Lattice local integrable regularization of the Sine-Gordon model.

A.Ovchinnikov[1]

[1] INR, Moscow

Quarks 2014

Introduction

- We study the local lattice integrable regularization of the Sine-Gordon model written down in terms of the lattice Bose- operators.
- We show that the local spin Hamiltonian obtained from the six-vertex model with alternating inhomogeneities in fact leads to the Sine-Gordon in the low-energy limit.
- We show that the Bethe Ansatz results for this model lead to the correct general relations for different critical exponents of the coupling constant.

Lattice Hamiltonian.

$$S_{12}(t_1 - t_2) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}_{12} (t), \quad t = t_1 - t_2,$$

with

$$a(t) = \text{sh}(t + i\eta), \quad c(t) = \text{sh}(t), \quad b(t) = \text{sh}(i\eta).$$

This S- matrix obeys the Yang-Baxter equation $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$. The transfer matrix $Z(t)$ acting in the quantum space $(1, \ldots L)$ has the form:

$$Z(t) = Z(t; \xi_1, \ldots \xi_L) = Tr_0 \left(S_{10}S_{20} \ldots S_{L0} \right), \tag{1}$$

$$\xi_{2k+1} = 0, \quad \xi_{2k} = \theta, \quad k \in Z. \tag{2}$$

The local Hamiltonian corresponding to the SG- model equals

$$H = H(0) + H(\theta) = \frac{\text{sh}(i\eta)}{2} \left(Z^{-1}(0)\dot{Z}(0) + Z^{-1}(\theta)\dot{Z}(\theta) \right), \tag{3}$$

Local Hamiltonian:

$$H = \sum_i \left(S^{-1}_{i+1,i+2} P_{i,i+2} \dot{S}_{i,i+2} S_{i+1,i+2} + \text{sh}(i\eta) S^{-1}_{i+1,i+2} \dot{S}_{i+1,i+2} \right).$$

We consider the Hamiltonian (**??**) at large θ and expand it in powers of $e^{-\theta}$. At $e^{-\theta}=0$ we get two coupled XXZ- spin chains:

$$H_0=\frac{1}{2}b_1^+b_3e^{i2\eta(n_2-1/2)}+h.c.+\Delta n_1n_3+\frac{1}{2}b_2^+b_4e^{-i2\eta(n_3-1/2)}+h.c.+\Delta n_2n_4+\ldots,\Delta=\cos(\eta), \quad (4)$$

where the dots stand for the next terms of the odd and even spin chains and the hard-core bosons b_i^+, b_i, $n_i=b_i^+b_i$ are introduced to describe the state at the site i in such a way that $n_i=1$ ($n_i=0$) corresponds to the spin-up (spin-down) state and b_i^+ (b_i) change the direction of spin. Now we can remove the interaction between two chains performing the transformation

$$\tilde{b}_{1x}^+=b_{1x}^+e^{i2\eta\Delta N_2(x)},\quad \tilde{b}_{2x}^+=b_{2x}^+e^{-i2\eta\Delta N_1(x)}, \quad (5)$$

where the notations $b_{1x}^+=b_{2x-1}^+$, $b_{2x}^+=b_{2x}^+$, $x=1,2,\ldots L/2$ are used and $\Delta N_1(x)=\sum_{i<x}(n_{1i}-1/2)$, $\Delta N_2(x)=\sum_{i<x}(n_{2i}-1/2)$. In terms of the new operators $\tilde{b}_{1x}^+$, $\tilde{b}_{2x}^+$ (5) the Hamiltonian (4) takes the form of two independent XXZ- spin chains and one can use the known results for the spin chain to study the Hamiltonian (**??**).

Now we calculate the interaction of the two spin chains of order $e^{-\theta}$. The task is simplified if one is extracting the terms which lead to the relevant interaction $\sim \cos(\beta\phi)$. This operator has the scaling dimension $d = \beta^2/4\pi$ and is relevant at the interval $0 < \beta^2 < 8\pi$ which corresponds exactly to the interval $\eta \in (0;\pi)$ where the spectrum of the single XXZ- spin chain is gapless. For example, the terms which contain the factor $(n_i - 1/2) \sim \partial_x \phi(x)|_{x=i}$, where $\phi(x)$- is some Bose field, lead to the irrelevant operators and can be omitted. REsult:

$$\hat{V} = 2(\sin(\eta))^2 e^{-\theta} \left(\sum_x (b^+_{1x} b_{2x} + h.c.) + \sum_x (b^+_{1(x+1)} b_{2x} + h.c.) \right) \sim hC \sum_x b^+_{1x} b_{2x} + h.c.$$

The operators of the separate XXZ spin are known:

$$\tilde{b}^+_{1x} \simeq (-1)^x e^{-i\pi\sqrt{\xi}(\hat{N}_1 - \hat{N}_2)^{(1)}(x)}, \quad \tilde{b}_{2x} \simeq (-1)^x e^{i\pi\sqrt{\xi}(\hat{N}_1 - \hat{N}_2)^{(2)}(x)}, \qquad (6)$$

where the operators $\hat{N}^{(i)}_{1,2}(x)$ for the two chains $i = 1, 2$ are expressed through the initial Fermi-operators of the Luttinger model.

Bosonization.

$$\hat{N}_1 - \hat{N}_2 = (1/\sqrt{\xi})(N_1 - N_2), \quad \hat{N}_1 + \hat{N}_2 = \sqrt{\xi}(N_1 + N_2),$$

where ξ- is the standard Luttinger liquid paramemeter and

$$N_{1,2}(x) = \frac{i}{L'} \sum_{p \neq 0} \frac{\rho_{1,2}(p)}{p} e^{-ipx}, \quad \rho_{1,2}(p) = \sum_k a^+_{1,2}(k+p) a_{1,2}(k).$$

where $L' = L/2$. The standard Bose fields $\phi_1(x)$, $\phi_2(x)$ are connected with the fields $\hat{N}^{(i)}_{1,2}(x)$ in the following way:

$$(\hat{N}^{(i)}_1 + \hat{N}^{(i)}_2)(x) = (1/\sqrt{\pi})\phi_i(x), \quad (\hat{N}^{(i)}_1 - \hat{N}^{(i)}_2)(x) = (1/\sqrt{\pi})\tilde{\phi}_i(x), \quad i = 1, 2,$$

where the dual fields $\tilde{\phi}_i(x)$ are defined according to the equations

$$\tilde{\phi}_i(x) = \int^x dy \pi_i(y), \quad \pi_i(x) = \dot{\phi}_i(x), \quad i = 1, 2,$$

where $\pi_i(x)$- are the conjugated momenta. Now from the equations (5) one can see that the operator b^+_{1x} take the following form:

$$b^+_{1x} \simeq \exp\left(-i\pi\sqrt{\xi}(\hat{N}_1 - \hat{N}_2)^{(1)}(x) + i(2\pi - 2\eta)\frac{1}{\sqrt{\xi}}(\hat{N}_1 + \hat{N}_2)^{(2)}(x)\right) = e^{i\sqrt{\pi}\sqrt{\xi}(-\tilde{\phi}_1(x) + \phi_2(x))}, \quad (7)$$

where the value $\xi = 2(\pi - \eta)/\pi$ was substituted. Analogously for the operator b_{2x} we obtain the expression

$$b_{2x} \simeq \exp\left(i\pi\sqrt{\xi}(\hat{N}_1 - \hat{N}_2)^{(2)}(x) + i(2\pi - 2\eta)\frac{1}{\sqrt{\xi}}(\hat{N}_1 + \hat{N}_2)^{(1)}(x)\right) = e^{i\sqrt{\pi}\sqrt{\xi}(\tilde{\phi}_2(x) + \phi_1(x))}. \quad (8)$$

Note that in the process of the derivation of (7), (8) we have inserted the additional factors equal to unity of the form $\prod_{i<x} e^{i2\pi n_i} = (-1)^x \prod_{i<x} e^{i2\pi(n_i - 1/2)}$ which cancels the factors $(-1)^x$ in the equations (6). Combining the equations (7) and (8) we get for the interaction density the expression:

$$b^+_{1x} b_{2x} \simeq e^{i\sqrt{\pi}\sqrt{\xi}(-\tilde{\phi}_1(x)+\tilde{\phi}_2(x)+\phi_1(x)+\phi_2(x))} = e^{i2\sqrt{\pi}\sqrt{\xi}\phi(x)}, \tag{9}$$

where we have introduced two new fields $\phi(x)$ and $\chi(x)$ defined according to the equations

$$\phi(x) = \sqrt{\pi}(\hat{N}^{(1)}_2 + \hat{N}^{(2)}_1)(x), \quad \chi(x) = \sqrt{\pi}(\hat{N}^{(1)}_1 + \hat{N}^{(2)}_2)(x),$$

or in terms of the dual fields

$$\phi(x) + \chi(x) = \phi_1(x) + \phi_2(x), \quad \phi(x) - \chi(x) = -\tilde{\phi}_1(x) + \tilde{\phi}_2(x).$$

In terms of this new fields we get exactly the Lagrangian of the SG- model:

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 + C\mu^{\xi} h\cos(\beta\phi) \tag{10}$$

The Bethe Ansatz equations for the parameters t_α, $\alpha = 1, \dots M$, which determine the common eigenstates of the transfer matrix (1) and the Hamiltonian (3) have the standard form:

$$\left(\frac{\mathrm{sh}(t_\alpha - i\eta/2)}{\mathrm{sh}(t_\alpha + i\eta/2)}\right)^{L/2}\left(\frac{\mathrm{sh}(t_\alpha - \theta - i\eta/2)}{\mathrm{sh}(t_\alpha - \theta + i\eta/2)}\right)^{L/2} = \prod_{\gamma\neq\alpha}\frac{\mathrm{sh}(t_\alpha - t_\gamma - i\eta)}{\mathrm{sh}(t_\alpha - t_\gamma + i\eta)} \qquad (11)$$

The solution of the equations (11) is similar to the solution of the corresponding equations for the XXZ- spin chain. In terms of the parameters $t_1, \dots t_M$ the energy and the momentum of the eigenstates of the operator (**??**) are

$$E = (\sin(\eta)/2)\sum_\alpha(\phi'(t_\alpha) + \phi'(t_\alpha - \theta)), \quad P = \sum_\alpha(\phi(t_\alpha) + \phi(t_\alpha - \theta)), \qquad (12)$$

where the function $\phi(t) = (1/i)\ln(-\mathrm{sh}(t - i\eta/2)/\mathrm{sh}(t + i\eta/2))$. For the ground state the roots t_α are real and the corresponding density of roots $R(t)$ equals

$$R(t) = \frac{1}{2}(R_0(t) + R_0(t-\theta)), \quad R_0(t) = \frac{1}{2\eta\mathrm{ch}(\pi t/\eta)}. \qquad (13)$$

The calculation of the energy and the momentum of the single hole is quite standard and the result is analogous to that for the XXZ- spin chain:

$$\epsilon(t) = (\sin(\eta)/2)2\pi(R_0(t) + R_0(t-\theta)), \quad p'(t) = 2\pi(R_0(t) + R_0(t-\theta)), \qquad (14)$$

where t is the rapidity of the hole and the prime means the derivative over t. From the equation (14) in the limit $\theta \to \infty$ one can easily obtain the relativistic dispersion relation for the soliton:

$$\epsilon(t) = M\mathrm{ch}(\pi t/\eta), \quad p(t) = M\mathrm{sh}(\pi t/\eta), \quad M = 4\sqrt{v}e^{-\pi\theta/2\eta}. \qquad (15)$$

Now we confirm our expression for β and the critical behaviour of the mass gap and the ground state energy found from the exact solution. Consider the system of free massless Bose field (H_0) perturbed by the relevant operator $V = h\sum_x V_x$. The perturbation theory in the coupling constant h has the infrared divergences. To take them into account one has to sum up the whole perturbation theory series. In general for the ground and excited states we have the expression of the type

$$E(h) = V\frac{1}{E_0 - H_0}V\left(1 + U + U^2 + \dots\right), \quad U = \frac{1}{E_0 - H_0}V\frac{1}{E_0 - H_0}V.$$

To estimate the first term we write

$$\langle V\frac{1}{E_0 - H_0}V\rangle \sim h^2\frac{1}{(1/L)}\sum_{i,j}\langle V_i V_j\rangle \simeq L^2 h^2 \sum_x \frac{1}{x^d} \sim h^2 L^{3-2d},$$

where d- is the scaling dimension of the operator V_x. Analogously we obtain $U \sim h^2 L^{4-2d}$. We see that the operator V is relevant provided $d < 2$. Thus the ground state energy has the form $E_0(h) = h^2 L^{3-2d} f(h^2 L^{4-2d})$ with some unknown function $f(y)$. From the condition $E_0 \sim L$ one can find the behaviour of $f(y)$ at large y. Thus we obtain the result:

$$E_0(h) \sim h^{\frac{2}{2-d}}. \tag{16}$$

Analogously for the mass gap we find:

$$M \sim h^{\frac{1}{2-d}}. \tag{17}$$

The equations (16), (17) are known also from the conformal line of arguments [**?**]. In our case the scaling dimension $d = \beta^2/4\pi = \xi$ and we see that the equations (16), (17) are in agreement with the predictions (15), (**??**) (in our case $h \sim e^{-\theta}$ and $1/(2-d) = 1/(2-\xi) = \pi/2\eta$). Thus the complete agreement of the perturbation theory

Conclusion.

- We have shown that the six-vertex model with alternating inhomogeneity parameters can be used to construct the local lattice integrable regularization of the Sine-Gordon model. We have shown *directly* that the system is equivalent to the two weakly coupled XXZ- spin chains and up to the irrelevant operators the interaction gives exactly the Sine-Gordon Lagrangian with the correct value of the parameter β.
- We compare the soliton mass and the vacuum energy obtained in the framework of the Bethe Ansatz with the general predictions of the perturbation theory for their power-law behaviour in the coupling constant.