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Path Integral Bosonization of Three-flavor Quark Dynamics and Ising Model Landau Magnetization

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Remake of B.O. Kerbikov and Yu.A. Simonov

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Plan of the talk

1. BCS inspired Vaks-Larkin and Nambu-Jona-Lasinio model
2. Instanton induced effective Lagrangian and 't Hooft determinant
3. Hubbard-Stratonovich three-flavor Bosonization
4. Simularity to infinite range Ising model

1-2 – a quick reminder

3 – a possibly new approach

4 – a promising analogy

(1) BCS inspired effective quark model

From 1961 to nowadays

1961 {
- Vaks-Larkin, ЖЭТФ, **40**, 282 (1961)
- subm. July 1960 – publ. Jan. 1961
- Nambu-Jona-Lasinio, Phys. Rev. **122**, 345 (1961)
- subm. Oct. 1960 – publ. Apr 1961 > 8000 ref-s...

V-L: $$\mathscr{L} = -\frac{1}{2}\bar{\psi}\hat{p}\psi + \frac{1}{16}\lambda\left[\left(i\bar{\psi}\gamma_5\psi\right)^2 + (\bar{\psi}\psi)^2\right]$$

N-J-L: $$\mathscr{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + g_0\left[(\bar{\psi}\psi)^2 - \left(\bar{\psi}\gamma_5\psi\right)^2\right]$$

We shall follow commonly accepted abbreviation $\mathscr{L}_{NJL}$ keeping Vaks-Larkin in mind.

Both **V-L** and **NJL** explicitly start from BCS.

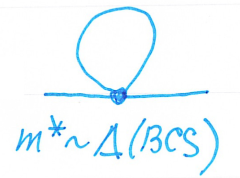
(1a) The canonical form of $N_f = 1$ NJL

$$\mathscr{L}_{\text{NJL}} = \bar{q} i\gamma\partial q + g\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5 q)^2\right]$$

$[q] = 1/\mathrm{m}^2$, point-like non renormalizable interaction

$\pm$ before γ_5 is our sign convention:

$$U_L(1) \otimes U_R(1) : \quad \begin{aligned} \frac{1}{2}(1+\gamma_5)\, q &= q_L \to e^{i\theta_L/2} q_L \\ \frac{1}{2}(1-\gamma_5)\, q &= q_R \to e^{i\theta_R/2} q_R \end{aligned}$$



$U_V(1)$ inv $\theta_L = \theta_R = \alpha$

$U_A(1)$ inv $\theta_L = -\theta_R = \beta$

No confinement. Gap equation – see Fig.

$$m^* = g \int \frac{d^4 p}{(2\pi)^4} \frac{1}{\gamma p - m^*} \qquad g > g_c \to m^* \neq 0.$$

(1b) BCS inspired Lagrangians

$$\boxed{N_f = 2}$$

$$\mathscr{L}_{\rm NJL} = \bar{q} i\gamma \cdot \partial q + g \left[(\bar{q}q)^2 + (\bar{q}_i i\gamma_5 \vec{\tau} q)^2 \right]$$

$$SU_L(2) \otimes SU_R(2)$$

$$\boxed{N_f = 3}$$

$$\mathscr{L}_{NJL} = \bar{q} i\gamma \cdot \partial q + q \sum_{a=0}^{8} \left[(\bar{q}\lambda_a q)^2 + (\bar{q} i\gamma_5 \lambda^a q)^2 \right]$$

$$\lambda_0 = \sqrt{\frac{2}{N_f}} I = \sqrt{\frac{2}{3}} I$$

$$SU(3)_{\rm color} \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A$$

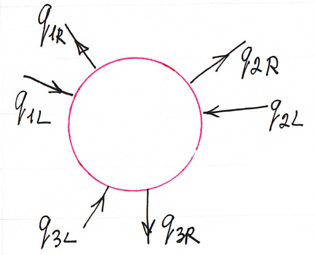
Effective Lagrangian in $N_c \gg 1$ limit of QCD

$U_A(1)$ breaking term is needed

(1c) $N_f = 3$ Anomaly term

$$\mathscr{L}_{\det} \sim \mathscr{L}_{KMT} \sim \mathscr{L}_H \to G\,[\det_{i,j}\bar{q}_i\,(1+\gamma_5)\,q_j + \text{ h.c. }] \quad \text{6-fermion}$$

$$[G] = 1/\mathrm{m}^5 \qquad \mathrm{VS} \qquad [\mathrm{g}] = 1/\mathrm{m}^2$$



Kobayashi, Maskawa 1970

't Hooft 1976

Shifman, Vainshtein, Zakharov 1980

$$N_f = 2 \to \det\bar{q}_i\,(1+\gamma_5)\,q_j =$$

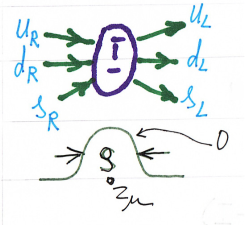
$$= 4\,[(\bar{q}_{uR}q_{uL})\,(\bar{q}_{dR}q_{dL}) - (\bar{q}_{uR}q_{dL})\,(\bar{q}_{dR}q_{uL})]$$

$$N_f = 2 \to G[\det(...) + \text{ h.c. }] = \frac{1}{2}G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2 - (\bar{q}\vec{\tau}q)^2 - (\bar{q}_i\gamma_5 q)^2\right]$$

<u>resembles NJL but not</u> $U_A(1)$

(2) Instanton Induced Effective Lagrangian $N_f = 3$

(a brief reminder of Dmitri Diakonov approach)

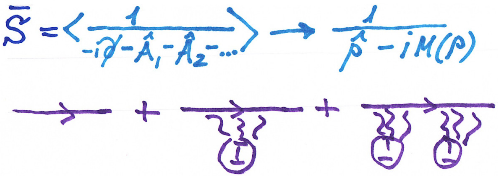


$$A_\mu^a(x) = \frac{2\rho^2 0^{ab} \bar{\eta}^b_{\mu\nu}(x-z)_\nu}{(x-z)^2\left[(x-z)^2+\rho^2\right]}$$

(O – orientation, z – centre, ρ – size)

6q vertex, $\mathscr{L}_{eff} \sim (\bar{q}\partial\varphi_0)(\bar{\varphi}_0\partial q)$, where φ_0 and $\bar{\varphi}_0$ are zero modes

$(I-\bar{I})$ ensemble $A_\mu = \sum_{I=1}^{N_+} A_\mu^I + \sum_{\bar{I}=1}^{N_-} A_\mu^{\bar{I}} + ($ B background $)$



(2a) Partition function corresponding to the Green's function in the field of instantons

$$Z = \int \mathcal{D}\psi\,\mathcal{D}\psi^+ \exp\left\{ \int d^4u \left[\psi^+ \not{\partial} \psi + g_+ \det \mathcal{J}_+(u) + g_- \det \mathcal{J}_-(u) \right] \right\}$$

$$(\mathcal{J}_\pm(u))_{fr} = \int \frac{dk\,dl}{(2\pi)^4} e^{i(k-l)u} \psi_f^+(k) k \varphi_0(k) \left(\frac{1 \pm \gamma_5}{2} \right) \varphi_0^+(l) l \psi_r(l)$$

$$\varphi_0(u) = \frac{1}{\pi} \frac{\rho}{\pi \left(u^2 + \rho^2\right)^{3/2}} \frac{u_\mu \gamma_\mu}{\sqrt{u^2}}$$

six-quark interaction $\rightarrow \int \exp(\det \mathcal{J}) \rightarrow \int d\xi \exp\left(a\xi^3 + b\xi^2 + e\xi\right)$

It is non gaussian integral. Bosonization is needed!

(3) Bosonization-Hubbard-Stratonovich Transformation

$$\int \mathcal{D}q\mathcal{D}\bar{q}e^{\bar{q}Aq} = \det A = \exp(\operatorname{Tr}\ln A), \quad \int \mathcal{D}q\mathcal{D}\bar{q}e^{(\bar{q}Aq)^n} = ?\,(n > 1)$$

Example of bosonization: $\int \mathcal{D}q\mathcal{D}\bar{q}\exp\left\{\int \bar{q}(i\not{D})q - g^2(\bar{q}q)^2\right\} =$

$$= \int \mathcal{D}q\mathcal{D}\mathcal{D}M\delta(M - g(\bar{q}q))\exp\{\ldots\} =$$

$$= \int \mathcal{D}q\mathcal{D}\mathcal{D}M\mathcal{D}L\exp\left\{\int \bar{q}(i\not{D})q - M^2 - iL(M - g(\bar{q}q))\right\} =$$

$$= \int \underbrace{\mathcal{D}M\exp\left(-M^2 - iLM\right)}_{\exp(-L^2/4)} \int \mathcal{D}q\mathcal{D}\bar{q}\exp\left\{\int \bar{q}(i\not{D})q + g\bar{q}(iL)q\right\} =$$

$$= \int \mathcal{D}L\exp\left\{-\frac{L^2}{4} + \operatorname{Tr}\ln(i\not{D} + igL)\right\}$$

M, L – auxiliary bosonic fields: σ, π-mesons, Cooper pairs

(3a) Bosonization

$$\mathrm{N_f} = 2, \quad \textbf{NJL:} \quad Z = \int \mathcal{D}\bar{q}\mathcal{D}q \exp\left\{\int \bar{q}(i\gamma\partial)q + g\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right]\right\} =$$

$$= \int \mathcal{D}\sigma\mathcal{D}\vec{\pi} \exp\left\{\int d^4x \left[\mathrm{Tr}\ln\left(i\gamma\partial - (\sigma + i\vec{\tau}\vec{\pi}\gamma_5)\right) + \frac{1}{4g}\left(\sigma^2 + \pi^2\right)\right]\right\}$$

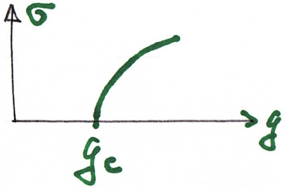
CSB: $\rightarrow \langle\sigma\rangle \neq 0, \quad \vec{\pi} = 0$

$$\frac{\delta S}{\delta\sigma} = 0 \Rightarrow \sigma + g\,\mathrm{Tr}\frac{\sigma}{\partial^2 + g^2\sigma^2} = 0$$ gap eq-n

$\sigma \sim \langle\bar{q}q\rangle$

$\sim \mathrm{BCS}$

$\mathcal{L}_{BCS} \propto g\,\psi^{+}_{\uparrow}\psi^{+}_{\downarrow}\psi_{\uparrow}\psi_{\downarrow}$



(3b) Bosonization of the 't Hooft determinant

$N_f = 2$ nearly trivial, $\underline{N_f = 3 \text{ intricate}}$

$$\boxed{N_f = 2}$$

$$\det \mathcal{J}_\pm = \frac{1}{2}\left[(\mathrm{Tr}\,\mathcal{J}_\pm)^2 - \mathrm{Tr}\,\mathcal{J}_\pm^2\right], \ \tau^a = (-i\hat{1}, \vec{\tau}), \ a = 0,1,2,3$$

$$\mathcal{J}_\pm = \sum_{a=0}^{3} \tau^a c_a^\pm, \ c_{a=1,2,3}^\pm = \frac{1}{2}\mathrm{Tr}\left(\tau^a \mathcal{J}_\pm\right), \ c_0 = -\frac{1}{2}\mathrm{Tr}\left(\tau^0 \mathcal{J}_\pm\right)$$

$$\Rightarrow \det \mathcal{J}_\pm = -\sum_{a=0}^{3}\left(c_a^\pm\right)^2. \text{ Auxiliary fields } \xi_a^\pm, \ a = 0 \sim 3$$

$$\xi_a \equiv \xi_a^+ \ \left(\text{the same for } \xi_a^-\right)$$

$$Z = N\int \mathcal{D}\psi\mathcal{D}\psi^+ \prod_{a=0}^{3}\delta\left(\xi_a - gC_a\right)\exp\left\{\int dx\left[\psi^+ ig\partial\psi + g\det\mathcal{J}\right]\right\}$$

$$= N\int \mathcal{D}\psi\mathcal{D}\psi^+ \prod \mathcal{D}\xi_a \prod \mathcal{D}L_a \exp\left\{\int dx\left[\psi^+ i\gamma\partial\psi + g\det\mathcal{J} + iL_a\left(\xi_a - gC_a\right)\right]\right\}$$

(3c) $N_f = 2$ det again

$$\int \mathcal{D}\psi\mathcal{D}\psi^+ \to Z_\psi = \int \mathcal{D}\psi\mathcal{D}\psi^+ \exp\left\{ i \int dx \left[\psi^+ \gamma\partial\psi - gL_a c_a \right] \right\}$$

$$a > 0: \quad L_a C_a = \quad \frac{1}{2}\sum_{a=1}^{3} L_a \sum_{fg} (\tau^a)_{fg}\, \psi_g^+ \,(1+\gamma_5)\, \psi_f$$

$$a = 0: \quad L_0 C_0 = \quad \frac{i}{2} L_0 \sum_f \psi_f^+ \,(1+\gamma_5)\, \psi_f$$

with $(1-\gamma_5)$ included, integrating $\mathcal{D}\xi^+\mathcal{D}\xi^-$ and collecting the pieces together

$$Z = N \int \prod_a \mathcal{D}L_a \mathcal{D}R_a \exp\left\{ \int dx \left[\mathrm{Tr}\ln\left(i\gamma\partial + g_+\left(\frac{1+\gamma_5}{2}\right)\left(L_0 + \vec{\tau}\vec{L}\right) + \right.\right.\right.$$

$$\left.\left.\left. + g_-\left(\frac{1-\gamma_5}{2}\right)\left(R_0 + \vec{\tau}\vec{R}\right) + g_+ L_a^2 + g_- R_a^2 \right]\right\}, \quad \text{compare to } \sigma, \vec{\pi} \text{ NJL}$$

Diakonov & Petrov, Simonov and other authors

(3d) $N_f = 3$ det Non Gaussian Airy's Integral

A note on literature: Reinhardt and Alkofer 1988,
Klimtetal, Osipov and coauthors (several works),
Simonov and B.K. 1995 (preprint)

Discussion on SPA (R & A, S & BK) – see Osipov and coauthors

Volkov, Ebert, ... SPA – leading term of Coleman & Weinberg?

$$N_f = 3 \det \mathcal{J}_\pm = \frac{1}{6}\left(\mathrm{Tr}\,\mathcal{J}_\pm\right)^3 - \frac{1}{2}\left(\mathrm{Tr}\,\mathcal{J}_\pm\right)\left(\mathrm{Tr}\,\mathcal{J}_\pm^2\right) + \frac{1}{3}\,\mathrm{Tr}\,\mathcal{J}_\pm^3$$

$$\mathcal{J}_\pm = \sum_{a=0}^{8} \lambda^a C_a^\pm,\ C_{a>0}^\pm = \frac{1}{2}\,\mathrm{Tr}\,\lambda^a \mathcal{J}_\pm,\ \lambda^0 = \hat{1},\ C_0^\pm = \frac{1}{3}\lambda^0 \mathcal{J}_\pm$$

$$\det \mathcal{J}_\pm = \left(C_0^\pm\right)^3 - \frac{5}{2} C_0^\pm \sum_{a=1}^{8}\left(C_a^\pm\right)^2 + \frac{1}{3}\sum_{a,b,d=1}^{8} \mathrm{Tr}\left(\lambda^a \lambda^b \lambda^d\right) C_a^\pm C_b^\pm C_d^\pm$$

$$\text{Integration } \mathcal{D}\psi^+ \mathcal{D}\psi^- \Rightarrow Z_\psi \Rightarrow \int \mathcal{D}\psi^+ \mathcal{D}\psi \prod_{a=0}^{8} \mathcal{D}\sigma_a^\pm \delta\left(\sigma_a^\pm - C_a^\pm\right) \rightarrow$$

$$\rightarrow \mathcal{D}\psi^+ \mathcal{D}\psi \prod \mathcal{D}\sigma_a^\pm \prod \mathcal{D}M_a^\pm \exp\left\{ i \int dx \left[M_a^+ \left(\sigma_a^+ - c_a^+\right) + + M_a^- \left(\sigma_a^- - c_a^-\right)\right]\right\}$$

$$L = \frac{i}{2} M^+ = L_a \lambda^a,\ R = \frac{i}{2} M^- = R_a \lambda^a \rightarrow \text{18 bosonie fields}$$

(3e) $N_f = 3$ SPA and Airy Integral

SPA – stationary phase approach

$$Z = N \int \prod_a \mathcal{D}\sigma_a^+ \mathcal{D}\sigma_a^- \mathcal{D}L_a \mathcal{D}R_a Z_\psi \times$$

$$\times \exp\left\{ \int dx \left[\frac{1}{g_\pm^2} \left(\sigma_0^{\pm 3} - \frac{5}{2}\sigma_0^\pm \sum_{a=1}^{8} \sigma_a^{\pm 2} + \frac{1}{3} \sum_{abd=1}^{8} \mathrm{Tr}\left(\lambda^a \lambda^b \lambda^d\right) \sigma_a^\pm \sigma_b^\pm \sigma_d^\pm \right) + \right.\right.$$

$$\left.\left. + \sum_a \left(iL_a \sigma_a^\pm + iR_a \sigma_a^\pm \right) \right] \right\}$$

$$Z_\sigma = \int \prod_a \mathcal{D}\sigma_a \exp\left\{ S\left(\sigma_a\right) \right\} \to \sigma_a = f\left(L_a, R_a\right)$$

Saddle point method (steepest descent) (SPA)

$$N_f = 2 \to \text{ exact Gaussian } = \mathrm{SPA},\ \langle \sigma \rangle \neq 0,\ \langle \vec{\pi} = 0 \rangle$$

$$\frac{\delta S}{\delta \sigma_0} = 3\sigma_0^2 - \frac{5}{2} \sum_{a=1}^{8} \sigma_a^2 + ig^2 L_0 = 0$$

$$\frac{\delta S}{\delta \sigma_{a>0}} = -5\sigma_0 \sigma_a + \frac{1}{3} \sum_{b,d=1} \mathrm{Tr}\left(\lambda^a \lambda^b \lambda^d\right) \sigma_b \sigma_d + igL_a = 0$$

(3f) SCSB and Ising Model Analogy

SCSB – spontaneous and chiral symmetry breaking

$$N_f = 2: \qquad \langle\sigma\rangle \neq 0, \langle\vec{\pi}\rangle = 0$$

$$N_f = 3: \quad \sigma_0 \neq 0, \sigma_a = 0, a = 1, 2, \dots, 8$$

$$\Rightarrow Z_\sigma = \int \mathcal{D}\sigma \exp\left\{\int dx \left(2L\sigma - \frac{\sigma^3}{\varkappa^2}\right)\right\} \ (g = i\varkappa) \rightarrow$$

$\rightarrow$ Airy integral $\rightarrow$ steepest descent $\rightarrow$

$$\rightarrow Z_\sigma = \exp\left\{-\frac{4}{3}\sqrt{\frac{2}{3}}\varkappa \int dx\, L^{3/2}(x)\right\}$$

Infinite Range Ising Model $H = -\frac{J}{2N}\sum_{i,j} s_i s_j - B\sum_i s_i \ (s_i = \pm 1)$

$$Z = \sum_{\text{spins}} e^{-\beta H} \rightarrow Hubbard - Stratonovich \rightarrow$$

$$\rightarrow \int d\mu \exp\left\{(\beta J)_\mu - \frac{1}{3}(\beta J)^3\mu^3\right\},$$ where μ is auxiliary field