Was there a Big Bang?

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D. Afanasev, Great Bay University, China

M. Katanaev, Steklov Mathematical Institute, Moscow

Main result: new exact solution of GR+scalar field

$(\mathbb{M}, g)$ - space-time = four dimensional manifold with Lorentzian signature metric.

$x = (x^\alpha), \quad \alpha = 0,1,2,3$ - local coordinates

The Liouville metric (1849):

$$g_{\alpha\beta} := \Phi(x)\eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} := \mathrm{diag}(+---)$$

$$\Phi := \phi_0(x^0) + \phi_1(x^1) + \phi_2(x^2) + \phi_3(x^3)$$ - the conformal factor

There is no Killing vector field for nontrivial functions $f_\alpha(x^\alpha)$

The advantage: the geodesic Hamilton-Jacobi equation admits complete separation of variables $\Rightarrow$ Geodesic equations are integrable for arbitrary $f_\alpha(x^\alpha)$.

The Stackel problem (1883): which metric admits complete separation of variables in the geodesic Hamilton-Jacobi equation?

It was solved for metrics of arbitrary signature in any number of dimensions:

Katanaev arXiv2305.02222, PhysicaScripta (2023), TMΦ (2024)

Two cases: $\Phi > 0$ signature $(+---)$
$\Phi < 0$ signature $(-+++)$

The Weyl tensor vanishes, and the curvature is of type 0 in Petrov's classification

The curvature and Ricci tensors and scalar curvature

$$R_{\alpha\beta\gamma}{}^{\delta} = \frac{1}{2\Phi}\left(\Phi''_{\alpha\gamma}\delta^{\delta}_{\beta} - \Phi''^{\delta}_{\alpha}\eta_{\beta\gamma}\right) + \frac{3}{4\Phi^2}\left(-\phi'_{\alpha}\phi'_{\gamma}\delta^{\delta}_{\beta} + \phi'_{\alpha}\phi'^{\delta}\eta_{\beta\gamma}\right) +$$
$$+\frac{1}{4\Phi^2}\phi'_{\varepsilon}\phi'^{\varepsilon}\eta_{\alpha\gamma}\delta^{\delta}_{\beta} - (\alpha \leftrightarrow \beta),$$

$$R_{\alpha\gamma} = \frac{1}{2\Phi}\left(2\Phi''_{\alpha\gamma} + \Phi''^{\varepsilon}_{\varepsilon}\eta_{\alpha\gamma}\right) - \frac{3}{2\Phi^2}\phi'_{\alpha}\phi'_{\gamma},$$

$$R = \frac{3}{\Phi^2}\Phi''^{\alpha}_{\alpha} - \frac{3}{2\Phi^3}\phi'_{\alpha}\phi'^{\alpha}.$$

Raising and lowering of indices is performed by Lorentz metric $\eta_{\alpha\beta}$

No symmetry assumptions on 4D metric

The action

$$S := \int dx \sqrt{|g|}\left(R - 2\Lambda + \frac{1}{2} g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\right), \qquad g := \det g_{\alpha\beta}$$

$\varphi(x)$ - scalar field

Λ - cosmological constant

$V(\varphi)$ - potential for a scalar field is bounded from below

There are no ghosts and tachyons for $\Phi > 0$. If $\Phi < 0$, then we have to change signs in front of the scalar curvature and the kinetic term of the scalar field

First, consider the case $\Phi > 0$. Equations of motion:

$$R_{\alpha\beta} = -\frac{1}{2}\partial_\alpha\varphi\partial_\beta\varphi + \frac{1}{2}\Phi\,\eta_{\alpha\beta}\left(V + 2\Lambda\right) \qquad (1)$$

$$\eta^{\alpha\beta}\partial^2_{\alpha\beta}\varphi + \frac{1}{\Phi}\Phi'^{\alpha}\partial_\alpha\varphi + \Phi V' = 0 \qquad (2)$$

The additional assumption $\varphi(x) := \Psi\left(\Phi(x)\right)$

Einstein's equations: $\Psi'^2 = \dfrac{3}{\Phi^2}$ $\Rightarrow$ $\boxed{\varphi = \pm\sqrt{3}\,\ln(C\Phi)}$

$C > 0$ is an integration constant

The full system of equations:

$$\Phi''_{\alpha\beta} + \frac{1}{2}\Phi''^{\gamma}_{\gamma}\eta_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta}\Phi^2\left(V + 2\Lambda\right)$$

$$\Phi''^{\gamma}_{\gamma} \pm \frac{\Phi^2}{\sqrt{3}}V' = 0$$

$\Rightarrow$ $\boxed{V = -2\Lambda + \mathrm{e}^{\pm\frac{2}{\sqrt{3}}(\varphi+\varphi_0)}}$

φ_0 is an integration constant

The only equation which is to be solved: $\Phi''_{\alpha\beta} = \dfrac{1}{6}\eta_{\alpha\beta}\Phi^2\left(V + 2\Lambda\right)$ $\Rightarrow$

$\boxed{\Phi = s + c,}$ where $s := \eta_{\alpha\beta}x^\alpha x^\beta$ and $c \in \mathbb{R}$ is an integration const

$\boxed{V = -2\Lambda + 12C^2\mathrm{e}^{\mp\frac{2}{\sqrt{3}}\varphi}}$ - bounded from below

Similar solution is obtained for $\Phi < 0$

The solution

$$g_{\alpha\beta} = (s+c)\eta_{\alpha\beta}, \qquad s := \eta_{\gamma\delta}x^\gamma x^\delta$$

6 noncommuting Killing vector fields $\Longrightarrow$ Spontaneous symmetry emergence

Afanasev, Katanaev. Phys.Rev.D (2019), ibid. (2020)

Curvature tensor $R_{\alpha\beta\gamma}{}^{\delta} = \frac{1}{\Phi}\left(3 - \frac{c}{\Phi}\right)\eta_{\alpha\gamma}\delta^\delta_\beta - \frac{3}{\Phi^2}(x_\alpha x_\gamma \delta^\delta_\beta - x_\alpha x^\delta \eta_{\beta\gamma}) - (\alpha \leftrightarrow \beta)$

Curvature invariants: $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{12}{\Phi^4}\left(9 - \frac{6c}{\Phi} + \frac{5c^2}{\Phi^2}\right)$

$$R^{\alpha\beta}R_{\alpha\beta} = \frac{36}{\Phi^4}\left(3 + \frac{c^2}{\Phi^2}\right)$$

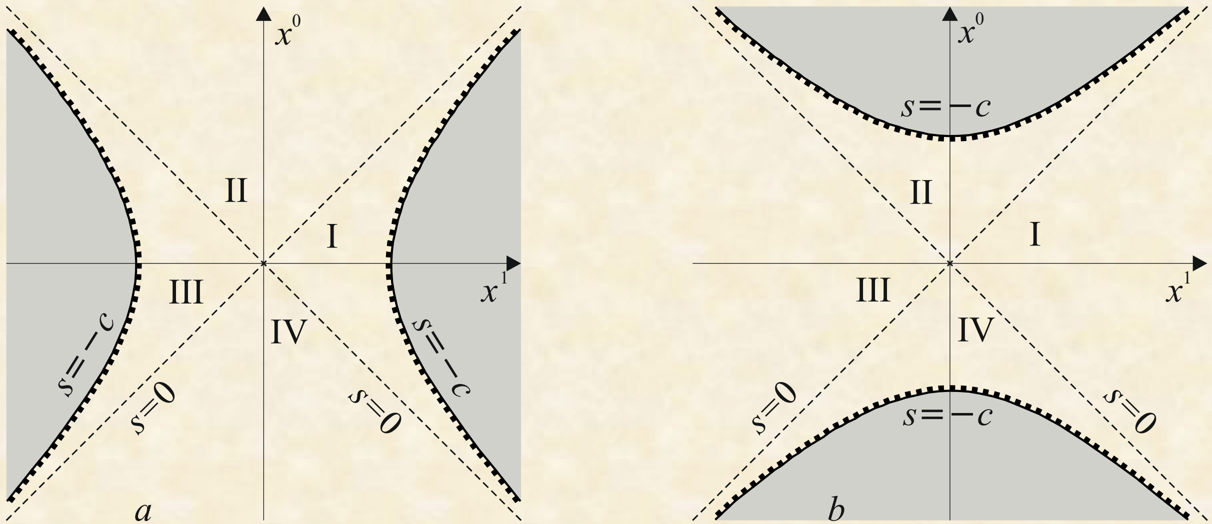
$$R = \frac{6}{\Phi^2}\left(3 + \frac{c}{\Phi}\right)$$

The only singularity is located at $\Phi = 0$

Global solutions

Naked singularity $\Phi > 0, \quad c > 0$

Formation of a black hole $\Phi < 0, \quad c < 0$



The forbidden regions are shaded

The pictures must be rotated in two extra space dimensions around x^0 axis

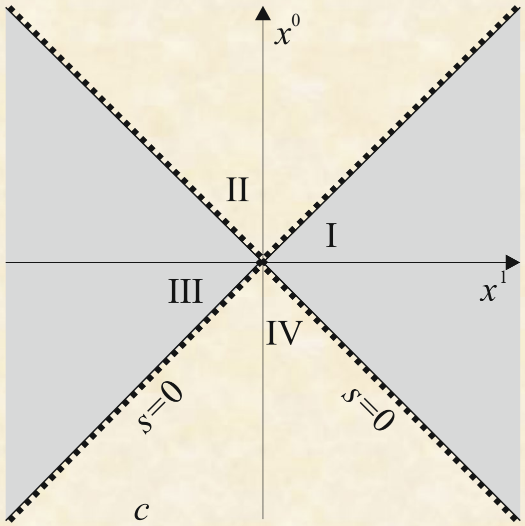
Naked singularity: test particles can live forever going through the throat or fall into the singularity at a finite proper time

Formation of the black hole: at a finite proper time the horizon appears and afterwards the singularity is formed

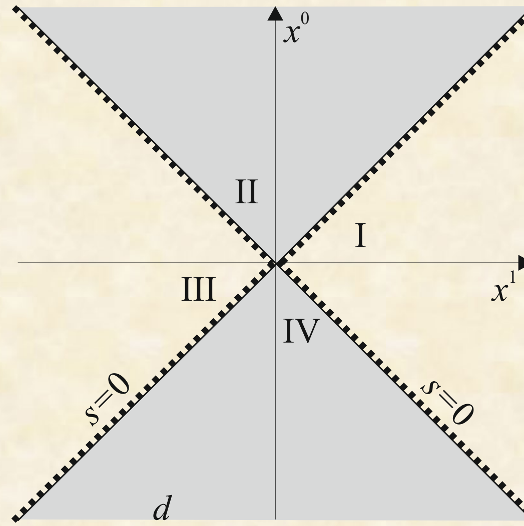
Global solutions

Two cones:

$\Phi > 0, \quad c = 0$

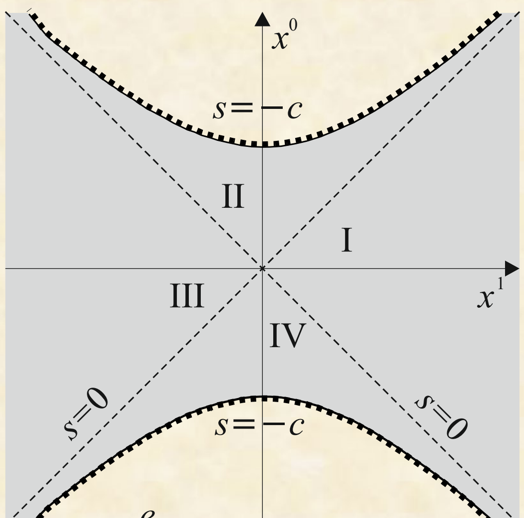


$\Phi < 0, \quad c = 0$

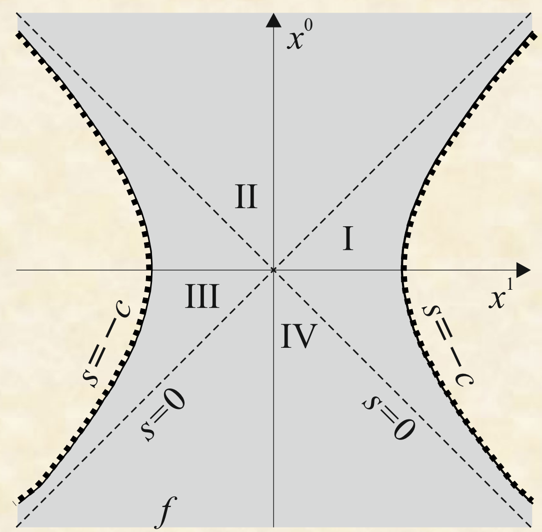


Two hyperboloids:

$\Phi > 0, \quad c < 0$



$\Phi < 0, \quad c > 0$



Friedmann-like form of the Liouville metric

Pseudospherical coordinates in quadrant II:

$$x^0 := t\cosh\chi$$
$$x^1 := t\sinh\chi\sin\vartheta\cos\psi$$
$$x^2 := t\sinh\chi\sin\vartheta\sin\psi$$
$$x^3 := t\sinh\chi\cos\vartheta$$

$\Longrightarrow \quad s = t^2 > 0$

$$ds^2 = (t^2 + c)\left(dt^2 - d\Omega\right)$$

$$d\Omega := d\chi^2 + \sinh^2\chi\,(d\vartheta^2 + \sin^2\vartheta\, d\psi^2)$$ - constant negative curvature metric

Next coordinate transformation $t \to \tau(t)$: $\dfrac{d\tau}{dt} = \sqrt{t^2 + c}$

$$\boxed{ds^2 = d\tau^2 - a(\tau)\, d\Omega}$$

$$a(\tau) := t\sqrt{t^2 + c}$$

$$\frac{d\,a}{d\tau} = \frac{2t^2 + c}{t^2 + c}$$

$$\frac{d^2 a}{d\tau^2} = \frac{2ct}{\left(t^2 + c\right)^{5/2}} > 0$$ - accelerated expansion

The expansion starts at the “horizon” $t = 0$, and there is no singularity in global coordinates. There are no Friedmann-like coordinates in quadrants I and III because lines $s = \mathrm{const}$ are timelike.

Geodesics $x^\alpha(\tau)$ for the Liouville space-time $(+---)$

Lagrangian and Hamiltonian for geodesics:

$$L := \frac{1}{2}(s+c)\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta,$$

$$H = \frac{1}{2(s+c)}\eta^{\alpha\beta} p_\alpha p_\beta, \quad p_\alpha = (s+c)\eta_{\alpha\beta}\dot{x}^\beta$$

Geodesic equations:

$$\ddot{x}^\alpha = -\Gamma_{\beta\gamma}{}^\alpha \dot{x}^\beta \dot{x}^\gamma = \frac{1}{s+c}(x^\alpha \dot{x}_\beta \dot{x}^\beta - 2x_\beta \dot{x}^\beta \dot{x}^\alpha)$$

The Hamilton-Jacobi equation for the action function $W(x)$:

$$\frac{1}{s+c}\eta^{\alpha\beta}\partial_\alpha W \partial_\beta W = 2m^2$$

m - mass of a test particle

Complete separation of variables: $W(x,d) = \sum_{\alpha=0}^{3} W_\alpha(x^\alpha, d), \qquad \det \frac{\partial^2 W}{\partial x^\alpha \partial d_\beta} \neq 0$

$$W_\alpha'^2 = 2m^2 x_\alpha^2 + d_\alpha \qquad d := (d_0, d_1, d_2, d_3)$$

$$d_0 - d_1 - d_2 - d_3 = 2m^2 c$$

Four independent parameters: d_1, d_2, d_3, m

We put $2m^2 = 1$ for timelike geodescs by stretching the canonical parameter τ

Lightlike (null) geodesics $\dot{x}^\alpha \dot{x}_\alpha = 0 \Leftrightarrow p_\alpha p^\alpha = 0$

The Hamilton-Jacobi equation $\eta^{\alpha\beta}\partial_\alpha W \partial_\beta W = \sum_{\alpha=0}^{3} \eta^{\alpha\alpha} W_\alpha'^2 = 0$

The complete integral $W(x,d) = \sum_{\alpha=0}^{3} W_\alpha(x^\alpha, d)$

where $W_\alpha'^2 = d_\alpha > 0, \quad d_0 - d_1 - d_2 - d_3 = 0$

$$d_\alpha := k_\alpha^2$$

Separating functions: $W_\alpha = k_\alpha x^\alpha + \tilde{b}_\alpha, \quad \tilde{b}_\alpha = \text{const}$ (no summation)

Four involutive conservation laws: $p_\alpha = k_\alpha$

$$(s+c)\dot{x}_\alpha = k_\alpha \quad \Rightarrow \quad \frac{dx^i}{dx^0} = k^i, \quad i = 1,2,3$$

$$\boxed{x^i = k^i x^0 + b^i, \quad b^i = \text{const}}$$

Lightlike geodesics are straight lines going through each point of the spacetime in the null direction like on the Minkowskian spacetime $\mathbb{R}^{1,3}$.

The canonical parameter $\tau - \tau_0 = k\left(x^0\right)^2 + bx^0, \quad k, b = \text{const}$

Timelike geodesics for $\Phi > 0$

The Hamilton-Jacobi equation $$g^{\alpha\beta}\partial_\alpha W \partial_\beta W = \sum_{\alpha=0}^{3} g^{\alpha\alpha} W_\alpha'^2 = 1$$

The complete integral $$W(x,d) = \sum_{\alpha=0}^{3} W_\alpha(x^\alpha, d)$$

where $$W_\alpha'^2 = \left(x^\alpha\right)^2 + d_\alpha > 0, \qquad d_0 - d_1 - d_2 - d_3 = c$$

Four involutive quadratic conservation laws: $\left(p_\alpha\right)^2 = \left(x^\alpha\right)^2 + d_\alpha > 0$

The metric belongs to class $[0,4,0]_2$ according to the classification of separable metrics in Katanaev arXiv2305.02222.

Four first integrals of the geodesic equations: $(s+c)^2\left(\dot{x}^\alpha\right)^2 = \left(x^\alpha\right)^2 + d_\alpha$

Timelike geodesics for $\Phi > 0$

The form of geodesics of general type $$\left(\frac{dx^i}{dx^0}\right)^2 = \frac{\left(x^i\right)^2 + d_i}{\left(x^0\right)^2 + d_0}, \quad i = 1,2,3$$

$$x^i = \left(\frac{C^i}{2} + \frac{d_i}{2C^i d_0}\right) x^0 + \left(\frac{C^i}{2} - \frac{d_i}{2C^i d_0}\right)\sqrt{\left(x^0\right)^2 + d_0} \qquad C^i > 0$$

Three degenerate cases:

1) $d_0 = 0,\ d_i \neq 0$

$$x^i = \frac{\left(C^i x^0\right)^2 - d_i}{2C^i x^0}$$

3) $d_0 = d_i = 0$

$$x^0 = C^i x^i, \qquad x^0 = \frac{C^i}{x^i}$$

2) $d_0 \neq 0,\ d_i = 0$

$$x^0 = \frac{\left(C^0 x^i\right)^2 - d_0}{2C^0 x^i} \qquad C^0 \neq 0$$

Conclusion

1) New global solution in General Relativity with a scalar field is found. The scalar field has exponential potential bounded from below, and there are no ghosts and tachyons.

2) The Liouville metric without a Killing vector is used as the initial ansatz. However the solution of Einstein's equations is invariant with respect to global Lorentz transformations and therefore has 6 noncommuting Killing vector fields (spontaneous symmetry emergence).

3) There are six global solutions depending on the integration constant. Solutions with the naked singularity and the black hole are of great interest.

4) Metric for the naked singularity can be transformed into the Friedmann form only inside the light cone, where it describes accelerated expansion of the Universe. The corresponding global solution is infinitely smooth and geodesically complete except the naked singularity.

5) The solution with the black hole cannot be brought into the static form outside horizon and describe the black hole formation by the scalar field.