Compactification scenario in Gauss-Bonnet gravity

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We start from the Einstein-Gauss-Bonnet action in $(N+1)$-dimensional spacetime:

$$S = \frac{1}{16\pi} \int d^{N+1}x \sqrt{|\det(g)|}\Big(\mathcal{L}_E + \alpha \mathcal{L}_{GB} + \mathcal{L}_m\Big), \quad \mathcal{L}_E = R,$$

$\mathrm{L}_{GB} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2, (1)$
where $R, R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor and Riemann tensor respectively; $\mathcal{L}_m$ is the Lagrangian of a matter; $\det(g)$ is the determinant of a metric tensor g. We introduce the metric components as

$$g_{00} = -1, \quad g_{kk} = \mathrm{e}^{2H_k t}, \quad g_{ij} = 0, \ \ i \neq j. \tag{2}$$

In the following we interest in the case of $H_i \equiv \mathrm{const}$. In this work we deal with a perfect fluid with the equation of state $p = \omega\rho$ as a matter source. Let $\varkappa = 8\pi\rho$; the equations of motions now can be written as follows:

$$2\sum_{i\neq j} H_i^2 + 2\sum_{\{i>k\}\neq j} H_i H_k + 8\alpha \sum_{i\neq j} H_i^2 \sum_{\{k>l\}\neq\{i,j\}} H_k H_l + 24\alpha \sum_{\{i>k>l>m\}\neq j} H_i H_k H_l H_m = -\omega\varkappa, \tag{3}$$

$$2\sum_{i>j} H_i H_j + 24\alpha \sum_{i>j>k>l} H_i H_j H_k H_l = \varkappa. \tag{4}$$

Since we have $H_i \equiv \mathrm{const}$, it follows from Eqs. (3)–(4) that $\rho \equiv \mathrm{const}$, so that the continuity equation

$$\dot\rho + (\rho + p)\sum_i H_i = 0 \tag{5}$$

reduces to

$$(\rho + p) \sum_i H_i = 0, \tag{6}$$

which allows several different cases: a) $\rho \equiv 0$ (vacuum case), b) $\rho + p = 0$ (Λ-term case), c) $\sum_i H_i = 0$ (constant volume case) and their combinations: d) $\sum_i H_i = 0$ vacuum and e) $\sum_i H_i = 0$ with Λ-term. In present paper we do not consider constant volume solutions (CVS), leaving their description for a separate paper. So we left with only options – either vacuum ($\rho = 0$) or Λ-term ($\rho + p = 0$) and further we consider only these two cases.

Subtracting i-th dynamical equation from j-th one we obtain:

$$(H_j - H_i) \left(\frac{1}{4\alpha} + \sum_{\{k>l\} \neq \{i,j\}} H_k H_l \right) \sum_k H_k = 0 \tag{7}$$

It follows from (7) that

$$H_i = H_j \quad \vee \quad \sum_{\{k>l\} \neq \{i,j\}} H_k H_l = -\frac{1}{4\alpha} \quad \vee \quad \sum_k H_k = 0 \tag{8}$$

These are necessary conditions for a given set $H_1, \ldots, H_N$ to be a solution of Eqs. (3)–(4). The case $\sum_k H_k = 0$ is CVS and will be considered in a separate paper; in this section we deal with the following possibilities only:

$$H_i = H_j \quad \vee \quad \sum_{\{k>l\} \neq \{i,j\}} H_k H_l = -\frac{1}{4\alpha} \tag{9}$$

We call the left equality as type I condition, the right equality as type II condition. In this case we have 6 pairs of the type I and type II conditions. The table 1 lists all such pairs. We assign numbers from 1 to 6 to each type I condition and letters from $\mathcal{A}$ to $\mathcal{F}$ to each type II condition. There are the following combinations of type I and type II conditions:

1. 0 type I conditions, 6 type II conditions $\Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-4\alpha}}$;
2. 1 (any) type I conditions, 5 type II conditions $\Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-4\alpha}}$;
3. 2 type I conditions, 4 type II conditions:
   - 3.1 type I conditions has no identical parameters, for example, (see table 1) $1 - 2 - \mathcal{B} - \mathcal{D} - \mathcal{E} - \mathcal{F} \Longrightarrow H_1 = H_2 \equiv H,\ H_3 = H_4 \equiv h,\ Hh = -\frac{1}{4\alpha}$;
   - 3.2 both of type I conditions include one the same parameter, for example, $1 - 3 - \mathcal{C} - \mathcal{D} - \mathcal{E} - \mathcal{F} \Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-4\alpha}}$;
4. 3 type I conditions, 3 type II conditions:
   - 4.1 a chain of conditions does not include one of the parameters, for example, $1 - 3 - 5 - \mathcal{C} - \mathcal{D} - \mathcal{F}$ (H_4 is absent) $\Longrightarrow H_1 = H_2 = H_3 \equiv H = \frac{1}{\sqrt{-\alpha}},\ H_4 \equiv h \in \mathbb{R}$,
   - 4.2 a chain of conditions includes all the parameters $\Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-4\alpha}}$;
5. 4 (or 5) type I conditions, 2 (or 1) type II conditions $\Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-4\alpha}}$;
6. 6 type I conditions, 0 type II conditions $\Longrightarrow H_1 = H_2 = H_3 = H_4 \equiv H \in \mathbb{R}$.

Table: The necessary conditions.

	type I		type II		type I		type II
1	$H_1 = H_2$	$\mathcal{A}$	$H_3 H_4 = -\frac{1}{4\alpha}$	4	$H_1 = H_4$	$\mathcal{D}$	$H_2 H_3 = -\frac{1}{4\alpha}$
2	$H_3 = H_4$	$\mathcal{B}$	$H_1 H_2 = -\frac{1}{4\alpha}$	5	$H_2 = H_3$	$\mathcal{E}$	$H_1 H_4 = -\frac{1}{4\alpha}$
3	$H_1 = H_3$	$\mathcal{C}$	$H_2 H_4 = -\frac{1}{4\alpha}$	6	$H_2 = H_4$	$\mathcal{F}$	$H_1 H_3 = -\frac{1}{4\alpha}$

Summarizing aforesaid we see that there are three cases: i) $H_1 = H_2 = H_3 = H_4 \equiv H \in \mathbb{R}$ (clearly, $H = \frac{1}{\sqrt{-4\alpha}}$ is the subcase of that case); ii) $H_1 = H_2 = H_3 \equiv H = \frac{1}{\sqrt{-4\alpha}}$, $H_4 \equiv h \in \mathbb{R}$; iii) $H_1 = H_2 \equiv H$, $H_3 = H_4 \equiv h,\ Hh = -\frac{1}{4\alpha}$.

In the (5+1) dimensional case we have 10 pairs of the type I and type II conditions (9). Note that, as distinct from the (4+1)-dimensional case, type II conditions are the sums of three pairwise products of Hubble parameters but it does not affect follow-up reasoning: as in the (4+1)-dimensional case one can check that there are the following necessary conditions for a given set of the Hubble parameters $H_1, \dots, H_5$ to be a solution of the dynamical equations: i) $H_1 = H_2 = H_3 = H_4 = H_5 \equiv H$; ii) $H_1 = H_2 = H_3 = H_4 \equiv H$, $H_5 \equiv h$; iii) $H_1 = H_2 = -H_3 = -H_4 \equiv H$, $H_5 \equiv h$; iv) $H_1 = H_2 = H_3 \equiv H,\ H_4 = H_5 \equiv h$.

Explicit form of these solutions have been written down in

Dmitry Chirkov, Sergey Pavluchenko, Alexey Toporensky, Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology, Mod. Phys. Lett. A 29, 1450093 (2014).

The same procedure works also for higher order Lovelock gravity, see
Dmitry Chirkov, Sergey Pavluchenko, Alexey Toporensky, Non-constant volume exponential solutions in higher-dimensional Lovelock cosmologies, General Relativity and Gravitation 47, 137 (2015).
The question arises: can we consider exponential solution as a typical outcome of cosmological evolution which starts from totally anisotropic initial conditions? The answer is "Yes". See
Dmitry Chirkov, Alexey Toporensky, Splitting into two isotropic subspaces as a result of cosmological evolution in Einstein-Gauss-Bonnet gravity, Grav. Cosmol. Vol. 25, No. 3, p. 243 (2019)
From Conclusions of this paper:
We have considered a cosmological evolution of a flat 5+1 and 6+1 dimensional anisotropic Universe in Gauss-Bonnet gravity. We started from an arbitrary anisotropic initial conditions and study the outcome of corresponding cosmological evolution. Three possible outcomes have been identified. About a half of trajectories traced (we have about 10^4 trajectories for each dimensionality) ends in a non-standard singularity. A few percents of trajectories represent a periodic in Hubble parameters solution. And the rest of them (roughly a half of overall number) lead to exponential solutions.
Our results show that a situation when a space is splitted into a warped product of isotropic subspaces can be a natural result in cosmological evolution of a flat Universe in Gauss-Bonnet gravity.

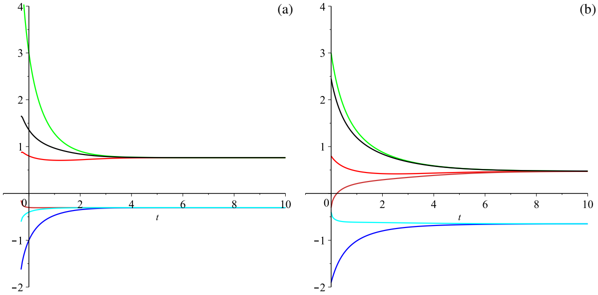


Figure: A typical behavior of Hubble parameters resulting in [3+3] (left) and [4+2] (right) splitting into isotropic subspaces

It is shown that spatial curvature in the "inner" subspace can stabilize the size of extra dimensions.

Dmitry Chirkov, Alex Giacomini, Sergey A. Pavluchenko, Alexey Toporensky, Cosmological solutions in Einstein-Gauss-Bonnet gravity with static curved extra dimensions, arXiv:2012.03517

These results allow us to construct a scenario of compactification which satisfy two important requirements:

- the evolution starts from a rather general anisotropic initial conditions,
- the evolution ends in a state with three isotropic big expanding dimensions and stabilized isotropic extra dimensions.

This scenario consists from two steps.

1. An anisotropic flat Universe evolves to a product of two isotropic spaces.
2. Contracting subspace stabilizes due to spatial curvature.

We take metric to be of the form

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right) +$$

b(t)^2 $\left[d\psi^2 + \sinh^2\psi \left(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2\right)\right]$ (10)Action under consideration reads

$$S = \int_{\mathcal{M}} d^7x \sqrt{|g|} \left\{ \frac{\dot{\phi}^2}{2} - V(\phi) + \left(m_{\rm Pl}^2 + \xi\phi^2\right) R - 2\Lambda + \alpha L_{GB} \right\}, \qquad (11)$$

where $m_{\rm Pl}$ is the 7-dimensional Planck mass, g is the determinant of metric tensor; ϕ is a spatially homogeneous scalar field with potential energy $V(\phi)$; Λ is a bare cosmological constant; α and ξ are the coupling constants; L_{GB} is quadratic Lovelock term:

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\zeta\eta}R^{\mu\nu\zeta\eta} \qquad (12)$$

where $R, R_{\mu\nu}, R_{\mu\nu\zeta\eta}$ are the $(D+4)$-dimensional scalar curvature, Ricci tensor and Riemann tensor, respectively

Equations of motion that follow from the action take the form

$$\ddot{\phi} + \frac{\dot{g}}{2g}\dot{\phi} + V' - 2\xi\phi R = 0 \qquad (13)$$

$$\left(m_{\rm Pl}^2+\xi\phi^2\right)G^{\mu}{}_{\nu}+\alpha E^{\mu}{}_{\nu}+$$
$$+g_{\nu\sigma}\left\{2\xi\left[\phi\dot\phi\left(2\frac{d}{dt}\frac{\partial R\sqrt{|g|}}{\partial\ddot g_{\sigma\mu}}-\frac{\partial R\sqrt{|g|}}{\partial\dot g_{\sigma\mu}}\right)+\left(\dot\phi^2+\phi\ddot\phi\right)\frac{\partial R\sqrt{|g|}}{\partial\ddot g_{\sigma\mu}}\right]-\right.\tag{14}$$
$$\left.-\frac{\partial}{\partial g_{\sigma\mu}}\left[\left(2\Lambda+\tfrac{1}{2}g^{\rho\eta}\nabla_\rho\phi\nabla_\eta+V\right)\sqrt{|g|}\right]\right\}=0$$

where

$$G^{\mu}{}_{\nu}=R^{\mu}{}_{\nu}-\frac{1}{2}R\delta^{\mu}{}_{\nu}\tag{15}$$

and

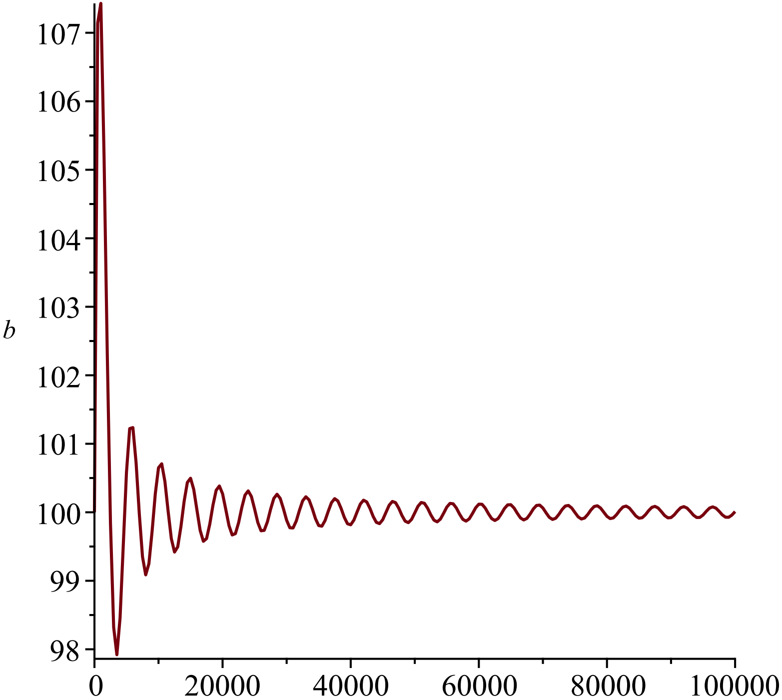
$$E^{\mu}{}_{\nu}=2\left(R^{\mu}{}_{\gamma\zeta\eta}R_{\nu}{}^{\gamma\zeta\eta}-2R^{\mu}{}_{\gamma\nu\eta}R^{\gamma\eta}-2R^{\mu}{}_{\gamma}R_{\nu}{}^{\gamma}+RR^{\mu}{}_{\nu}\right)-\frac{1}{2}L_{GB}\delta^{\mu}{}_{\nu}\tag{16}$$

.

Substituting (**??**) into (13) and (14) we get

$$\ddot\phi+\left(3H+\frac{4\dot b}{b}\right)\dot\phi+V'-2\xi\phi\left(12H^2+6\dot H+\frac{24\dot bH}{b}+\frac{8\ddot b}{b}+\frac{12\dot b^2}{b^2}-\frac{12}{b^2}\right)=0\tag{17}$$

where prime stands for derivative with respect to ϕ,



107
106
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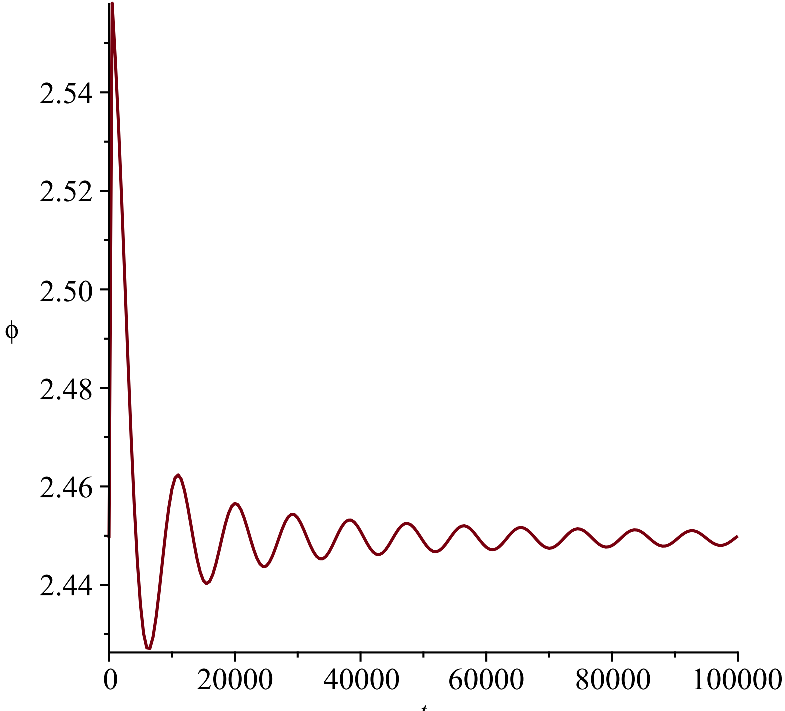

2.54
2.52
2.50
2.48
2.46
2.44
ϕ
0
20000
40000
60000
80000
100000

Table:

	$\xi > 0$	$\xi < 0$
$H_a > \frac{1}{b_a}$	$\phi_a = \sqrt{\frac{6\xi}{\lambda}\left(H_a^2 - \frac{1}{b_a^2}\right)}$	$\phi_a = 0$
$H_a < \frac{1}{b_a}$	$\phi_a = 0$	$\phi_a = \sqrt{\frac{6\xi}{\lambda}\left(H_a^2 - \frac{1}{b_a^2}\right)}$

Realistic compactification regime assume that the asymptotic value of the Hubble parameter $H(t)$ is extremely small in natural units. So that, substituting $H(t) = 0$ and $b(t) = b_a$ we get that the evolution of the scalar field is governed by an effective potential having this simple form

$$V_{\mathrm{eff_a}} = \lambda\phi^4 + \frac{12\xi}{b_a^2}\phi^2 \tag{18}$$

.

The point of minimum $\phi_{\min}$ is solution to $V'_{\mathrm{eff_a}} = 0$ equation. A non-zero solution exists for negative values of ξ only:

$$\phi_{\min} = \frac{1}{b_a}\sqrt{\frac{6|\xi|}{\lambda}} \tag{19}$$

So that, we got an effective Mexican hat potential for $\xi < 0$ starting from a simple quartic bare potential.
This means that a usual self-interacting scalar field behaves as a Higgs field!
More details see in (Dmitry Chirkov, Alex Giacomini, AT, Petr Tretyakov, Spontaneous symmetry breaking as a result of extra dimensions compactification, arXiv:2407.20409).