QED amplitudes at high energies: factorization formulae and Frobenius expansion in mass[a]

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[a]Partly based on joint paper with **V.S. Fadin**, [arXiv:2308.09479]

Motivation

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- Ongoing and planned experiments require the calculations of the amplitudes and differential cross sections with NNLO precision. This requirement is especially relevant in the context of New Physics searches.
- Thanks to the development of modern methods of multi-loop calculations, achieving NNLO precision of theoretical predictions becomes a doable task. Using computer algebra for NNLO calculations is a must.
- The presence of massive lines in the diagrams essentially complicates the calculations.
- In collider experiments the particles are accelerated almost up to the speed of light, therefore their masses are small compared to the energies. Typically, the mass is also small compared to momentum transfers.

Multiloop calculations: state of the art

Complexity crucially depends on # of loops L and on # of scales S.

S \ L	1 loop	2 loops	3 loops	4 loops	5 loops	> 5
1	✓	✓	✓	✓	a few	
2	✓	✓	some	a few		
3	✓	some	a few			
> 3	✓	a few				

Massive calculations are typically 1 or 2 loops behind the corresponding massless calculations:

- QED and QCD form factors:
  - massless: 4 loops analytically,
  - massive: 2 loops analytically, 3 loops numerically.
- Self-energy:
  - massless: 4 loops analytically,
  - massive: 3 loops: spectral density.

Even when m can be considered as small parameter, it can not be completely neglected due to the collinear (or mass) logarithms $\ln(s/m^2)$.

Calculation path

1. Diagram generation ✓

Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001], `tapir` [Gerlach et al., 2022],...

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], ...

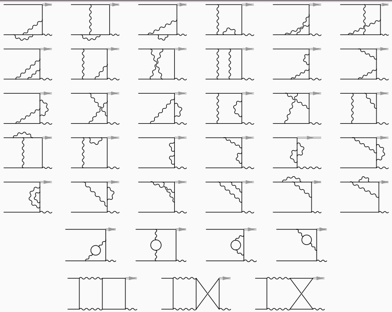
3. DE Solution

Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

Example: Process $e^+e^- \to \gamma\gamma^*$ at two loops.

- Diagrams: 37 distinct diagrams.
- IBP reduction
  - $m = 0$: reduction is easy, leads to 60 masters.
  - $m \neq 0$: reduction is hardly doable, leads to $\gtrsim$ 400 masters.
- DE reduction
  - $m = 0$: easily reducible to ϵ-form. All masters are polylogarithmic.
  - $m \neq 0$: irreducible to ϵ-form, many masters are non-polylogarithmic.



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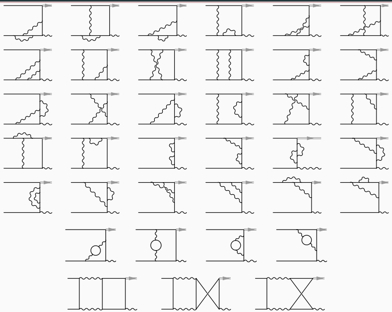

It is easy to calculate the amplitude at $m = 0$, but almost impossible for $m \neq 0$.

How to account for small $m \neq 0$?

1. Physical approach: use factorization formula relating massive to massless amplitude.
2. Perform IBP reduction exactly in m, solve DE via Frobenius expansion.
3. Use expansion by regions technique and IBP reduction in parametric representation.

Factorization formulae in QED

Factorization of soft radiation [Yennie et al., 1961] in massive QED

It is well known that soft radiation factorizes:

$$d\sigma^{(n)} = \frac{W^n(\omega_0)}{n!} d\sigma^{(0)} ,$$
$$d\sigma^{inc} = \sum_n d\sigma^{(n)} = e^{W(\omega_0)} d\sigma^{(0)}$$

$d\sigma^{(n)}$ — cross section with n additional soft photons, $d\sigma^{inc}$ — inclusive cross section. $W(\omega_0)$ — probability of soft (with $\omega < \omega_0$) photon radiation, which depends on charges Q_i and momenta p_i of incoming and outgoing particles:

$$W(\omega_0) = -\sum_{i<j} Q_i Q_j W(p_i, p_j|\omega_0)$$

In dimensional regularization ($d = 4 - 2\epsilon$) we have

$$W(p_i, p_j|\omega_0) = -e^2 \int\limits_{\omega<\omega_0} \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon} 2\omega} \left(\frac{p_i}{k\cdot p_i} - \frac{p_j}{k\cdot p_j} \right)^2$$

Both the elastic cross section $d\sigma_0$ and the probability W contains infrared divergencies which are canceled in the inclusive cross section $d\sigma^{inc}$. In the dimensional regularization the divergencies correspond to the poles in ϵ with the pole order being equal to the number of loops l.

Soft radiation probability

Exact expression for W up to ϵ^0 [RL, 2020]

$$W(p_1, p_2|\omega_0) = \frac{2\alpha}{\pi}\frac{(2\omega_0)^{-2\epsilon}}{(-2\epsilon)}\frac{1}{\beta_3}\Bigg\{ -\ln x_3 \quad \Longleftarrow \text{[Berestetskii et al., 1982]}$$

$$+\epsilon\left[f\left(\frac{x_1x_3}{x_2}\right) + f\left(\frac{x_2x_3}{x_1}\right) + f\left(x_1x_2x_3\right) - f\left(\frac{x_1x_2}{x_3}\right) - f\left(x_3^2\right)\right] + O\left(\epsilon^2\right)\Bigg\},$$

$$f(x) = \mathrm{Li}_2(1-x) + \frac{1}{4}\ln^2 x\,, \qquad x_k = \sqrt{\frac{1-\beta_k}{1+\beta_k}},$$

$\beta_{1,2}$ — velocities of the 1st and 2nd particles in lab frame,
β_3 — velocity of one particle in the rest frame of another (relative velocity).

Higher orders in ϵ have much more complicated form.
Question: up to which order in ϵ one needs to know W for L-loop calculations?

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Let

$$W = \alpha\left(w_{-1}/\epsilon + w_0 + w_1\epsilon + \ldots\right),$$

$$\sigma = \sigma_0 + \alpha\sigma_1 + \alpha^2\sigma_2 + \ldots \qquad \Longleftarrow \sigma_k \text{ starts from } O(\epsilon^{-k})$$

Then

$$e^W\sigma = \sigma_0 + \alpha\left[\underline{\sigma_1 + \sigma_0 w_{-1}/\epsilon} + w_0 + \mathcal{O}(\epsilon)\right]$$

$$+\alpha^2\left[\sigma_2 + \sigma_1\left(w_0 + w_{-1}/\epsilon\right) + \sigma_0\left(w_0 + \frac{w_{-1}}{\epsilon}\right)^2 + (\underline{\sigma_1 + \sigma_0 w_{-1}/\epsilon})\epsilon w_1 + \mathcal{O}(\epsilon)\right]$$

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Claim: It is sufficient to know W up to ϵ^0 for any number of loops.

Factorization of soft singularities in massive QED amplitudes [Yennie et al., 1961]

Soft divergencies in the elastic cross section are due to the virtual corrections. They are known to factorize in the amplitude $A(in \to out)$ as [Yennie et al., 1961]

$$A(in \to out) = \exp\left\{ -\sum_{i<j} Q_i Q_j V\left(p_i, p_j\right) \right\} H(in \to out)\,,$$

where $H(in \to out)$, sometimes called “hard amplitude” in massive QED, is **finite** and

$$V\left(p_1, p_2\right) = -\frac{e^2}{2} \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2+i0} \left(\frac{2p_i - k}{k^2 - 2(kp_i) + i0} + \frac{2p_j + k}{k^2 + 2(kp_j) + i0} \right)^2 ,$$

NB: $V = \alpha\left(\frac{v_{-1}}{\epsilon} + v_0 + \ldots\right)$ can be expanded in ϵ in terms of polylogs.

Observation

- To derive H from A it is sufficient to know v_{-1} and v_0 only.
- To recover A from H we need to know also the higher-order terms. The more loops, the more terms.

Thus, it looks like we loose something when passing from A to H. Nevertheless, $d\sigma^{inc} = e^W |A|^2 d\Phi = e^{W+2\,\mathrm{Re}\,V} |H|^2 d\Phi$ and it is sufficient to know $e^{W+2\,\mathrm{Re}\,V}$ at $\epsilon = 0$.

Factorization of soft and collinear singularities in massless QED amplitudes

In massless QED in addition to soft singularities there are also collinear singularities. In particular, in the l-loop amplitude the leading pole in ϵ has the order $2l$, to be compared with l for massive QED. Nevertheless, the factorization of soft and collinear singularities is well-understood in QCD, starting from works [Catani, 1998; Sterman and Tejeda-Yeomans, 2003] and other. Therefore, there is a temptation to translate the corresponding formulae to QED. Then the amplitude (in massless QED) $\mathcal{A}(in \to out)$ is related to *finite* "hard massless" amplitude $\mathcal{H}(in \to out)$ via

$$\mathcal{A}(in \to out) = \mathcal{Z}(in \to out)\mathcal{H}(in \to out)\,,$$

where

$$\mathcal{Z}(in \to out) = \exp\left\{\int_0^{\bar{a}} \frac{da_1}{a_1\beta(\epsilon, a_1)}\left[-\frac{1}{2}\sum_i \gamma_i(a_1) + \frac{1}{4}\sum_{i<j} Q_i Q_j \left(\gamma_K(a_1)\ln\left(\frac{-(p_i+p_j)^2 - i0}{\mu^2}\right) + \int_0^{a_1} \frac{da_2\, \gamma_K(a_2)}{a_2\beta(\epsilon, a_2)}\right)\right]\right\}.$$

Here $\bar{a} = \alpha_{\overline{\text{MS}}}(\mu)/4\pi$ is an $\overline{\text{MS}}$ coupling constant, γ_K is a light-cone cusp anomalous dimension, γ_i is the collinear anomalous dimension of i-th external particle and $\beta = \frac{d\ln\bar{a}}{d\ln\mu^2} = -\epsilon - \sum_{l=0}^{\infty} \beta_l \bar{a}^{l+1}$ is the beta-function in $4-2\epsilon$ dimension.

Relation between massless and massive QED amplitudes

Finally, the most important factorization formula introduced in Ref. [Becher and Melnikov, 2007] relates the amplitude $\mathcal{A}(in \to out)$ in massless QED to the amplitude $A(in \to out)$ in massive QED up to power corrections wrt m. It reads

$$A(in \to out) = \left[Z_3^{\mathrm{OS}}\right]^{k/2} [Z_J]^{n/2}\, S(in \to out)\mathcal{A}(in \to out) + \mathcal{O}(m^2)\,. \qquad (\star)$$

Here k and n are the numbers of external photons and electrons/positrons, respectively, Z_3^{OS} is the on-shell renormalization constant of photon field, S is a **soft function**,

$$\ln S(in \to out) = -\sum_{i<j} Q_i Q_j \delta S\left(-(p_i + p_j)^2 - i0, m^2\right)\,,$$

$$\delta S(Q^2, m^2) = \bar{a}^2 \left(\tfrac{\mu}{m}\right)^{4\epsilon} n_f \left[-\tfrac{4}{3\epsilon^2} + \tfrac{20}{9\epsilon} - \tfrac{112}{27} - \tfrac{4}{3}\zeta_2 + O\left(\epsilon^1\right)\right] \ln(Q^2/m^2) + \mathcal{O}\left(\bar{a}^3\right)\,,$$

and $Z_J = Z_J(\mu/m, \epsilon)$ is a universal **jet function**.

The practical idea of using the factorization formula $(\star)$ is to obtain Z_J from the same relation for simplest possible amplitude — electron form factor and then use this expression for more complicated amplitudes.

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Open question

Is it possible to express Z_J via anomalous dimensions, similar to $\mathcal{Z}$?

Consequences of factorization

Let us consider the factorization relation of the form

$$X(\epsilon, z) = F(\epsilon, z) Y(\epsilon)\,,$$

where z is a set of small parameters and quantities X, Y and F admit perturbative expansions

$$X(\epsilon, z) = \sum_{l=0}^{\infty} X_l(\epsilon, z)\tilde{a}^l\,, \qquad Y(\epsilon) = \sum_{l=0}^{\infty} Y_l(\epsilon)\tilde{a}^l\,, \qquad \ln F(\epsilon, z) = \sum_{l=1}^{\infty} f_l(\epsilon, \ln z)\tilde{a}^l$$

with $f_l(\epsilon, \ln z)$ being polynomial in $\ln z$.

Examples (specialized to $e^+e^- \to \gamma\gamma^*$ process)

- Relation between hard massive and hard massless amplitudes ($z = \{m\}$):

$$H(e^+e^- \to \gamma\gamma^*) = e^{-V(p_+, p_-)} \left(Z_3^{\mathrm{OS}}\right)^{1/2} Z_J\, S\, \mathcal{Z}\, \mathcal{H}(e^+e^- \to \gamma\gamma^*)$$

- Relation between inclusive cross section and hard massless amplitudes: ($z = \{m, \omega_0\}$):

$$d\sigma^{inc} = e^{W(p_+, p_- | \omega_0)} Z_3^{\mathrm{OS}}\, |Z_J S \mathcal{Z}|^2 |\mathcal{H}|^2\, d\Phi\,.$$

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with $f_l(\epsilon, \ln z)$ being polynomial in $\ln z$.

Suppose that $X(\epsilon, z)$ is finite at $\epsilon = 0$.

Consequences

1. In order to determine $X(0, z)$, we need to know factors $f_l(\epsilon, \ln z)$ only up to ϵ^0 terms.
2. The terms in $X_l(0, z)$ amplified by powers of $\ln z$ are entirely expressed via lower-loop results $X_k(0, z)$, $k < l$.

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- So far, the factorization of mass logarithms via factor $Z_J S$ were checked only for $e^+e^- \to \gamma^*$ (form factor) through two loops. Meanwhile, this factorization formula "relies on the assumption that only hard, collinear and soft momentum modes are relevant in the effective theory computation. However, as was explicitly shown in Ref. [Smirnov, 1999], this assumption is invalid for some diagrams that contribute to the form factor." (citation from [Becher and Melnikov, 2007]).

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Therefore direct *ab initio* calculations for various processes is highly desirable.

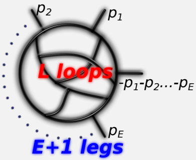
Frobenius method and DE for boundary constants

IBP reduction [Chetyrkin and Tkachov, 1981]

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = j(n_1, \ldots, n_N) = \int d\mu_L \boldsymbol{D}^{-\boldsymbol{n}} = \int \prod_{i=1}^{L} d^d l_i \prod_{k=1}^{N} D_k^{-n_k},$$

$l_1, \ldots l_L$ –loop momenta, $p_1, \ldots p_E$ — external momenta.



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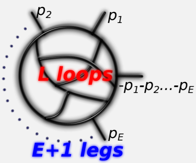

There are $N = L(L+1)/2 + L \cdot E$ scalar products involving loop momenta:

$$s_{ij} = l_i \cdot q_j, \qquad q_j = \begin{cases} l_j & j \leqslant L \\ p_{j-L} & j > L \end{cases} \qquad (1 \leqslant i \leqslant L, i \leqslant j \leqslant L + E)$$

$D_1, \ldots, D_M$ — denominators of the diagram, $D_{M+1}, \ldots, D_N$ — irreducible numerators, such that $D_1, \ldots, D_N$ form a basis, i.e. any scalar product can be uniquely expressed via linear function of D_k.

IBP identities

In dim. reg. integral of divergence is zero (no surface terms):

$$0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_j \boldsymbol{D}^{-\boldsymbol{n}} = \sum_s c_s(\boldsymbol{n}) j(\boldsymbol{n} + \boldsymbol{\delta}_s).$$

Explicitly differentiating, we obtain relations between integrals.

Differential equations and dimensional recurrences

As a result of IBP reduction we express amplitudes via a finite set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. What is even more important, we can obtain closed equations for the master integrals. To obtain these equations we simply apply the dimensional shifts and/or differentiate the master integrals and then IBP-reduce the result. Then the dimension shifts and/or derivatives of the master integrals is expressed as linear combination of the same set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. We obtain

Differential equations

[Kotikov, 1991; Remiddi, 1997]

$$\partial_x \boldsymbol{j} = M(x, d)\boldsymbol{j}$$

Dimensional recurrences

[Tarasov, 1996; Derkachov et al., 1990]

$$\boldsymbol{j}(d-2) = R(x, d)\boldsymbol{j}(d)$$

It is usually easier to solve these equations than to use direct methods for calculation of the master integrals.

Differential equations for master integrals

- IBP reduction provides differential equations for master integrals

$$\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$$

- [Henn, 2013]: there is often a "canonical" basis $\boxed{\boldsymbol{J} = T^{-1}\boldsymbol{j}}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution for d.e. in ϵ-form is easily expanded in ϵ:

$$U(x,x_0) = \mathrm{Pexp}\left[\epsilon\int_{x_0}^{x} dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

- The algorithm of finding transformation to ϵ-form was presented in [RL, 2015]. It is implemented at least in 3 publicly available codes: `Fuchsia`, `epsilon`, `Libra`.
- Criterion of reducibility to ϵ was presented in [RL and Pomeransky, 2017]. With some reservations, the reducibility mean that master integrals are polylogarithmic.

Unfortunately, many differential systems for massive integrals are irreducible to ϵ-form!

Frobenius expansion

- If x is a small parameter, we can think of solving DE

$$\partial_x \boldsymbol{j} = M(x, \epsilon) \boldsymbol{j}$$

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- Indeed, the regularized path-ordered exponent (the fundamental matrix)

$$U(x, \underline{0}) = \lim_{x_0 \to 0} \mathrm{Pexp}\left[\int_{x_0}^{x} M(x)dx\right] x_0^{M_0}, \quad M_0 = \mathrm{res}_{x=0} M(x)$$

can be expanded in generalized power series:

$$U(x, \underline{0}) = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C\,(n+\lambda, k) x^n \ln^k x\,.$$

Note that for expansion around singular point (which we usually want) non-integer powers x^{λ} (depending on ϵ) and explicit $\ln x$ might appear. **Those will be the sources of mass logarithms in the amplitude.**

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- Indeed, the regularized path-ordered exponent (the fundamental matrix)

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can be expanded in generalized power series:

$$U(x, \underline{0}) = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C\,(n+\lambda, k) x^n \ln^k x\,.$$

Note that for expansion around singular point (which we usually want) non-integer powers x^λ (depending on ϵ) and explicit $\ln x$ might appear. **Those will be the sources of mass logarithms in the amplitude.**

- The coefficient matrices $C\,(n+\lambda, k)$ up to a fixed n can be routinely found, e.g., with `Libra`. Therefore, we "only" need to fix the boundary constants $\boldsymbol{J}$ in $\boldsymbol{j} = U(x, \underline{0})\boldsymbol{J}$.

Differential equation for boundary coefficients [RL et al., 2021]

Note that the master integrals for amplitudes, apart from m, depend also on other parameters (e.g. s, t). The dependence of the boundary constants $\boldsymbol{J}$ on the ratios of these parameters is nontrivial. Therefore, we would like to apply the DE approach for the calculation of these constants. How can we derive DE for $\boldsymbol{J}$?

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Suppose that x is small parameter (in particular, we will be interested in $x = m^2/s$) and y is another parameter(s), e.g., $y = t/s$. Let the master integrals satisfy DE wrt x and y:

$$\partial_x \boldsymbol{j} = M_x \boldsymbol{j}\,, \quad \partial_y \boldsymbol{j} = M_y \boldsymbol{j}$$

We write $\boldsymbol{j} = U\boldsymbol{J} = U(x, \underline{0})\boldsymbol{J}$, where the column of "constants" $\boldsymbol{J}$ depends on y, and substitute in the second system to obtain DE for $\boldsymbol{J}$ as function of y. We have

$$\partial_y \boldsymbol{J} = \widetilde{M}_y \boldsymbol{J}\,,$$

where $\widetilde{M}_y = U^{-1} M_y U$.

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Observation

Although we are able to obtain only a finite (although a rather big) number of terms in x-expansion of U, it is sufficient to obtain $\widetilde{M}_y$ **exactly** as by construction it does not depend on x (while M_y does).

Massive factorization vs Frobenius expansion

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- In order to simplify the IBP reduction step, one can use the approach based on expansion by regions and IBP reduction in parametric representation (not described in this talk).

Conclusion and outlook

- In many physical applications masses of some particles can be considered as small parameters. However, due to the presence of collinear regions they can not be simply neglected.
- There is an approach based on soft-collinear factorization which allows to obtain small-mass asymptotics of massive amplitude via the corresponding massless amplitude.
- Another approach is to perform IBP reduction exactly in m and then to apply Frobenius method. This approach has a few advantages.
- Currently, the calculation of NNLO amplitudes for $e^+e^- \to \gamma\gamma$ and $e^+e^- \to \mu^+\mu^-$ processes is in progress. The master integrals for both processes are already calculated using the Frobenius method wrt electron mass. Stay tuned!

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Thank you!

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