Multiparticle amplitudes in $\lambda\phi^4$ theory

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Motivation and introduction. I

- Scalar field theory model in $d = 3 + 1$

$$S[\varphi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \right], \quad \boxed{\lambda \ll 1}$$

- Multiparticle production at *threshold*

$$\mathcal{A}_{1\to n} = \langle n, \mathbf{p} = \mathbf{0} \,|\, T\left(\hat{\mathcal{S}}\hat{\varphi}(0)\right) |0\rangle, \quad n \gg 1$$

- Perturbation theory result (n – odd), *Brown '92, Voloshin '92*

$$\mathcal{A}^{\text{loop}}_{1\to n} = \frac{\mathcal{A}_{1\to n}}{\mathcal{A}^{\text{tree}}_{1\to n}} = 1 + \underbrace{\lambda B (n-1)(n-3)}_{} + \ldots, \quad \mathcal{A}^{\text{tree}}_{1\to n} = n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

1-loop + remormalization conditions

Motivation and introduction. II

Libanov et al '94

- Resummation of leading n parts of loop corrections,

$$\mathcal{A}^{\text{loop}}_{1\to n} = 1 + \lambda(Bn^2 + ...) + \lambda^2(\frac{B^2 n^4}{2} + ...) + ... = \mathrm{e}^{B\lambda n^2} + ...$$

- Generalization: double scaling limit $n \to \infty, \lambda n =$const

$$\mathcal{A}^{\text{loop}}_{1\to n} = P_n(\lambda n)\, \mathrm{e}^{\frac{1}{\lambda}F_{-1}(\lambda n)} \equiv \exp\left(\frac{1}{\lambda}F_{-1}(\lambda n) + F_0(\lambda n) + \lambda F_1(\lambda n) + ...\right)$$

with $F_{-1} = B(\lambda n)^2 + \mathcal{O}((\lambda n)^3)$

Can we calculate $F_{-1}, F_0, ...$ in a systematic way?

This requires a resummation of perturbation theory series!

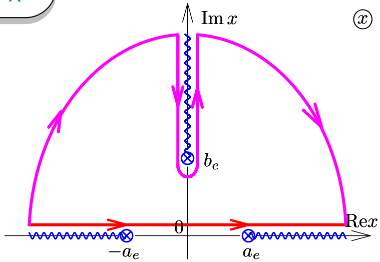
Quantum anharmonic oscillator – Landau method.

$$\hat{H} = \frac{1}{2}\hat{p}^2 + U(x)\,, \quad U(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4\,, \quad \lambda \ll 1$$

$$A_n = \langle n|\hat{x}|0\rangle\,, \quad n \gg 1$$

$$x = \frac{1}{\sqrt{\lambda}}y \quad E = \frac{1}{\lambda}e$$

$$\Psi(y) = \Psi^+(y) + \Psi^-(y)$$



Semiclassical solutions:

$$\Psi_e^{\pm}(y) = \frac{C_e}{\sqrt{p_e(y)}}\mathrm{e}^{\pm\frac{i}{\lambda}\int_{-a}^{y} dy\sqrt{p_e(y)} \mp i\frac{\pi}{4}}$$

$$A_n = \frac{2}{\lambda}\,\mathrm{Re}\int_{-\infty}^{+\infty} dy\, \Psi_e^+(y) y \Psi_0(y)$$

$$p_e(y) = \sqrt{\frac{(a_e^2-y^2)(y^2-b_e^2)}{2}} = \pm\frac{1}{\sqrt{2}}\left(y^2 + 1 - \frac{4e+1}{y^2} + \mathcal{O}(y^{-4})\right)$$

Anharmonic oscillator – result for the matrix element

Semiclassical exponent and leading correction:

$$A_n = -\sqrt{\frac{2}{\lambda}}\pi C_0 C_e \mathrm{e}^{F_L/\lambda}, \quad F_L = \int_{a_e}^{\infty} dy|p_y| - \int_{a_0}^{\infty} dy|p_0|$$

Quantization condition: $\int_{-a_e}^{a_e} dy\, p_e(y) = \pi\lambda(n+\frac{1}{2})$

$$A_n = \underbrace{\sqrt{\frac{n!}{2}}\left(\frac{\lambda}{16}\right)^{(n-1)/2}}_{A_n^{\mathrm{tree}}} \underbrace{\exp\left(\frac{1}{\lambda}F_{-1}(\lambda n) + F_0(\lambda n) + ...\right)}_{A_n^{\mathrm{loop}}}$$

$$F_{-1} = -\frac{17}{32}\lambda^2 n^2 + \frac{125}{256}\lambda^3 n^3 + \mathcal{O}(\lambda^4 n^4)$$

$$F_0 = -\frac{5}{32}\lambda n + \mathcal{O}(\lambda^2 n^2)$$

Expanding in powers of λ:

$$A_n^{\mathrm{loop}} = 1 + \frac{\lambda}{32}(-17n^2 - 5n + ...) + \frac{\lambda^2}{2048}(289n^4 + 1170n^3 + ...) + ...$$

Agree with results by *Jaekel, Schenk, 2018*

Matrix element → Path integral

Switch off interaction: $\boxed{\lambda \to \lambda(\epsilon t)\,,\ \lambda(0) = \lambda_0\,,\ \lambda(\infty) = 0\,,\ \epsilon \ll 1}$

Instantaneous basic: $\hat{H}(t)|n(t)\rangle = E_n(t)|n(t)\rangle$

Adiabatic theorem:

$$T\mathrm{e}^{-i\int_0^t dt H(t)}|n(0)\rangle \approx \mathrm{e}^{-i\int_0^t dt E_n(t)}|n(t)\rangle$$

$$A_n \equiv \langle n(0)|T\hat{x}(0)|0(0)\rangle \approx \mathrm{e}^{i\phi}\frac{\langle n^{(0)}|T\hat{\mathcal{S}}\hat{x}(0)|0^{(0)}\rangle}{\langle n^{(0)}|T\hat{S}|0^{(0)}\rangle}$$

with $\phi = \int_0^\infty dt(E_n(t) - E_0(t) - n)$

$$\hat{\mathcal{S}} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \mathrm{e}^{i\hat{H_0}t_f} T\mathrm{e}^{-i\int_{t_i}^{t_f} dt\hat{H}(t)}\mathrm{e}^{-i\hat{H}_0 t_i}$$

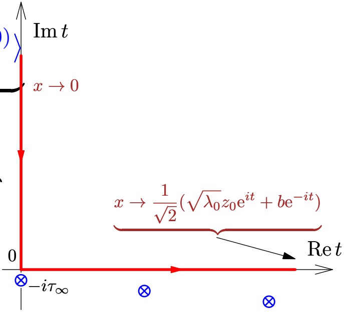
$$|z_0\rangle = \frac{1}{\sqrt{n!}} \mathrm{e}^{z_0 a^\dagger} |0^{(0)}\rangle$$

$$\langle n^{(0)}|T\hat{S}\hat{x}(0)|0^{(0)}\rangle = \underbrace{\frac{\sqrt{n!}}{2\pi i} \oint \frac{dz_0}{{z_0}^{n+1}} \langle z_0|T\hat{S}\hat{x}(0)|0^{(0)}\rangle}$$

$$z_0 = \frac{4}{\sqrt{\lambda_0}} \mathrm{e}^{-\tau_\infty} \qquad \swarrow\swarrow\swarrow \qquad \lambda = \lambda_0 \mathrm{e}^{-2\epsilon t}$$

$$\overbrace{\frac{\sqrt{n!}}{4} \left(\frac{\lambda_0}{16}\right)^{\frac{n-1}{2}} \int \frac{d\tau_\infty}{2\pi i} \mathrm{e}^{n\tau_\infty} \int \mathcal{D}x x(0)\, \mathrm{e}^{\frac{1}{\lambda_0}(i\tilde{S}+B_f)}}$$

$$\begin{cases} \tilde{S} = \int dt \left(-\frac{1}{2} x\ddot{x} - \frac{1}{2}x^2 - \frac{\mathrm{e}^{-2\epsilon t}}{4} x^4\right) \\ B_f = \sqrt{8}\mathrm{e}^{-\tau_\infty} b \end{cases}$$



Saddle-point solution:
$$\boxed{x_B(t) = \frac{i\sqrt{2}\mathrm{e}^{\epsilon t}}{\sin(t + i\tau_\infty + i\epsilon t)} + \mathcal{O}(\epsilon)}$$

LO: $$A_n^{\mathrm{loop}} = \mathcal{N} \oint_{|\tau_\infty| = r} \frac{d\tau_\infty}{2\pi i} \mathrm{e}^{n\tau_\infty} \frac{\sqrt{2}}{\sinh \tau_\infty} = \frac{1}{2}(1 - (-1)^n)$$

Introduce the source term

$$1 = \int_{-\infty}^{+\infty} dx_0 \delta(x_0 - x(0)) = \int_{-\infty}^{+\infty} dx_0 \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i \lambda_0} \mathrm{e}^{\frac{j}{\lambda_0}(x_0 - x(0))}$$

$$A_n^{\mathrm{loop}} = \int dx_0\, x_0 \int dj d\tau_\infty \mathrm{e}^{\frac{1}{\lambda_0}(j x_0 + \lambda_0 n \tau_\infty)} \frac{Z(j,\tau_\infty)}{Z(0,\tau_\infty)}$$

Generating functions $Z(j,\tau_\infty)$ and $W(j,\tau_\infty)$:

$$Z(j,\tau_\infty) = \int \mathcal{D}x \mathrm{e}^{\frac{i}{\lambda_0}(\tilde{S} + B_f - ijx(0))} = \mathrm{e}^{\frac{1}{\lambda_0}W(j,\tau_\infty)}$$

Perturbative (loop) expansion in theory with the source:

$$W(j,\tau_\infty) = \underbrace{W_0(j,\tau_\infty)}_{\text{tree part}} + \underbrace{\lambda_0 W_1(j,\tau_\infty) + \lambda_0^2 W_2(j,\tau_\infty) + \ldots}_{\text{loop corrections}}$$

Calculation of F_{-1}, F_0

$$A_n^{\text{loop}} = \int dx_0 x_0 \int dj d\tau_\infty \mathrm{e}^{\frac{1}{\lambda_0}(jx_0 + \lambda_0 n \tau_\infty + W_0(j,\tau_\infty)) + W_1(j,\tau_\infty) + \ldots}$$

Saddle equations: $\left\{ \begin{array}{l} \lambda_0 n + \frac{\partial W_0}{\partial \tau_\infty} = 0 \\ x_{cl}(0) - x_0 = 0 \end{array} \right\}$ $\int dx_0$ – residue at $x_0 = \infty$

$$\ddot{x}_{cl} + x_{cl} + \mathrm{e}^{-2\epsilon t} x_{cl}^3 = -ij\delta(t)$$

$$F_{-1} = (jx_0 + \lambda_0 n \tau_\infty + W_0)\Big|_{x_0 \to \infty}$$

$$\mathrm{e}^{F_0} = \lim_{x_0 \to \infty} \frac{\lambda_0 x_0^2}{\sqrt{2}} \sqrt{-\frac{dj}{dx_0} \frac{d\tau_\infty}{d\lambda_0 n}} \mathrm{e}^{W_1}$$

with $\tau_\infty = \frac{dF_{-1}}{d(\lambda_0 n)}\Big|_j$ and $x_0 = -\frac{dW}{dj}\Big|_{\lambda_0 n}$ or, equivalently,

$$F_{-1} = \int_0^{\lambda n} d(\lambda_0 n)\, \tau_\infty(\lambda_0 n) \quad \text{and} \quad W = -\int_0^j dj\, x_{cl}(0)$$

$\lambda\varphi^4$: Amplitude → Path integral → Saddle-point solution

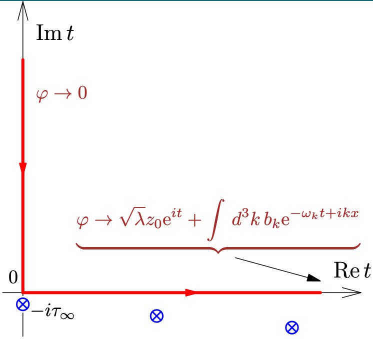
$$|z_0\rangle = \mathrm{e}^{z_0 a^\dagger_{\mathbf{p}=\mathbf{0}}}|0^{(0)}\rangle\,, \qquad m=1$$

$$\mathcal{A}_{1\to n} = \frac{n!}{2\pi i}\oint \frac{dz_0}{{z_0}^{n+1}}\langle z_0|T\hat{S}\hat{\varphi}(0)|0\rangle$$

$$z_0 = \left(\frac{8}{\lambda}\right)^{1/2}\mathrm{e}^{-\tau_\infty}$$

$$\mathcal{A}^{\text{loop}}_{1\to n} \propto \int d\tau_\infty \mathrm{e}^{n\tau_\infty}\int \mathcal{D}\varphi\,\varphi(0)\mathrm{e}^{\frac{1}{\lambda}(\tilde{S}+B_f)}$$

$$\begin{cases} \tilde{S} = \int d^4x\left(-\frac{1}{2}\varphi\Box\varphi - \frac{1}{2}\varphi^2 - \frac{\mathrm{e}^{-2\epsilon t}}{4}\varphi^4\right) \\ B_f = \sqrt{8}\mathrm{e}^{-\tau_\infty}b_{\mathbf{k}=\mathbf{0}} \end{cases}$$



Saddle-point: $$\boxed{\varphi_B(t) = \frac{i\sqrt{2}\mathrm{e}^{\epsilon t}}{\sin\left(t+i\tau_\infty+i\epsilon t\right)}}$$

At leading order: $$\mathcal{A}^{\text{loop}}_{1\to n} = \mathcal{N}\int d\tau_\infty \mathrm{e}^{n\tau_\infty}\frac{\sqrt{2}}{\sinh\tau_\infty} = \frac{1}{2}(1-(-1)^n)$$

$\lambda\varphi^4$: Calculation of F_{-1}, F_0

$$\mathcal{A}^{\rm loop}_{1\to n} = \mathcal{N}\int d\varphi_0\,\varphi_0 \int dj d\tau_\infty\, \mathrm{e}^{\frac{1}{\lambda}(j\varphi_0+\lambda n\tau_\infty)} Z(j,\tau_\infty)$$

$$Z(j,\tau_\infty) = \int \mathcal{D}\varphi \mathrm{e}^{\frac{i}{\lambda}(\tilde{S}+B_f-ij\varphi(0))} = \mathrm{e}^{\frac{1}{\lambda}W(j,\tau_\infty)}$$

$$W(j,\tau_\infty) = \underbrace{W_0(j,\tau_\infty)}_{\text{tree part}} + \underbrace{\lambda W_1(j,\tau_\infty) + \lambda^2 W_2(j,\tau_\infty) + \ldots}_{\text{loop corrections}}$$

$$F_{-1} = (j\varphi_0 + \lambda n\tau_\infty + W_0)\Big|_{\varphi_0\to\infty}$$

$$\mathrm{e}^{F_0} = \mathcal{N}\lim_{\varphi_0\,\to\,\infty} \varphi_0^2\sqrt{-\frac{dj}{d\varphi_0}\,\frac{d\tau_\infty}{d\lambda n}}\,\mathrm{e}^{W_1}$$

$F_{-1} = \int_0^{\lambda n} d(\lambda n)\,\tau_\infty(\lambda n)$ and $W = -\int_0^j dj\,\varphi_{cl}(0)$

F_1, F_2 etc. can be calculated as corrections to the saddle approximation.

W_0: Solve classical e.o.m. : $\Box\varphi_{cl} + \varphi_{cl} + \varphi_{cl}^3 = -ij\delta^{(4)}(x)$

$$\varphi_B(t) = \frac{\sqrt{2}i}{\sin(y)}, \quad \varphi_{cl}(x) = \varphi_B(t) + \delta\varphi(x)$$

$$y = t + i\tau_\infty$$

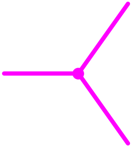
$G_B(y, y', \mathbf{x})$

$$G(y, y', \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} G(y, y'; \mathbf{p}) \mathrm{e}^{i\mathbf{p}\mathbf{x}}$$

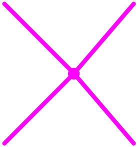
$$G(y, y'; \mathbf{p}) = \frac{1}{W_p}\left(f_1^{\omega_\mathbf{p}}(y) f_2^{\omega_\mathbf{p}}(y')\theta(y'-y) + f_2^{\omega_\mathbf{p}}(y) f_1^{\omega_\mathbf{p}}(y')\theta(y-y')\right)$$

$$f_1^{\omega_\mathbf{p}}(y) = f_2^{-\omega_\mathbf{p}}(y) = \mathrm{e}^{i\omega_\mathbf{p} y}\left(\omega_\mathbf{p}^2 + 2 + 3i\omega_\mathbf{p}\,\mathrm{ctg}\,y - \frac{3}{\mathrm{ctg}^2\,y}\right)$$

$$W_p = 2\omega_\mathbf{p}(\omega_\mathbf{p}^2 - 1)(\omega_\mathbf{p}^2 - 4)$$



$-6i\varphi_B(t)$



$-6i$

$-j\delta(x)$

Leading exponent $F_{-1}(\lambda n)$ from the classical solution

$$\tau_\infty = \tau_0 + (\lambda n)\tau_1 + \dots \qquad \varphi_{cl}(x) = \varphi_B(t) + j\delta\varphi_1(x) + \dots$$

$$W_0(j,\tau_\infty) = \underbrace{-\phi_B(0)j}_{W_{0,1}} \underbrace{-\delta\varphi_1(0)j^2/2}_{W_{0,2}} - .. \qquad \lambda n + \frac{\partial W_0}{\partial \tau_\infty} = 0$$

$\mathcal{O}(j^1)$

$$W_{0,1} = \otimes\!\!-\!\!-\!\!-\!\!- = -\frac{\sqrt{2}ij}{\sin(i\tau_\infty)}\,, \quad \lambda n = \frac{\sqrt{2}j\cos(i\tau_\infty)}{\sin^2(i\tau_\infty)}\,, \quad \tau_\infty = \tau_0 + \mathcal{O}(\lambda n)$$

$\Rightarrow j = -\frac{\lambda n}{\sqrt{2}}\tau_0^2 + \mathcal{O}(\tau_0^3)$. As $\varphi(0) \to \infty$, $\tau_0 \to 0$ and $F_{-1}(\lambda n) = 0$

$\mathcal{O}(j^2)$

$$W_{0,2} = \frac{1}{2}\otimes\!\!-\!\!-\!\!-\!\!-\!\!\otimes = \frac{j^2}{2}\Big(\frac{2B}{\tau_\infty^4} + \overbrace{\dots}^{\text{less singular terms}}\Big)\,, \qquad \tau_\infty = \tau_0 + (\lambda n)\tau_1 + \mathcal{O}(\lambda^2 n^2)$$

Saddle equation: $\frac{\partial}{\partial \tau_\infty}\left(W_{0,1} + W_{0,2}\right) = \lambda n \Rightarrow \tau_1 = 2B + \mathcal{O}(\tau_0^2)$

$$F_{-1}(\lambda n) = B\lambda^2 n^2$$

Perturbation theory for $W(j,\tau_\infty)$ – power counting

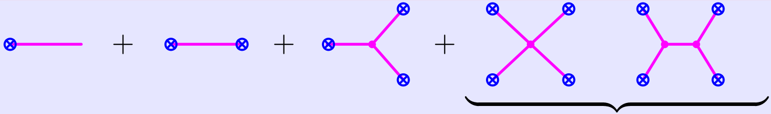
$$Z(j,\tau_\infty) = \mathrm{e}^{\frac{1}{\lambda}W(j,\tau_\infty)} = \exp\left(\frac{1}{\lambda}W_0(j,\tau_\infty) + W_1(j,\tau_\infty) + \ldots\right)$$

L – number of loops, N – number of external legs

$$W(j,\tau_\infty) = \sum_{L,N} \lambda^{L-1} j^N W_{L,N}(\tau_\infty)$$

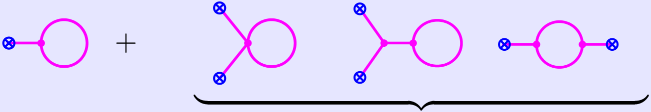
$$j = -\frac{\lambda n}{\sqrt{2}}\tau_0^2 + \mathcal{O}(\tau_0^3)$$

W_0 – determines $F_{-1}(\lambda n)$



$\mathcal{O}(\lambda^2 n^2)$ $\mathcal{O}(\lambda^3 n^3)$ $\mathcal{O}(\lambda^4 n^4)$

W_1 – determines $F_0(\lambda n)$

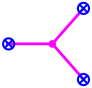


$\mathcal{O}(\lambda n)$ $\mathcal{O}(\lambda^2 n^2)$

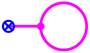
$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} + \frac{\lambda(\epsilon t)\hat{x}^4}{4}, \quad \lambda = \lambda_0 \mathrm{e}^{-2\epsilon t}$$

$$\langle n|\hat{x}|0\rangle = \mathrm{e}^{i\phi} A_n^{\mathrm{tree}} \mathrm{e}^{\frac{1}{\lambda_0}F_{-1}+F_0+\dots}, \quad \phi = \int_0^\infty dt(E_n(t) - E_0(t) - n)$$

Perturbation theory around $x_B(t) = \frac{i\sqrt{2}\mathrm{e}^{\epsilon t}}{\sin\left(t + i\tau_\infty + i\epsilon t\right)}$

$F_{-1} =$  $+$  $+$ 

$$= \lambda_0^2 n^2\left(-\frac{3i}{16\epsilon} - \frac{17}{32}\right) + \lambda_0^3 n^3\left(\frac{17i}{256\epsilon} + \frac{125}{256}\right)$$

$F_0 =$  $+\log\left(\frac{\lambda_0 x_0^2}{\sqrt{2}}\sqrt{-\frac{dj}{dx_0}\frac{d\tau_\infty}{d\lambda_0 n}}\right) = \lambda_0 n\left(-\frac{3i}{16\epsilon} - \frac{5}{32}\right)$

$$E_n(t) = n + \frac{1}{2} + \lambda\left(\frac{3n^2}{8} + \frac{3n}{8} + \dots\right) + \lambda^2\left(-\frac{17n^3}{64} + \dots\right) + \mathcal{O}(\lambda^3)$$

Agreement at tree and 1-loop order!

Semiclassical exponent – numerically.

Method of singular solutions *Son, 1995*

- Inclusive probability

$$\mathcal{P}_n(E) \equiv \sum_f \left|\langle f; E, n \middle| \hat{\mathcal{S}} \hat{\mathcal{O}} \middle| 0\rangle\right|^2 \sim \mathrm{e}^{F(\lambda n, \epsilon_k)/\lambda}, \quad \epsilon_k \equiv \frac{E}{n} - m$$

- Leading exponent does not depend on a few-particle $\mathcal{O}$
  $\mathcal{O} = \exp\left(-\int d^3\mathbf{x} J(\mathbf{x}) \hat{\varphi}(0, \mathbf{x})\right)$
- Numerically find saddle-point solution with $J \neq 0$
- Calculate $F_J(\lambda n, \epsilon_k)$ and extrapolate $J \to 0$

Numerical results.

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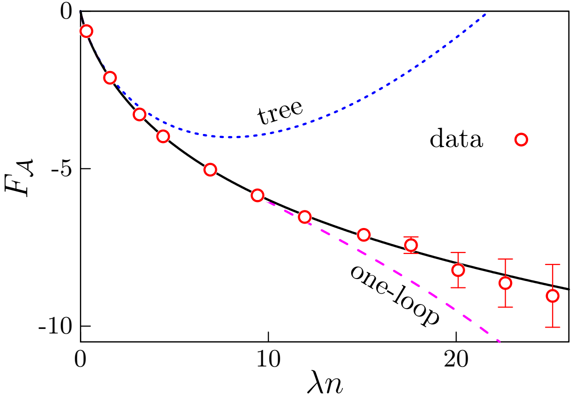
$$|\mathcal{A}_{1\to n}|^2 \sim \lim_{\epsilon_k\to 0} \frac{n!}{\mathcal{V}_n} \mathrm{e}^{F/\lambda} \sim n! \mathrm{e}^{2F_{\mathcal{A}}(\lambda n)/\lambda}$$

$$F_{\mathcal{A}} = -\frac{1}{2}\lambda n \log 8 + \frac{1}{2}(\lambda n \log(\lambda n) - \lambda n) + \mathrm{Re}\, F_{-1}(\lambda n)$$



Can be used for verification of the theoretical method!

Conclusions and future plans

- Threshold multiparticle amplitudes in $\lambda\phi^4$ theory in the double scaling limit $\lambda \to 0$, $\lambda n =$const can be obtained from the same theory with the source and perturbative expansion around a singular solution (Brown solution).
- This procedure has been verified for $\langle n|x|0\rangle$ in QM anharmonic oscillator at tree and 1-loop levels.
- We plan to calculate contribution $\mathcal{O}(\lambda^3 n^3)$ to F_{-1} and compare with numerical results.

Thank you!