with respect to the origin. Unlike the above mentioned cases given by piecewise smooth x-steps, physical quantities calculated for the analytic Sauter field are presented by elementary functions, which makes this case especially convenient for physical interpretations.



Here we present a new example of exactly solvable case in which the external field is given by the following analytic function:

$$A_0(x) = \frac{\sigma E_0}{\sqrt{1+\exp\left(\frac{x}{\sigma}\right)}}, \quad E_0 > 0, \; \sigma > 0,$$
$$E(x) = \frac{E_0}{8}\sqrt{1+\exp\left(\frac{x}{\sigma}\right)} \cosh^{-2}\left(\frac{x}{2\sigma}\right). \tag{68}$$

Regularizations of Klein step (x-case) I



The magnitude
δU of the potential step is given by the difference

$$\delta U = U_{\mathrm{R}} - U_{\mathrm{L}} = eE_0\sigma. \tag{69}$$

Depending on the magnitude
δU, the step is called noncritical or critical one,

$$\begin{array}{ll} \delta U < \delta U_c = 2m\,, & \text{noncritical step} \\ \delta U > \delta U_c, & \text{critical step} \end{array} . \tag{70}$$

Let us study characteristics of the vacuum instability caused by the asymmetric field with σ sufficiently small, $\sigma \to 0$. If $U_{\mathrm{L/R}}$ are given constant and

$$\delta U\sigma \ll 1 \tag{71}$$

the field imitates sufficiently well the asymmetric Klein step and coincides with the latter as $\sigma \to 0$.

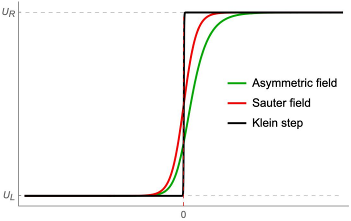
Regularizations of Klein step (x-case) II

The vacuum instability is due to contributions formed in the Klein zone $\Omega_3 = \{U_\mathrm{L} + \pi_\perp \leq p_0 \leq U_\mathrm{R} - \pi_\perp\}$. In this range the important characteristic of all the processes are differential mean numbers. For Sauter electric field we have

$$N_n = \left|g_n\left({}_+|{}^-\right)\right|^{-2} = \frac{\sinh(\pi L|p^\mathrm{L}|)\sinh(\pi L|p^\mathrm{R}|)}{\left|\sinh\left(\pi L\frac{\delta U+|p^L|-|p^R|}{2}\right)\sinh\left(\pi L\frac{\delta U-|p^L|+|p^R|}{2}\right)\right|}. \quad (72)$$

For asymmetric electric field we obtain:

$$N_n = \left|g_n\left({}_+|{}^-\right)\right|^{-2} = \sinh\left(2\pi\left|p^\mathrm{L}\right|\sigma\right)\sinh\left(4\pi\left|p^\mathrm{R}\right|\sigma\right)\left|\beta_+\beta_-\right|^{-1},$$
$$\beta_\pm = \sinh\left\{\pi\sigma\left[\sqrt{2\delta U^2 + 2\left|p^\mathrm{R}\right|^2 - \left|p^\mathrm{L}\right|^2} \pm \left(2\left|p^\mathrm{R}\right| - \left|p^\mathrm{L}\right|\right)\right]\right\}. \quad (73)$$



In the Klein step limit for both fields we have

$$N_n \approx \frac{4\left|p^\mathrm{L}\right|\left|p^\mathrm{R}\right|}{\delta U^2 - \left(|p^\mathrm{L}| - |p^\mathrm{R}|\right)^2}, \quad (74)$$

where $p^{\mathrm{L/R}} = \sqrt{\left(p_0 - U_{\mathrm{L/R}}\right)^2 - \pi_\perp^2}$.

A. I. Breev, S. P. Gavrilov and D. M. Gitman, Phys. Rev. D **109** (2024)

Final remarks

- One of the important result of the work is the original approach to solving the Dirac equation in the asymmetric electric field and the construction of the corresponding in- and out-solutions.
- With the help of these solutions, basic characteristics of the vacuum instability are calculated nonperturbatively, namely, the vacuum-to-vacuum transition probability P_{v}, differential N_n and total N mean numbers of created pairs.
- We analyze the dependence of the calculated quantities on the time scale parameter σ, which determines the shape of the analytic asymmetric electric field, for example, for the strong field and in the case when the increment ΔW of the longitudinal momentum is large enough (this is the case of a rapidly changing electric field) the differential mean numbers N_n reach their maximum possible for fermions values $N_n \approx 1$ in wide ranges of the momenta p_x' and $\mathbf{p}_\perp$.
- The influence of the asymmetry of the field under consideration on the particle production is studied. It was demonstrated that differential mean numbers behave differently as functions of positive and negative longitudinal momenta p_x.
- The obtained characteristics of the vacuum instability in the analytic asymmetric electric field were compared with ones obtained in other exactly solvable cases.