Peculiarities of Schwinger – DeWitt technique: one-loop double poles, surface terms, and determinant anomalies

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Heat kernel method in background field formalism

Heat kernel

- Functional method in QFT
- Allows for both UV and IR approximations on an arbitrary background
- Used in QFT in curved spacetimes, quantum gravity, cosmology, etc. [Vassilevich (2003), Avramidi (2000)]

$$K_F(s\,|\,x,y) \equiv e^{sF}\,\delta(x,y), \qquad F^{-1} = -\int_0^\infty ds\, K_F(s), \qquad \log \mathrm{Det}\, F = -\int_0^\infty \frac{ds}{s}\, \mathrm{Tr}\; K_F(s) \tag{1}$$

Asymptotic UV expansion

Schwinger – DeWitt technique for hermitian operator F, $\mathrm{ord}\, F = 2k$, $\dim \mathcal{M} = d$ [DeWitt (1965), Schwinger (1961), Gilkey (1984), McKean, Singer (1967),...]

$$\mathrm{Tr}\, e^{sF} \equiv \int\limits_{\mathcal{M}} d^d x\, K_F(s\,|\,x,y)\,\Big|_{y=x} = \frac{1}{s^{d/2k}} \sum_{n=0}^{\infty} A_n\, s^{n/2k}, \quad s \to 0. \tag{2}$$

A_n contains both volume and surface terms [McKean, Singer (1967)]

$$A_n = \int\limits_{\mathcal{M}} d^d x\, E_n(x) + \int\limits_{\partial\mathcal{M}} d^{d-1} x\, B_n(x), \tag{3}$$

with non-zero $E_{2m}(x) = \frac{a_m(x,x)}{(4\pi)^{d/2k}}$ for even $n = 2m$.

Two types of surface terms:

- $B_n(x)$ are heavily dependent on boundary conditions and properties of $\partial\mathcal{M}$ [Kirsten (2002), Gilkey (2004)]
- total derivative terms in $E_{2m}(x)$

Schwinger – DeWitt technique

The operator

$$F_{AB}(\nabla)\delta(x,y) \stackrel{\text{def}}{=} \frac{\delta^2 S[\varphi]}{\delta\varphi^A(x)\delta\varphi^B(y)}, \qquad \hat{F}(\nabla)\hat{G}(x,y) = -\hat{1}\,\delta(x,y), \tag{4}$$

$$\Gamma^{1-\mathrm{loop}} = \frac{1}{2}\log \mathrm{Det}\,\hat{F}(\nabla) = \frac{1}{2}\,\mathrm{Tr}\log\hat{F}(\nabla). \tag{5}$$

Principal symbol

For $F_{AB}(\nabla)$ of order $2k$:

$$F_{AB}(\nabla) = D_{AB}^{\alpha_1\dots\alpha_{2k}}\nabla_{\alpha_1}\dots\nabla_{\alpha_{2k}} + \Pi_{AB}(\nabla), \tag{6}$$

where $\mathrm{ord}\,\Pi_{AB}(\nabla) \le 2k-1$. Matrix $D_{AB}(n)$

$$D_{AB}(n) \equiv D_{AB}^{\alpha_1\dots\alpha_{2k}} n^{\alpha_1}\dots n^{\alpha_{2k}} \tag{7}$$

for some n^α is the *principal symbol* of $F_{AB}(\nabla)$

Minimal, nonminimal, and causal operators

Operator $F_{AB}(\nabla)$ is *minimal*, if its principal symbol has the form

$$D_{AB}(n) = C_{AB}(g_{\alpha\beta}n^\alpha n^\beta)^k, \tag{8}$$

where (a non-zero) matrix C_{AB} is invertible and independent of n^α.
Operator $F_{AB}(\nabla)$ is *causal* if

$$\det D_{AB}(n) = C(g_{\alpha\beta}n^\alpha n^\beta)^{k\,\mathrm{tr}\,\delta_{AB}} \neq 0. \tag{9}$$

Schwinger – DeWitt technique

Minimal (second order) operator

$$\hat{F}_2(\nabla) = \hat{\Box} + \hat{P}, \tag{10}$$

Early-time asymptotic expansion $s \to 0$ [DeWitt (1965)]

$$\hat{K}_{F_2}(s\,|x,y) = \frac{1}{(4\pi)^{d/2}} \frac{\Delta^{1/2}(x,y)}{s^{d/2}} g^{1/2}(y) \exp\left[-\frac{\sigma(x,y)}{2s}\right] \sum_{n=0}^{\infty} s^n \hat{a}_n^{F_2}(x,y), \quad s \to 0. \tag{11}$$

where $\sigma(x,y)$ is the Synge world function

$$\sigma(x,y) = \frac{1}{2}(\text{geodesic distance between } x \text{ and } y)^2, \tag{12}$$

$\Delta(x,y)$ is the dedensitized Pauli – van Vleck – Morette determinant

$$\Delta(x,y) = g^{-1/2}(x) g^{-1/2}(y) \left| \det \partial^x_\alpha \partial^y_\beta \sigma(x,y) \right| \tag{13}$$

$\hat{a}_n^{F_2}(x,y)$ are the *Schwinger – DeWitt coefficients*

Hence, $\mathrm{Tr}\,\hat{K}_{F_2}$ is given in terms of coincidence limits $[\hat{a}_n](x) \overset{\text{def}}{=} \hat{a}_n(x,y)\Big|_{y=x}$ which are calculated iteratively

$$\mathrm{Tr}\,\hat{K}_{F_2}(s) = \int d^d x \frac{g^{1/2}}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} s^n\, \mathrm{tr}[\hat{a}_n^{F_2}](x), \tag{14}$$

$$\log \mathrm{Det}\, \hat{F}_2\big|_d^{\mathrm{div}} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\omega - d/2} \int d^d x\, g^{1/2}\, \mathrm{tr}[\hat{a}_{d/2}^{F_2}](x), \qquad \omega \to d/2 - 0. \tag{15}$$

Generalization for nonminimal operators

Perturbation theory

$$\hat{F}(\nabla|\lambda) = \Box\delta^{\mu}_{\nu} - \lambda\nabla^{\mu}\nabla_{\nu} \tag{16}$$

Let $\hat{F}(\nabla) \equiv \hat{F}(\nabla|\lambda)$ be a λ-family of causal operators for $\lambda \in [0, \lambda_0)$ with $\hat{F}(\nabla|0)$ being minimal. Choose $\hat{K}(n)$ as

$$\hat{D}(n)\hat{K}(n) = \hat{1}\,(n^2)^m, \qquad m \leq k\,\mathrm{tr}\,\hat{1}, \tag{17}$$

Going back to operators ($n^{\alpha} \mapsto \nabla^{\alpha}$), an additional term $\hat{K}_1(\nabla) = O[\ell_{\mathrm{bg}}^{-1}]$ of order $2m-1$ appears:

$$\hat{D}(\nabla)\hat{K}(\nabla) = \hat{\Box}^m + \hat{K}_1(\nabla), \tag{18}$$

and for $\hat{F}(\nabla) = \hat{D}(\nabla) + \hat{\Pi}(\nabla)$:

$$\hat{F}(\nabla)\hat{K}(\nabla) = \hat{\Box}^m + \hat{M}(\nabla), \qquad \hat{M}(\nabla) = \hat{\Pi}(\nabla)\hat{K}(\nabla) + \hat{K}_1(\nabla) = O[\ell_{\mathrm{bg}}^{-1}]. \tag{19}$$

Perturbation theory in $\hat{M}$ [Barvinsky, Vilkovisky (1983, 1985)]:

$$\hat{G} = \hat{K}(\nabla)\frac{\hat{1}}{\Box^m}\sum_{p=0}^{p_{\max}}(-1)^{p+1}\left(\hat{M}(\nabla)\frac{\hat{1}}{\Box^m}\right)^p + O\left[\ell_{\mathrm{bg}}^{-(p_{\max}+1)}\right]. \tag{20}$$

$$\log \mathrm{Det}\,\hat{F}(\nabla|\lambda) = \log \mathrm{Det}\,\hat{F}(\nabla|0) + \mathrm{Tr}\int_0^{\lambda} d\lambda'\,\frac{d\hat{F}(\nabla|\lambda')}{d\lambda'}G(\lambda') + \delta(0)(\ldots). \tag{21}$$

Series (20) effectively reduces the nonminimal Green function to the sum of *universal functional traces*

$$\nabla_{\mu_1}\ldots\nabla_{\mu_\ell}\frac{\hat{1}}{\Box^m}\,\delta(x,y)\Big|_{y=x}. \tag{22}$$

P.T. is insensitive to total-derivative terms!

Proca heat kernel and 1-loop double poles

Proca 1-loop effective action from perturbation theory

Proca operator

$$F(m^2|\nabla) \equiv F^\mu_\nu(m^2|\nabla) = \Box\delta^\mu_\nu - \nabla^\mu\nabla_\nu - m^2\delta^\mu_\nu - R^\mu_\nu. \tag{23}$$

Its GF is expressed in terms of minimal massive GF:

$$\frac{\delta^\alpha_\beta}{F(m^2|\nabla)} = \left(\delta^\alpha_\nu - \frac{1}{m^2}\nabla^\alpha\nabla_\nu\right)\frac{\delta^\nu_\beta}{H(\nabla) - m^2}, \tag{24}$$

where $H(\nabla) - m^2 \equiv H^\mu_\nu(\nabla) - m^2\delta^\mu_\nu = \Box\delta^\mu_\nu - R^\mu_\nu - m^2\delta^\mu_\nu$.

$$\frac{1}{2}\,\mathrm{Tr}\log\hat{F}(m^2|\nabla) = \frac{1}{2}\,\mathrm{Tr}\int_{m^2}^\infty d\mu^2\,\frac{\delta^\mu_\nu}{F(\mu^2|\nabla)} \tag{25}$$

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\log F^\mu_\nu(m^2|\nabla)\Big|^{\mathrm{div}} &= \frac{1}{\omega-2}\int d^4x\frac{g^{1/2}}{32\pi^2}\Big\{-\frac{11}{180}R^2_{\alpha\beta\mu\nu} + \frac{43}{90}R^2_{\alpha\beta} - \frac{1}{9}R^2 + \frac{m^2}{2}R + \frac{3}{2}m^4\\ &- \frac{1}{12}(\gamma_E + \log m^2)\Box R - \frac{1}{30}\Box R\Big\} - \frac{1}{(\omega-2)^2}\int d^4x\frac{g^{1/2}}{32\pi^2}\frac{1}{12}\Box R.\end{aligned} \tag{26}$$

Its principal symbol is degenerate:

$$\det\left(n^2\delta^\mu_\nu - n^\mu n_\nu\right) = 0 \tag{27}$$

But if mass is included explicitly, a perturbation theory can be constructed:

$$D^\mu_\nu(n) = n^2\delta^\mu_\nu - n^\mu n_\nu - m^2\delta^\mu_\nu, \quad K^\mu_\nu(n) = \delta^\mu_\nu - \frac{1}{m^2}n^\mu n_\nu, \quad F^\mu_\alpha(m^2|\nabla)K^\alpha_\nu(\nabla) = (\Box - m^2)\delta^\mu_\nu + M^\mu_\nu. \tag{28}$$

With perturbation operator being purely potential $M^\mu_\nu = -R^\mu_\nu$

Proca heat kernel and 1-loop double poles

Proca heat kernel

Heat kernel is given via Green function's Mellin transform:

$$\begin{aligned}\left[e^{\tau F(m^2|\nabla)}\right]^{\mu}_{\ \nu} &= -\frac{1}{2\pi i}\int_{\mathcal{C}} dz\left[\frac{e^{z\tau}}{F(m^2|\nabla)-z}\right]^{\mu}_{\ \nu}\\ &= -\frac{1}{2\pi i}\int_{\mathcal{C}} dz\left[\delta^{\mu}_{\lambda}-\frac{1}{m^2+z}\nabla^{\mu}\nabla_{\lambda}\right]\left[\frac{e^{z\tau}}{H(\nabla)-m^2-z}\right]^{\lambda}_{\ \nu}\\ &= \left[e^{\tau(H(\nabla)-m^2)}\right]^{\mu}_{\ \nu}+\nabla^{\mu}\nabla_{\lambda}\left[\frac{e^{-m^2\tau}}{H(\nabla)}-\frac{e^{\tau(H(\nabla)-m^2)}}{H(\nabla)}\right]^{\lambda}_{\ \nu}.\end{aligned} \tag{29}$$

Using the Ward identity ($\Box_{\rm s}$ is the scalar d'Alembertian)

$$\frac{\delta^{\lambda}_{\nu}}{H(\nabla)}\nabla_{\lambda}\nabla^{\mu} = \frac{1}{\Box_{\rm s}}\nabla_{\nu}\nabla^{\mu} \tag{30}$$

$$\begin{aligned}\nabla_{\beta}\left[e^{\tau(H(\nabla)-m^2)}\right]^{\beta}_{\ \nu} &= e^{-\tau m^2}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\nabla_{\beta}\left[H(\nabla)^n\right]^{\beta}_{\ \nu} = e^{-\tau m^2}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\Box_{\rm s}^n\nabla_{\nu} = e^{\tau(\Box_{\rm s}-m^2)}\nabla_{\nu},\\ \nabla^{\mu}\nabla_{\alpha}\left[e^{\tau(H(\nabla)-m^2)}\right]^{\alpha}_{\ \beta}\left[\frac{1}{H(\nabla)}\right]^{\beta}_{\ \nu} &= \nabla^{\mu}\, e^{\tau(\Box_{\rm s}-m^2)}\frac{1}{\Box_{\rm s}}\nabla_{\nu}.\end{aligned} \tag{31}$$

The heat kernel then reads

$$K_{F(m^2)}{}^{\mu}_{\ \nu}(\tau) = K_H{}^{\mu}_{\ \nu}(\tau)\,e^{-\tau m^2} - e^{-\tau m^2}\nabla^{\mu}\frac{e^{\tau\Box_{\rm s}}-1}{\Box_{\rm s}}\nabla_{\nu}, \tag{32}$$

which satisfies the heat equation.

Proca heat kernel and 1-loop double poles

$\Gamma^{1-\mathrm{loop}}_{\mathrm{div}}$ from the heat kernel

Does our heat kernel yield the same determinant?

$$\Gamma^{1-\mathrm{loop}} = -\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K_{F(m^2)}{}^{\mu}_{\nu}(\tau), \tag{33}$$

$$\mathrm{Tr}\,K_{F(m^2)}{}^{\mu}_{\nu} = \mathrm{Tr}\,K_{H}{}^{\mu}_{\nu}\,e^{-\tau m^2} - \mathrm{Tr}\left[e^{-\tau m^2}\,\nabla^\mu\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\nabla_\nu\right]. \tag{34}$$

Functional trace cyclicity involves *integration by parts*:

$$\begin{aligned}\mathrm{Tr}\,\nabla^\mu\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\nabla_\mu &= \int d^dx\,d^dy\,\nabla^\mu\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\delta(x,y)\nabla_\mu\delta(y,x)\\ = -\int d^dx\,d^dy\,\nabla^\mu_{(x)}\left[\nabla^{(x)}_\mu\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\delta(x,y)\delta(y,x)\right] &+ \int d^dx\,d^dy\,\Box_s\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\delta(x,y)\delta(y,x)\end{aligned} \tag{35}$$

The additional total-derivative term is to blame for the double pole

$$\begin{aligned}\mathrm{Tr}\,K_{F(m^2)}{}^{\mu}_{\nu} &= \mathrm{Tr}\,K_{H}{}^{\mu}_{\nu}\,e^{-\tau m^2} - \mathrm{Tr}\,K_{\Box_s}\,e^{-\tau m^2}\\ &+ \int d^dx\,\nabla_\mu\left[\nabla^\mu\,\frac{e^{\tau\Box_s}-1}{\Box_s}\,\delta(x,x')\big|_{x'=x}\right]\,e^{-\tau m^2}.\end{aligned} \tag{36}$$

Proca heat kernel and 1-loop double poles

$\Gamma^{1-\text{loop}}_{\text{div}}$ from the heat kernel

The surface term in the heat kernel yields the double pole

$$\begin{aligned}
&\int_0^\infty \frac{d\tau}{\tau} e^{-\tau m^2} \int d^d x\, \nabla_\mu \left[\nabla^\mu \frac{e^{\tau \Box_{\text{s}}} - 1}{\Box_{\text{s}}} \delta(x,x')\big|_{x'=x} \right] \\
&= \int_0^\infty \frac{d\tau}{\tau} \int_0^\tau ds \int d^d x\, \nabla_\mu \left[\nabla^\mu\, e^{s\Box_{\text{s}} - \tau m^2} \delta(x,x')\big|_{x'=x} \right] \\
&= \int \frac{d^d x}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left(m^2\right)^{-n+d/2-1} \frac{\Gamma(n-d/2+1)}{n-d/2+1} \nabla_\mu \left[\nabla^\mu a_n^{\Box}\big|_{x'=x} \right].
\end{aligned} \tag{37}$$

In $d=4$ the double pole is at $n=1$:

$$\frac{1}{2}\log \mathrm{Det}\, F(m^2|\nabla)\Big|^{\text{div}} = -\frac{1}{(\omega-2)^2} \int \frac{d^4x\, g^{1/2}}{32\pi^2} \nabla_\mu [\nabla^\mu a_1^{\Box}] + O((\omega-2)^{-1}), \tag{38}$$

where $[\nabla_\mu a_1^{\Box}] = \frac{1}{12} \nabla_\mu R$.
Including the rest of the terms in the heat kernel yields the same result as perturbation theory.

Proca heat kernel and 1-loop double poles

Boundary terms summation method

Alternative method: following [Barvinsky, Vilkovisky (1985)], make use of Ward identity

$$\nabla_\alpha(\Box\delta^\alpha_\beta - R^\alpha_\beta) = \Box_s \nabla_\beta , \tag{39}$$

$$F^\mu_\alpha(m^2)\left(\delta^\alpha_\nu - \frac{1}{m^2}\nabla^\alpha\nabla_\nu\right) = (\Box - m^2)\delta^\mu_\nu - R^\mu_\nu . \tag{40}$$

Whence:

$$\mathrm{Tr}\log F^\mu_\nu(m^2) = \mathrm{Tr}\log\left[(\Box - m^2)\delta^\mu_\nu - R^\mu_\nu\right] - \mathrm{Tr}\log\left[\delta^\mu_\nu - \frac{1}{m^2}\nabla^\mu\nabla_\nu\right] + \delta(0)(\dots). \tag{41}$$

First term is $\mathrm{Tr}\log$ of minimal operator, contains only simple poles. The second term:

$$\begin{aligned}\mathrm{Tr}\log\left[\delta^\mu_\nu - \frac{1}{m^2}\nabla^\mu\nabla_\nu\right] &= -\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\nabla^\mu\frac{\Box_s^{n-1}}{m^{2n}}\nabla_\nu = \mathrm{Tr}\,\nabla^\mu\frac{1}{\Box_s}\log\left(1 - \frac{\Box_s}{m^2}\right)\nabla_\nu \\ &= \int d^dx\nabla^\mu\frac{\log(1 - \Box_s/m^2)}{\Box_s}\nabla_\mu\delta(x,y)\Big|_{y=x}.\end{aligned} \tag{42}$$

Using the identity

$$\nabla^{(x)}f(x,x) = \nabla^{(x)}f(x,y)\big|_{y=x} + \nabla^{(y)}f(x,y)\big|_{y=x}, \tag{43}$$

we obtain the surface term:

$$\begin{aligned}&\int d^dx\nabla^\mu\frac{\log(1 - \Box_s/m^2)}{\Box_s}\nabla_\mu\delta(x,y)\Big|_{y=x} = \\ &= \mathrm{Tr}\log\left[1 - \frac{\Box_s}{m^2}\right] - \int d^dx\nabla_\mu\left[\nabla^\mu\frac{\log\left(1 - \Box_s/m^2\right)}{\Box_s}\delta(x,y)\Big|_{y=x}\right].\end{aligned} \tag{44}$$

Proca heat kernel and 1-loop double poles

Boundary terms summation method

In $d = 4$ we have:

$$
\begin{aligned}
&-\int d^d x \nabla_\mu \left[\nabla^\mu \frac{\log\left(1 - \square_{\mathrm{s}}/m^2\right)}{\square_{\mathrm{s}}} \delta(x,y)\Big|_{y=x} \right] = \\
&= \int d^4 x \nabla^\alpha \left\{ \nabla_\alpha \int_{m^2}^{\infty} \frac{d\mu^2}{\mu^2} \int_0^\infty ds\, e^{-s\mu^2} e^{s\square_{\mathrm{s}}} \delta(x,y)\Big|_{y=x} \right\} \\
&= \left(\frac{1}{(\omega-2)^2} + \frac{\gamma_{\mathrm{E}} + \log m^2}{\omega - 2} \right) \int \frac{d^4 x\, g^{1/2}}{16\pi^2} \frac{\square R}{12} + O\left((\omega-2)^0\right),
\end{aligned}
\tag{45}
$$

which also contains a double pole (and gives the same answer as the heat kernel method).

Determinant anomalies and surface terms

Determinant anomalies

Define multiplicative (or determinant) anomaly for operators $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$ [Wodzicki (1984, 1987), Kontsevich (1994)]

$$\hat{\mathcal{O}}_{12} \overset{\text{def}}{=} \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 , \tag{46}$$

$$\mathcal{A}(\hat{\mathcal{O}}_1 , \hat{\mathcal{O}}_2) \overset{\text{def}}{=} \log \mathrm{Det}\, \hat{\mathcal{O}}_{12} - \log \mathrm{Det}\, \hat{\mathcal{O}}_1 - \log \mathrm{Det}\, \hat{\mathcal{O}}_2 . \tag{47}$$

Relation with surface terms

Product determinant (order-preserving) deformation:

$$\delta \log \mathrm{Det}[\hat{\mathcal{O}}_{12}] \equiv \delta[\mathrm{Tr} \log \hat{\mathcal{O}}_{12}] = \mathrm{Tr}[\hat{\mathcal{O}}_{12}^{-1} \delta \hat{\mathcal{O}}_{12}] = \mathrm{Tr}\left[\hat{\mathcal{O}}_2^{-1} \hat{\mathcal{O}}_1^{-1} \left(\delta \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 + \hat{\mathcal{O}}_1 \delta \hat{\mathcal{O}}_2\right)\right] . \tag{48}$$

Cyclic permutations under Tr involves integration by parts:

$$\delta \log \mathrm{Det}[\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2] = \delta \log \mathrm{Det}\, \hat{\mathcal{O}}_1 + \delta \log \mathrm{Det}\, \hat{\mathcal{O}}_2 + \int_{\mathcal{M}} d^d x\, \partial_\mu [\ldots]^\mu . \tag{49}$$

Hence, expect only surface terms in $\mathcal{A}(\hat{\mathcal{O}}_1 , \hat{\mathcal{O}}_2)$. Verify this by explicit calculation of div parts of anomaly for minimal & nonminimal 2^{nd} order operators.

Determinant anomalies and surface terms

Notation

Compact manifold $\mathcal{M}$, $\dim \mathcal{M} = d$ with indices: $\alpha, \beta, \gamma, \ldots$
Bundle indices $A, B, C, \ldots$ are omitted: $\varphi \equiv \varphi^A$, $\hat{X} \equiv X^A_B$, $X \equiv \mathrm{tr}\,\hat{X} \equiv X^A_A$.
Connection ∇_α is torsionless and compatible with metric $g_{\alpha\beta}$ on $\mathcal{M}$.

$$[\nabla_\alpha, \nabla_\beta] v^\gamma = R^\gamma{}_{\lambda\alpha\beta} v^\lambda, \qquad [\nabla_\alpha, \nabla_\beta]\,\varphi = \hat{\mathcal{R}}_{\alpha\beta}\,\varphi. \tag{50}$$

Method of calculations

Let $\mathrm{ord}\,\hat{F}_{(2)} = 2$, $\mathrm{ord}\,\hat{F}_{(4)} = 4$, and $\omega \to d/2 - 0$. From Schwinger–DeWitt technique:

$$\log \mathrm{Det}\,\hat{F}_{(2)}\big|^{\mathrm{div}} = \frac{1}{\omega - d/2} \int d^d x\, g^{1/2}\, \mathrm{tr}\,\hat{E}_d^{\hat{F}(2)}(x),$$
$$\log \mathrm{Det}\,\hat{F}_{(4)}\big|^{\mathrm{div}} = \frac{2}{\omega - d/2} \int d^d x\, g^{1/2}\, \mathrm{tr}\,\hat{E}_d^{\hat{F}(4)}(x), \tag{51}$$

where $\hat{E}_{2m}^{\hat{F}(2)}$ and $\hat{E}_{2m}^{\hat{F}(4)}$ are Gilkey–Seeley coefficients for $\hat{F}_{(2)}$ and $\hat{F}_{(4)}$ from their HK expansions:

$$\hat{E}_{2m}(x) = \frac{1}{(4\pi)^{d/2}}\,\hat{a}_m(x,y)\,\big|_{y=x}. \tag{52}$$

Hence divergent part of anomaly:

$$\mathcal{A}_{12}^{d\to 2,4}\Big|_d^{\mathrm{div}} = \frac{1}{\omega - d/2} \int d^d x\, g^{1/2}\, \mathrm{tr}\left[2\hat{E}_d^{\hat{F}12} - \hat{E}_d^{\hat{F}1} - \hat{E}_d^{\hat{F}2}\right], \qquad \omega \to d/2 - 0. \tag{53}$$

Determinant anomalies and surface terms

Minimal determinant anomaly

$\mathcal{A}(\hat{F}_1, \hat{F}_2)$ for minimal operators:

$$\hat{F}_1 = \Box\hat{1} + \hat{A}_\alpha \nabla^\alpha + \hat{Q}, \qquad \hat{F}_2 = \Box\hat{1} + \hat{P}, \tag{54}$$

$$\hat{F}_{12} \overset{\text{def}}{=} \hat{F}_1 \hat{F}_2 = \Box^2\hat{1} + \hat{\Omega}_{\alpha\beta\gamma}\nabla^\alpha\nabla^\beta\nabla^\gamma + \hat{D}_{\alpha\beta}\nabla^\alpha\nabla^\beta + \hat{H}_\alpha\nabla^\alpha + \hat{U}, \tag{55}$$

with coefficients

$$\hat{\Omega}_{\alpha\beta\gamma} = \frac{1}{3}\left(g_{\beta\gamma}\hat{A}_\alpha + g_{\gamma\alpha}\hat{A}_\beta + g_{\alpha\beta}\hat{A}_\gamma\right), \qquad \hat{D}_{\alpha\beta} = \left(\hat{P} + \hat{Q}\right) g_{\alpha\beta},$$
$$\hat{H}_\alpha = -\frac{1}{3}\hat{A}^\beta\left(2R_{\alpha\beta} + 3\,\hat{\mathcal{R}}_{\alpha\beta}\right) + 2\nabla_\alpha\hat{P} + \hat{A}_\alpha\hat{P}, \quad \hat{U} = \frac{1}{3}\hat{A}_\alpha\nabla_\beta\hat{\mathcal{R}}^{\alpha\beta} + \hat{A}_\alpha\nabla^\alpha\hat{P} + \Box\hat{P} + \hat{Q}\hat{P}. \tag{56}$$

Using $\hat{E}_{2m}$ from [Barvinsky, Vilkovisky (1985)] for $\hat{F}_{1,2}$ and [Barvinsky, Wachowski (2022)][a] for $\hat{F}_{12}$, we obtain:

$$\mathcal{A}_{12}^{d\to 2}\big|^{\text{div}} = -\frac{1}{\omega - 1}\int \frac{d^2x\, g^{1/2}}{8\pi}\nabla_\alpha A^\alpha, \qquad \omega \to 1 - 0, \tag{57}$$

$$\begin{aligned}\mathcal{A}_{12}^{d\to 4}\big|^{\text{div}} = \frac{1}{\omega - 2}\int \frac{d^4x\, g^{1/2}}{16\pi^2}\nabla_\alpha\,\mathrm{tr}\Big\{ &-\frac{1}{4}\left(\hat{A}^\alpha(\hat{P} + \hat{Q})\right) - \frac{1}{12}(\hat{A}^\alpha R + \hat{A}_\beta R^{\alpha\beta}) \\ &- \frac{1}{9}\nabla_\alpha\nabla_\beta\hat{A}^\beta - \frac{7}{36}\Box\hat{A}^\alpha + \frac{2}{9}\nabla_\beta\nabla^\alpha\hat{A}^\beta + \frac{11}{72}\hat{A}^\alpha\nabla_\beta\hat{A}^\beta \\ &- \frac{1}{72}\left(\nabla^\alpha\hat{A}^\beta\hat{A}_\beta + \nabla_\beta\hat{A}^\alpha\hat{A}^\beta\right) + \frac{1}{24}(\hat{A}^\alpha\hat{A}_\beta\hat{A}^\beta)\Big\}, \qquad \omega \to 2 - 0.\end{aligned} \tag{58}$$

Only total-derivative terms, as expected.

[a] Ω-dependent part of $\hat{E}_4$ given in this work contained mistakes

Determinant anomalies and surface terms

Nonminimal determinant anomaly

Now 2$^{\mathrm{nd}}$ order nonminimal vector operators:

$$
\begin{aligned}
{F_1}^{\alpha}_{\beta}(\varkappa) &= \Box\delta^{\alpha}_{\beta} - \varkappa\nabla^{\alpha}\nabla_{\beta} + X^{\alpha}_{\beta}\,,\\
{F_2}^{\alpha}_{\beta}(\lambda) &= \Box\delta^{\alpha}_{\beta} - \lambda\nabla^{\alpha}\nabla_{\beta} + Y^{\alpha}_{\beta}\,.
\end{aligned}
\tag{59}
$$

Their product is minimal if $\varkappa = \frac{\lambda}{\lambda - 1}$:

$$
{F_1}^{\alpha}_{\gamma}(\lambda){F_2}^{\gamma}_{\beta}(\varkappa) = \Box^2\delta^{\alpha}_{\beta} + D^{\alpha\,\mu\nu}_{\beta}\nabla_{\mu}\nabla_{\nu} + H^{\alpha\,\mu}_{\beta} + U^{\alpha}_{\beta}\,,
\tag{60}
$$

with coefficients ($\varkappa = \frac{\lambda}{\lambda - 1}$):

$$
\begin{aligned}
D^{\alpha\,\mu\nu}_{\beta} &= (X^{\alpha}_{\beta} + Y^{\alpha}_{\beta})g^{\mu\nu} - \frac{\varkappa}{2}\Big[(Y^{\mu}_{\beta} + R^{\mu}_{\beta})g^{\alpha\nu} + (Y^{\nu}_{\beta} + R^{\nu}_{\beta})g^{\alpha\mu}\Big]\\
&\qquad - \frac{\lambda}{2}\Big[(X^{\alpha\mu} + R^{\alpha\mu})\delta^{\nu}_{\beta} + (X^{\alpha\nu} + R^{\alpha\nu})\delta^{\mu}_{\beta}\Big]\,,\\
H^{\alpha\,\mu}_{\beta} &= -\varkappa\nabla^{\alpha}(Y^{\mu}_{\beta} + R^{\mu}_{\beta}) - \varkappa(\frac{1}{2}\nabla_{\beta}R + \nabla_{\gamma}Y^{\gamma}_{\beta})g^{\mu\alpha} + 2\nabla^{\mu}Y^{\alpha}_{\beta}\,,\\
U^{\alpha}_{\beta} &= \varkappa R^{\alpha}_{\gamma}(R^{\gamma}_{\beta} + Y^{\gamma}_{\beta}) - \varkappa\nabla_{\gamma}\nabla^{\alpha}(R^{\gamma}_{\beta} + Y^{\gamma}_{\beta}) + X^{\alpha}_{\gamma}Y^{\gamma}_{\beta} + \Box Y^{\alpha}_{\beta}\\
&\qquad - \frac{\varkappa}{2}(Y^{\mu\gamma} + R^{\mu\gamma})R_{\gamma\beta\mu}{}^{\alpha} + \frac{\lambda}{2}(X^{\alpha\mu} + R^{\alpha\mu})R_{\beta\mu}\,.
\end{aligned}
\tag{61}
$$

Nonminimal determinant anomaly

Using E_{2m} from [Gusynin (1999)], [Barvinsky, Camargo, Kalugin, Ohta, Shapiro (2023)], and [Barvinsky, Wachowski (2022)] we obtain:

$$\mathcal{A}_{12}^{d\to 2}\Big|^{\rm div} = 0, \tag{62}$$

$$\begin{aligned}\mathcal{A}_{12}^{d\to 4}\Big|^{\rm div} = \frac{1}{\omega-2}\int\frac{d^4x\,g^{1/2}}{16\pi^2}\nabla_\alpha\Big\{&\frac{7(\lambda-2)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{24(\lambda-1)\lambda}\nabla^\alpha R\\
+\frac{(5\lambda-4)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12\lambda^2}\nabla_\beta X^{\alpha\beta}&+\frac{(\lambda+1)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12\lambda^2}\nabla^\alpha X\\
-\frac{(\lambda+4)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12(\lambda-1)\lambda^2}\nabla_\beta Y^{\alpha\beta}&-\frac{(2\lambda-1)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12(\lambda-1)\lambda^2}\nabla^\alpha Y\Big\}.\end{aligned} \tag{63}$$

Again, only total-derivative terms.

Conclusion

Main results

1. Double pole total-derivative term in the nonminimal Proca field 1-loop effective action is explained
2. Nonminimal massive Proca heat kernel is constructed
3. Double-pole term in Proca operator determinant is obtained via the heat kernel and the boundary term summation method
4. A connection between determinant anomalies and total-derivative terms is established
5. The determinant variational formula argument is verified via explicit calculations for both minimal and nonminimal operators

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