NLO solutions of DGLAP evolution equations in QED

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based on works with U. Voznaya:
JPG'2023, PRD'2024, PRD'2024
(supported by RSF grant N 22-12-00021)

Efim Fradkin Centennial Conference

2nd September 2024

Electron is as inexhaustible as atom

Outline

1. Motivation
2. e^+e^- colliders
3. QED
4. Higher order logs
5. Outlook

Motivation

- Development of physical programs for future high-energy HEP colliders
- Having high-precision theoretical description of basic e^+e^- and other HEP processes is of crucial importance
- as for solving problems of the Standard Model, as for new physics searches
- Two-loop calculations are still in progress, and higher-order QED corrections are also important
- The formalism of QED parton distribution functions gives a fast estimate of the bulk of higher-order effects
- Parallels between QCD and QED

Future e^+e^- collider projects

Linear Colliders

- ILC, CLIC

E_{tot}

- ILC: 91; 250 GeV — 1 TeV
- CLIC: 500 GeV — 3 TeV

$\mathcal{L} \approx 2 \cdot 10^{34}$ cm^{-2}s^{-1}

Stat. uncertainty $\sim 10^{-3}$

Circular Colliders

- FCC-ee, CEPC
- Z-factory (Protvino)
- $\mu^+\mu^-$ collider

E_{tot}

- 91; 160; 240; 350 GeV

$\mathcal{L} \approx 2 \cdot 10^{36}$ cm^{-2}s^{-1} (4 exp.)

Stat. uncertainty $\sim 10^{-6}$

Tera-Z mode!

Super Charm-Tau Factory Projects

Budker Institute of Nuclear Physics + Sarov and/or China

Colliding electron-positron beams with c.m.s. energies from 2 to 7 GeV with unprecedented high luminosity $10^{35} cm^{-2}c^{-1}$

The electron beam will be longitudinally polarized

The main goal of experiments at the Super Charm-Tau Factory is to study the processes charmed mesons and tau leptons, using a data set that is 2 orders of magnitude more than the one collected by BESIII

Estimated experimental precision

Now:

Quantity	Theory error	Exp. error
M_W [MeV]	4	15
$\sin^2\theta^l_{eff}[10^{-5}]$	4.5	16
Γ_Z [MeV]	0.5	2.3
$R_b[10^{-5}]$	15	66

Quantity	ILC	FCC-ee	CEPC	Projected theory error
M_W [MeV]	3–4	1	3	1
$\sin^2\theta^l_{eff}[10^{-5}]$	1	0.6	2.3	1.5
Γ_Z [MeV]	0.8	0.1	0.5	0.2
$R_b[10^{-5}]$	14	6	17	5–10

The estimated error for the theoretical predictions of these quantities is given, under the assumption that $O(\alpha\alpha_s^2)$, fermionic $O(\alpha^2\alpha_s)$, fermionic $O(\alpha^3)$, and leading four-loop corrections entering through the ρ-parameter will become available.

To do list for QED

- Compute 2-loop QED radiative corrections to differential distributions of key processes: Bhabha scattering, muon decay, $e^+e^- \to \mu^+\mu^-$, $e^+e^- \to \pi^+\pi^-$, $e^+e^- \to ZH$ etc.
- Estimate higher-order contributions within some approximations
- Account for interplay with QCD and electroweak effects
- Construct a reliable Monte Carlo code(s)

Perturbative QED (I)

Fortunately, in our case the general perturbation theory can be applied:

$$\frac{\alpha}{2\pi} \approx 1.2 \cdot 10^{-3}, \quad \left(\frac{\alpha}{2\pi}\right)^2 \approx 1.4 \cdot 10^{-6}$$

Moreover, other effects: hadronic vacuum polarization, (electro)weak contributions, hadronic pair emission, etc. are small in, e.g., Bhabha scattering and can be treated one-by-one separately

Nevertheless, there are some enhancement factors:

1) First of all, the large logarithm $L \equiv \ln \frac{\Lambda^2}{m_e^2}$ where $\Lambda^2 \sim Q^2$ is the momentum transferred squared, e.g., $L(\Lambda = 1\,\mathrm{GeV}) \approx 16$ and $L(\Lambda = M_Z) \approx 24$.

2) The energy region at the Z boson peak ($s \sim M_Z^2$) requires a special treatment since factor M_Z/Γ_Z appears in the annihilation channel

Perturbative QED (II)

Methods of resummation of higher-order QED corrections

- Resummation of vacuum polarization corrections (geometric series)
- Yennie–Frautschi–Suura (YFS) soft photon exponentiation and its extensions, see, e.g., PHOTOS
- Resummation of leading logarithms via QED structure functions or QED PDFs (E.Kuraev and V.Fadin 1985;
  A. De Rujula, R. Petronzio, A. Savoy-Navarro 1979)

N.B. Resummation of real photon radiation is good for sufficiently inclusive observables…

Leading and next-to-leading logs in QED

The QED leading (LO) logarithmic corrections

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^n \frac{s}{m_e^2}$$

were relevant for LEP measurements of Bhabha, $e^+e^- \to \mu^+\mu^-$ etc. for $n \leq 3$ since $\ln(M_Z^2/m_e^2) \approx 24$

NLO contributions

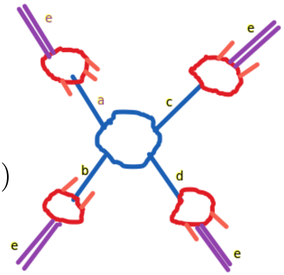
$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^{n-1} \frac{s}{m_e^2}$$

with at least $n = 3, 4$ are required for future e^+e^- colliders

In the collinear approximation we can get them within the NLO QED structure function formalism

- F.A.Berends, W.L. van Neerven, G.J.Burgers, NPB'1988
- A.A., K.Melnikov, PRD'2002; A.A. JHEP'2003

QED NLO master formula



The NLO Bhabha cross section reads

$$d\sigma = \sum_{a,b,c,d=e,\bar{e},\gamma} \int_{\bar{z}_1}^{1} dz_1 \int_{\bar{z}_2}^{1} dz_2 \mathcal{D}^{\mathrm{str}}_{ae}(z_1)\mathcal{D}^{\mathrm{str}}_{b\bar{e}}(z_2)$$

$$\times \left[d\sigma^{(0)}_{ab\to cd}(z_1,z_2) + d\bar{\sigma}^{(1)}_{ab\to cd}(z_1,z_2)\right]$$

$$\times \int_{\bar{y}_1}^{1} \frac{dy_1}{Y_1} \int_{\bar{y}_2}^{1} \frac{dy_2}{Y_2} \mathcal{D}^{\mathrm{frg}}_{ec}\left(\frac{y_1}{Y_1}\right) \mathcal{D}^{\mathrm{frg}}_{\bar{e}d}\left(\frac{y_2}{Y_2}\right)$$

$$+\mathcal{O}\left(\alpha^n L^{n-2}, \frac{m_e^2}{s}\right)$$

$\alpha^2 L^2$ and $\alpha^2 L^1$ terms are completely reproduced [A.A., E.Scherbakova, JETP Lett. 2006; PLB 2008] || $\bar{e} \equiv e^+$

High-order ISR in e^+e^- annihilation

$$\frac{d\sigma_{e^+e^-\to\gamma^*}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^-,\gamma,e^+} D_{ae^-} \otimes \tilde{\sigma}_{ab\to\gamma^*} \otimes D_{be^+}$$

$a\backslash b$	e^+	γ	e^-
e^-	$D_{e^-e^-}D_{e^+e^+}\sigma_{e^-e^+}$ LO (1)	$D_{\gamma e^-}D_{e^-e^-}\sigma_{e^-\gamma}$ NLO $(\alpha^2 L)$	$D_{e^-e^-}D_{e^-e^+}\sigma_{e^-e^-}$ NNLO $(\alpha^4 L^2)$
γ	$D_{\gamma e^-}D_{e^+\,e^+}\sigma_{e^+\gamma}$ NLO $(\alpha^2 L)$	$D_{\gamma e^-}D_{\gamma e^+}\sigma_{\gamma\gamma}$ NNLO $(\alpha^4 L^2)$	$D_{\gamma e^-}D_{e^-e^+}\sigma_{e^-\gamma}$ NLO $(\alpha^4 L^3)$
e^+	$D_{e^+e^-}D_{e^+e^+}\sigma_{e^+e^+}$ NNLO $(\alpha^4 L^2)$	$D_{e^+e^-}D_{\gamma e^+}\sigma_{e^+\gamma}$ NLO $(\alpha^4 L^3)$	$D_{e^+e^-}D_{e^-e^+}\sigma_{e^+e^-}$ LO $(\alpha^4 L^4)$

Contributions from $D_{e^-e^+}$ and $D_{e^+e^-}$ are missed in [J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald, "Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^*/Z^{0^*}$ to $O(\alpha^6 L^5)$," NPB 955 (2020) 115045]

QED NLO DGLAP evolution equations

$$\mathcal{D}_{ba}\left(x, \frac{\mu_R}{\mu_F}\right) = \delta_{ab}\delta(1-x) + \sum_{c=e,\gamma,\bar{e}} \int\limits_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int\limits_x^1 \frac{dy}{y} P_{bc}(y,t) \mathcal{D}_{ca}\left(\frac{x}{y}, \frac{\mu_R^2}{t}\right)$$

μ_F is a factorization (energy) scale

μ_R is a renormalization (energy) scale

D_{ba} is a parton density function (PDF)

P_{bc} is a splitting function or kernel of the DGLAP equation

N.B. In QED $\mu_R = m_e \approx 0$ is the natural choice

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x,\bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi}P_{ba}^{(0)}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi}\right)^2 P_{ba}^{(1)}(x) + \mathcal{O}(\alpha^3)$$

$$\text{e.g.}\quad P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x}\right]_+$$

They come from direct loop calculations, see, e.g., review "Partons in QCD" by G. Altarelli. For instance, $P_{ba}^{(1)}(x)$ comes from 2-loop calculations.

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED.

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\text{MS}}$ scheme

Running coupling constant

Compare QED-like

$$\bar{\alpha}(t) = \alpha\left\{1 + \frac{\alpha}{2\pi}\left(-\frac{10}{9} + \frac{2}{3}L\right) + \left(\frac{\alpha}{2\pi}\right)^2\left(-\frac{13}{27}L + \frac{4}{9}L^2 + \ldots\right) + \ldots\right.$$

and QCD-like

$$\bar{\alpha}(t) = \frac{4\pi}{\beta_0 \ln(t/\Lambda^2)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln[\ln(t/\Lambda^2)]}{\ln(t/\Lambda^2)} + \ldots\right]$$

Note that "$-10/9$" could have been hidden into Λ

In QED $\beta_0 = -4/3$ and $\beta_1 = -4$

$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma^{(1)}_{e\bar{e}\to\gamma^*} = \frac{\alpha}{2\pi}\left\{2LP^{(0)} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*} + 2d^{(1)}_{ee} \otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\right\} + d\,\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*} + \mathcal{O}\left(\alpha^2\right)$$

We know the massive $d\sigma^{(1)}$ and massless $d\,\bar{\sigma}^{(1)}$ ($m_e \to 0$ with $\overline{\mathrm{MS}}$ subtraction) results in $\mathcal{O}(\alpha)$. E.g.

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left[\frac{1+z^2}{1-z}\right]_+ \left(\ln\frac{s}{m_e^2} - 1\right) + \delta(1-z)(\ldots), \quad z \equiv \frac{s'}{s}$$

Scheme dependence comes from here

Factorization scale dependence is also from here

N.B. "Massification procedure”

Factorization scale choice

We apply the BLM-like prescription, i.e., hide the bulk of one-loop correction into the scale

For e^+e^- annihilation

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left[\frac{1+z^2}{1-z}\right]_+ \left(\ln\frac{s}{m_e^2} - 1\right) + \delta(1-z)(\ldots) \Rightarrow \mu_F^2 = s \quad \text{or } \mu_F^2 = \frac{s}{e}$$

Remind Drell-Yan where we usually take $\mu_F^2 = s' \equiv zs$, i.e., the enegry scale of the hard subprocess (?!)

For muon decay $\mu_F = m_\mu$ is good, but $\mu_F = m_\mu z(1-z)$ is better. It was cross-checked with the help of (partially) known two-loop results [K.Melnikov et al. JHEP'2007]

Iterative solution

Analytic expressions for NLO "e in e" and "γ in e" PDFs and fragmentation functions [A.A., U.Voznaya, JPG 2023]

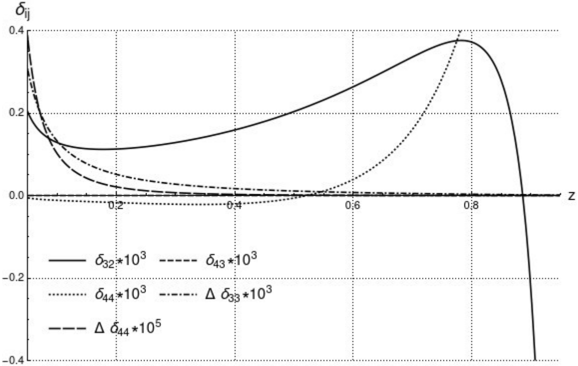
$$
\begin{aligned}
&\mathcal{D}_{ee}(x,\mu_F,m_e) = \delta(1-x) + \frac{\alpha}{2\pi} L P^{(0)}_{ee}(x) + \frac{\alpha}{2\pi} d^{(1)}_{ee}(x,m_e,m_e) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L \left(P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e}(x,m_e,m_e) + P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) - \frac{10}{9} P^{(0)}_{ee}(x) + P^{(1)}_{ee}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{6} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{6} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma e}(x) + \ldots\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(\frac{539}{27} + \frac{11}{3z} - 8\ln^3(1-z)\frac{1+z^2}{1-z} + \ldots\right) \\
&+ \mathcal{O}(\alpha^2 L^0, \alpha^3 L^1, \alpha^4 L^4)
\end{aligned}
$$

The large logarithm $L \equiv \ln \frac{\mu_F^2}{\mu_R^2}$ with factorization scale $\mu_F^2 \sim s$ or $\sim -t$; and renormalization scale $\mu_R = m_e$.

N.B. A mistake in $\mathcal{O}(\alpha^3 L^3)$ is corrected.

Higher-order effects in e^+e^- annihilation

$$d\sigma^{\rm NLO}_{e\bar{e}\to\gamma^*} = d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\left\{1+\sum_{k=1}^{\infty}\left(\frac{\alpha}{2\pi}\right)^k\sum_{l=k-1}^{k}\delta_{kl}L^l + \mathcal{O}(\alpha^k L^{k-2})\right\}$$



[A.A., U.Voznaya, arXiv:2405.03443 (PRD'2024)]

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = M_Z$)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corrections in % at the Z-peak and $z_{\min} = 0.1$

Type n	1	2	3	4	5
LO γ	-32.7365	4.8843	-0.3776	0.0034	0.0032
NLO γ	2.0017	-0.5952	0.0710	-0.0019	
LO pair	—	-0.3058	0.0875	0.0016	-0.0001
NLO pair	—	0.1585	-0.0460	0.0038	
Σ	-30.7348	4.1418	-0.2651	0.0069	0.0031

N.B. $\mathcal{O}(\alpha^2 L^0$ ISR corrections are known [Berends; Blümlein]

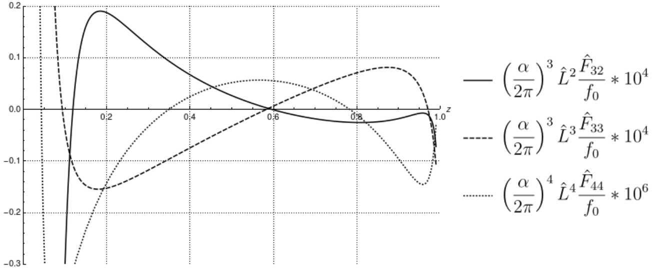
Impact of new corrections on LEP results?!

PRELIMINARY NUMBERS

Higher-order effects in muon decay spectrum

Example for unpolarized case

$$d\Gamma^{\mathrm{NLO}}_{\mu\to e\nu\bar{\nu}} = d\Gamma^{(0)}_{\mu\to e\nu\bar{\nu}} \left\{ 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha}{2\pi}\right)^k \sum_{l=k-1}^{k} \frac{\hat{F}_{kl}}{f_0} \hat{L}^l + \mathcal{O}(\alpha^k L^{k-2}) \right\}$$



[A.A., U.Voznaya, PRD'2024]

Applications

- ISR in electron-positron annihilation $e^+e^- \to \gamma^*, Z^*$
  "Higher-order NLO initial state radiative corrections to e^+e^- annihilation revisited" [A.A., U.Voznaya, arXiv:2405.03443 (to appear in PRD)]
- $\mathcal{O}(\alpha^3 L^2)$ corrections to muon decay spectrum: relevant for future experiments [A.A., U.Voznaya, PRD'2024]
- Implementation into ZFITTER, production of benchmarks, tuned comparisons with KKMC which uses YFS exponentiation for ISR
- Application to different e^+e^- annihilation channels and asymmetries within the SANC project
- $\mathcal{O}(\alpha^3 L^2)$ corrections to muon-electron scattering for MUonE experiment (in progress)

QED PDFs vs. QCD ones

Common properties:

- QED splitting functions = abelian part of QCD ones
- The same structure of DGLAP evolution equations
- The same Drell-Yan-like master formula with factorization
- Factorization scale and scheme dependence

Peculiar properties:

- QED PDFs are calculable
- QED PDFs are less inclusive
- QED renormalization scale $\mu_R = m_e$ is preferable
- QED PDFs can (do) lead to huge corrections
- Massification procedure

Outlook

- Parton picture is there also in QED
- QED PDF are similar to QCD ones, but with some differences
- QED cross-checks QCD
- Having high theoretical precision for the normalization processes $e^+e^- \to e^+e^-$, $e^+e^- \to \mu^+\mu^-$, and $e^+e^- \to 2\gamma$ is crucial for future e^+e^- colliders, especially for the Tera-Z mode
- We need complete two-loop QED results, but (sub)leading higher order corrections are also numerically important
- New Monte Carlo codes are required
- Semi-analytic codes are relevant for estimates and benchmarks

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = 350$ GeV)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corrections in % at $\sqrt{s} = 350$ GeV and $z_{\min} = 0.1$

Type n	1	2	3	4	5
LO γ	25.1041	0.6696	−0.0616	−0.0013	0.0001
NLO γ	−0.2861	0.1027	0.0318	−0.0003	
LO pair	—	1.4516	−0.0616	−0.0013	0.0001
NLO pair	—	0.1585	−0.0312	0.0003	
Σ	24.8180	2.3824	−0.1226	−0.0026	0.0002

PRELIMINARY NUMBERS