Nonequilibrium Schwinger-Keldysh formalism for mixed states: analytic properties and cosmological applications

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Problem statement I

- Consider general nonequilibrium field theory

$$S[\phi] = \frac{1}{2}\int dt\Big(\dot{\phi}^T A\dot{\phi} + \dot{\phi}^T B\phi + \phi^T B^T\dot{\phi} + \phi^T C\phi\Big)$$

ϕ^I — fields, $I = (\boldsymbol{x}, i)$ — multi-indices, A, B, C — time-dependent operator coefficients ($A = A_{IJ}(t), \ldots$).

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ϕ^I — fields, $I = (\boldsymbol{x}, i)$ — multi-indices, A, B, C — time-dependent operator coefficients ($A = A_{IJ}(t), \ldots$).

- Field equations read

$$F = -\frac{d}{dt}A\frac{d}{dt} - \frac{d}{dt}B + B^T\frac{d}{dt} + C,$$

and imply Klein-Gordon type inner product

$$(\phi_1, \phi_2) = i\phi_1^\dagger(W\phi_2) - i(W\phi_1)^\dagger\phi_2, \qquad W = A\frac{d}{dt} + B$$

Problem statement II

- **Goal** is to calculate in-in Green's correlation function generating functional
$$Z[J_1, J_2] = \mathrm{tr}\left[\hat{U}_{J_1}(T,0)\,\hat{\rho}\,\hat{U}^{\dagger}_{-J_2}(T,0)\right].$$
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where the Hamiltonian was modified by source term $-J^T(t)\phi(t)$.
- Density matrix is general Gaussian density matrix, defined in coordinate space
$$\langle\varphi_+|\,\hat{\rho}\,|\varphi_-\rangle = \mathrm{const}\times\exp\left\{-\frac{1}{2}\boldsymbol{\varphi}^T\boldsymbol{\Omega}\,\boldsymbol{\varphi}+\boldsymbol{j}^T\boldsymbol{\varphi}\right\},$$
where
$$\boldsymbol{\varphi}=\begin{bmatrix}\varphi_+\\ \varphi_-\end{bmatrix},\quad \boldsymbol{j}=\begin{bmatrix}j_+\\ j_-\end{bmatrix},\quad \boldsymbol{\Omega}=\begin{bmatrix}R & S\\ S^* & R^*\end{bmatrix}$$

Formal result I

- Result of functional Gaussian integration reads

$$Z[\boldsymbol{J}] = \text{const} \times \exp\left\{ -\frac{i}{2} \int_0^T dt\, dt'\, \boldsymbol{J}^T(t) \boldsymbol{G}(t,t') \boldsymbol{J}(t) \right.$$
$$\left. - \int_0^T dt\, \boldsymbol{J}^T(t) \boldsymbol{G}(t,0)\, \boldsymbol{j} + \frac{i}{2} \boldsymbol{j}^T \boldsymbol{G}(0,0)\, \boldsymbol{j} \right\},$$

where the source reads

$$\boldsymbol{J} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

Formal result II

- Green's function has block-matrix form

$$\boldsymbol{G}(t,t') = \begin{bmatrix} G_{\mathrm{T}}(t,t') & G_{<}(t,t') \\ G_{>}(t,t') & G_{\bar{\mathrm{T}}}(t,t') \end{bmatrix},$$

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- and satisfy inhomogeneous equation

$$\boldsymbol{F}\boldsymbol{G}(t,t') = \boldsymbol{I}\,\delta(t-t'),$$

supplemented by boundary condition

$$(i\boldsymbol{W} + \boldsymbol{\Omega})\boldsymbol{G}(t,t')\big|_{t=0} = 0, \qquad \begin{aligned} \begin{bmatrix} I & I \end{bmatrix} \boldsymbol{W}\boldsymbol{G}(t,t')\big|_{t=T} &= 0, \\ \begin{bmatrix} I & -I \end{bmatrix} \boldsymbol{G}(t,t')\big|_{t=T} &= 0. \end{aligned}$$

where

$$\boldsymbol{F} = \begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}, \qquad \boldsymbol{W} = \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix}$$

Basis functions I

- Explicit form of Green's functions can be expressed through basis functions

$$\boldsymbol{F}\boldsymbol{v}_{\pm}(t) = 0,$$

$$(i\boldsymbol{W} + \boldsymbol{\Omega})\,\boldsymbol{v}_{-}(t)\big|_{t=0} = 0, \qquad \begin{array}{l} \begin{bmatrix} I & I \end{bmatrix} \boldsymbol{W}\boldsymbol{v}_{+}(t)\big|_{t=T} = 0, \\ \begin{bmatrix} I & -I \end{bmatrix} \boldsymbol{v}_{+}(t)\big|_{t=T} = 0. \end{array}$$

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- We will construct it using the simpler ones

$$\boldsymbol{F}\boldsymbol{v}(t) = 0, \qquad (i\boldsymbol{W} - \boldsymbol{\omega})\boldsymbol{v}(t)\big|_{t=0} = 0, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega & 0 \\ 0 & \omega^* \end{bmatrix}.$$

having block-diagonal form

$$\boldsymbol{v} = \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}, \qquad \boldsymbol{v}^* = \begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix}.$$

Basis functions II

- Desired basis functions are obtained through Bogolyubov transformation

$$\boldsymbol{v}_+ = \boldsymbol{v} + \boldsymbol{v}^* \boldsymbol{X} = \begin{bmatrix} v & v^* \\ v & v^* \end{bmatrix}, \qquad \boldsymbol{X} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$\boldsymbol{v}_- = \boldsymbol{v}^* \boldsymbol{U}^T - \boldsymbol{v} \boldsymbol{V}^T, \qquad \begin{aligned} \boldsymbol{U} &= \frac{1}{\sqrt{2\boldsymbol{\Omega}_{\mathrm{re}}}} (\boldsymbol{\Omega} + \boldsymbol{\omega}) \frac{1}{\sqrt{2\boldsymbol{\omega}_{\mathrm{re}}}}, \\ \boldsymbol{V} &= \frac{1}{\sqrt{2\boldsymbol{\Omega}_{\mathrm{re}}}} (\boldsymbol{\Omega} - \boldsymbol{\omega}^*) \frac{1}{\sqrt{2\boldsymbol{\omega}_{\mathrm{re}}}}. \end{aligned}$$

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- Answer for $\boldsymbol{G}(t, t')$ has following final form

$$\begin{aligned} i\boldsymbol{G}(t, t') &= i\boldsymbol{G}_0(t, t') + \boldsymbol{v}_+(t)\, \boldsymbol{\nu}\, \boldsymbol{v}_+^T(t'), \\ i\boldsymbol{G}_0(t, t') &= \boldsymbol{v}_+(t)\, \boldsymbol{v}^\dagger(t')\, \theta(t - t') + \boldsymbol{v}^*(t)\, \boldsymbol{v}_+^T(t')\, \theta(t' - t) \\ \boldsymbol{\nu} &= \left[\boldsymbol{I} + \boldsymbol{X} - \sqrt{2\boldsymbol{\omega}_{\mathrm{re}}}\, \boldsymbol{X}\, (\boldsymbol{\omega} + \boldsymbol{\Omega})^{-1} \boldsymbol{X} \sqrt{2\boldsymbol{\omega}_{\mathrm{re}}} \right]^{-1} - \boldsymbol{X}. \end{aligned}$$

which has no “good” block structure.

Basis functions III

- **How to choose ω?**

Particle interpretation

- Decompose Heisenberg field operator

$$\hat{\phi}(t) = v(t)\hat{a} + v^*(t)\hat{a}^\dagger$$

define non-anomalous and anomalous averages

$$\nu = \mathrm{tr}\big[\hat{\rho}\,\hat{a}^\dagger\hat{a}\big], \qquad \kappa = \mathrm{tr}\big[\hat{\rho}\,\hat{a}\,\hat{a}\big],$$

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$$\omega = R^{1/2}\sqrt{I - \sigma^2}R^{1/2}, \quad \sigma \equiv R^{-1/2}SR^{-1/2}.$$

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- Block-matrix components $\boldsymbol{G}(t,t')$ have simple form, in particular

$$iG_>(t,t') = v(t)\,\big(\nu + I\big)\,v^\dagger(t') + v^*(t)\,\nu\,v^T(t').$$

Euclidean density matrix I

- Consider the particular type of the density matrix, defined by the Euclidean action

$$\rho_E(\varphi_+, \varphi_-; J_E] = \frac{1}{Z} \int\limits_{\phi(\tau_\pm)=\varphi_\pm} D\phi \, \exp\left\{ -S_E[\phi] - \int_0^\beta d\tau \, J_E(\tau)\phi(\tau) \right\},$$

where S_E is Euclidean action

$$iS[\phi(t)]\big|_{t=-i\tau} = -S_E[\phi_E(\tau)].$$

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$$iS[\phi(t)]\big|_{t=-i\tau} = -S_E[\phi_E(\tau)].$$

- Operator coefficients of the Euclidean action are defined by the initial one as

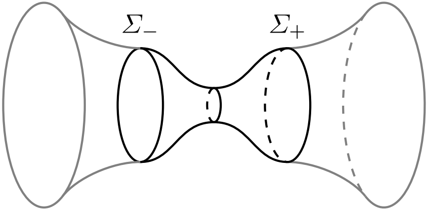
$$A_E(\tau) = A(-i\tau), \qquad B_E(\tau) = -iB(-i\tau), \qquad C_E(\tau) = -C(-i\tau),$$

and hermiticity implies

$$A_E(\beta-\tau) = A_E^*(\tau), \quad B_E(\beta-\tau) = -B_E^*(\tau), \quad C_E(\beta-\tau) = C_E^*(\tau)$$

Euclidean density matrix II

- This arise in the context of spatially closed cosmology



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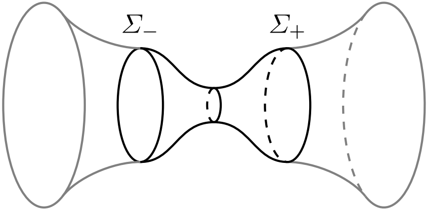

- Density matrix is Gaussian, which parameters read

$$\boldsymbol{\Omega} = \begin{bmatrix} -\overrightarrow{W}_E G_D(\beta,\beta)\overleftarrow{W}_E & \overrightarrow{W}_E G_D(\beta,0)\overleftarrow{W}_E \\ \overrightarrow{W}_E G_D(0,\beta)\overleftarrow{W}_E & -\overrightarrow{W}_E G_D(0,0)\overleftarrow{W}_E \end{bmatrix}$$

Euclidean density matrix III

- Special choice of ω implies following properties of basis functions

$$v(t - i\beta) = v(t)\frac{\nu + I}{\nu}, \qquad v^*(t - i\beta) = v^*(t)\frac{\nu}{\nu + I}.$$

Euclidean density matrix III

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$$v(t - i\beta) = v(t)\frac{\nu + I}{\nu}, \qquad v^*(t - i\beta) = v^*(t)\frac{\nu}{\nu + I}.$$

- that lead do famous Kubo-Martin-Schwinger condition

$$G_>(t - i\beta, t') = G_<(t, t')$$

despite non-stationary nature of system!

Итоги

- Generating functional of in-in Green's functions for general non-equilibrium system and initial state was calculated
- Special choice of basis functions, allowing particle interpretation was made
- For Euclidean initial state analytic structure of basis functions was examined and KMS condition was derived

Thank you for your attention!