Separable supergravity backgrounds

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Introduction

- Classical solutions of Einstein equation
- Spacetime isometries and dimensional reduction
- Stationary 4D spacetimes and 3D sigma-models on cosets G/H
- Target space isometries. Harrison transformations and Ehlers group
- Maximal $\mathcal{N} = 8$ sugra and G=SO(4,4) as generating symmetry. Constructing sugra black holes from Kerr. Type I from type D.
- Stationary axisymmetric spacetimes (SAS) and 2D integrable systems – indirect way to solve Einstein equations
- Hidden symmetries: Killing tensors, separability of Hamilton-Jacoby and Klein-Gordon equations
- Carter's vacuum and electrovacuum spacetimes (1968) admitting direct separation of Einstein equations (Petrov type D)
- Generalisation to supergravity: to find ansatz ensuring separability of Einstein equations for Petrov type I and sugra matter sources (multiple vector and scalar fields)

Generic model

General bosonic part of extended 4d sugras is

$$S=\int d^4x\left[\left(R-\frac{1}{2}f_{AB}\partial_\mu\Psi^A\partial^\mu\Psi^B-\frac{1}{2}K_{IJ}F^I_{\mu\nu}F^{J\mu\nu}\right)\sqrt{-g}-\frac{1}{2}H_{IJ}F^I_{\mu\nu}F^J_{\lambda\tau}\epsilon^{\mu\nu\lambda\tau}\right]$$

The scalar moduli parametrize a four-dimensional coset (e.g. $U(8)/E_{7(7)}$ for $\mathcal{N}=8, D=4$ supergravity) with an associated target metric $f_{AB}(\Psi)$. Vector fields transform under the same global symmetry implemented by real symmetric matrices K_{IJ}, H_{IJ} depending on scalar fields Ψ^A
The corresponding Einstein equations read:

$$R_{\mu\nu}=\frac{1}{2}f_{AB}\Psi^A_{,\mu}\Psi^B_{,\nu}-K_{IJ}\left(F^I_{\mu\lambda}F^{J\lambda}{}_\nu+\frac{1}{4}g_{\mu\nu}F^I_{\alpha\beta}F^{J\alpha\beta}\right)$$

The scalar fields depend only on r, y, so they contribute directly only in transverse part of the Ricci tensor.

Stationary spacetime

- Stationary metric in Kaluza-Klein form

$$ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1} h_{ij} dx^i dx^j,$$

- The equations reduce to those of the three-dimensional gravitating sigma-model for $n_\sigma = 2 + 2n_v + n_s$ scalars, gr-qc 2405.19196, $X^M = f, \chi, \Psi_A, v^I, u_I,\ M = 1, \ldots, n_\sigma$ and the three-dimensional metric h_{ij}:

$$S_3 = \int \left[\mathcal{R}(h) - \mathcal{G}_{MN}(X)\, \partial_i X^M\, \partial_j X^N\, h^{ij} \right] \sqrt{h}\, d^3x,$$

  where target space is a coset G/H, in particular, $G = SO(4,4,)$ for maximal 4d sugra. Using Harrison transformations from G one can generate sugra black holes from Kerr metric.

- This is indirect way to solve Einstein equations. It involves non-point-like transformations.

Stationary axisymmetric spacetime (SAS)

- Second way. Assuming in addition an axial symmetry, commuting with stationarity –> Lewis-Papapetrou form (two blocks)

$$ds^2 = \gamma_{ab}(\rho, z)dx^a dx^b + e^{2\nu}(d\rho^2 + dz^2)\,,$$

which leads to two-dimensional integrable systems. This is also an indirect way to solve Einstein equations, rather complicated.

- The third way: separation of variables in the Einstein equations of the SAS vacuum and electrovacuum was demonstrated by Carter (1968), who assumed the additional existence of a second-rank Killing tensor. In particular, he obtained Kerr-Newman metric of Petrov type D, ensuring stronger Killing-Yano hidden symmetry.
- Can one generalize this method to supergravity?
- Observation: sugra black holes metrics obtained by the first and the second methods are of Petrov type I, but they still possess second rank Killing tensor.

Killing tensor

Killing tensor is a symmetric tensor satisfying $\nabla_{(\alpha}K_{\mu\nu)} = 0$ which provides an additional integral

$$C = \frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\, K^{\mu\nu}$$

for the Hamilton-Jacobi (HJ) equation

$$\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\, g^{\mu\nu} = \mu^2$$

- **Benenti theorem** (1976):
  In n dimensions a necessary and sufficient condition for separability of HJ equation is the existence of a closed commuting system of Schouten-Nijenhuis brackets of n Killing vectors and Killing tensors, with additional conditions for eigenvectors of the latter.
- For SAS 4-dim spacetime one (trivial) Killing tensor is the metric, so one nontrivial is needed for HJ separability

Benenti-Francaviglia ansatz

Metric parametrization ensuring existence of non-trivial Killing tensor was given by Benenti and Francaviglia (BF, 1979). It includes ten arbitrary functions, each depending on one variable $A_k(r)$, $B_k(y)$, $k = 1..5$.

- Metric is SAS, block diagonal (orthogonal transitivity), $x^\mu = (x^a, x^i)$, where $x^a = t, \varphi$ correspond to the subspace spanned by the Killing vectors $K^{(t)} = \partial_t$ and $K^{(\varphi)} = \partial_\varphi$ and $x^i = r, y$, belong to orthogonal two-dimensional space whose metric without loss of generality can be assumed diagonal. BF ansatz initially is written in terms of the contravariant metric tensor $g^{\mu\nu} = \left(g^{ab}, g^{ij}\right)$:

$$g^{ab} = \Sigma^{-1} \begin{pmatrix} A_3 - B_3 & A_4 - B_4 \\ A_4 - B_4 & A_5 - B_5 \end{pmatrix}, \quad g^{ij} = -\Sigma^{-1} \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$$

- In order to ensure existence of an exact Killing tensor (EKT), the conformal factor $\Sigma = \Sigma(r, y)$ must be of the special form

$$\Sigma = A_1 + B_1.$$

BF Killing tensor

In terms of A, B, the BF Killing tensor reads $K^{\mu\nu} = \left(K^{ab},\ K^{ij}\right)$, where

$$\Sigma K^{ab} = \begin{pmatrix} A_1 B_3 + A_3 B_1 & A_1 B_4 + A_4 B_1 \\ A_1 B_4 + A_4 B_1 & A_1 B_5 + A_5 B_1 \end{pmatrix}, \quad \Sigma K^{ij} = \begin{pmatrix} -A_2 B_1 & 0 \\ 0 & A_1 B_2 \end{pmatrix}$$

The inverse metric and the Killing tensor have the following automorphism:

$$A_1 \leftrightarrow -B_1, \quad A_i \leftrightarrow B_i, \quad (i = 2..4), \quad g^{rr} \leftrightarrow -g^{yy}, \quad K^{rr} \leftrightarrow -K^{yy}.$$

For an arbitrary conformal factor Σ only a conformal Killing tensor exists. Two blocks of the covariant metric tensor $g_{\mu\nu} = (g_{ab},\ g_{ij})$: read

$$g_{ab} = \frac{\Sigma}{\mathcal{P}} \begin{pmatrix} A_5 - B_5 & -A_4 + B_4 \\ -A_4 + B_4 & A_3 - B_3 \end{pmatrix}, \qquad g_{ij} = -\Sigma \begin{pmatrix} A_2^{-1} & 0 \\ 0 & B_2^{-1} \end{pmatrix}$$

where

$$\mathcal{P} = (A_3 - B_3)(A_5 - B_5) - (A_4 - B_4)^2.$$

Null shear-free congruences

- By Goldberg-Sachs theorem, a vacuum spacetime is algebraically special if and only if it admits null geodesic shear-free congruence. This means that the corresponding principal null direction of the Weyl tensor is degenerate. If there are two such congruences, the metric is of type D.
- Type D guarantees existence of the second rank Killing tensor, and, moreover, Killing-Yano tensor
- For some supergravity black holes existence of null shear-free geodesic congruences was demonstrated, while metric type is I. There is no contradiction witgh GS theorem, since solutions are non-vacuum. Moreover, second rank Killing tensor also exists, though only in absence of scalar fields the black hole solution are type D.
- To clarify relationship between null geodesis shear-free congruences, Killing tensors and Petrov types we proceed with the Newman-Penrose description of BF spacetimes.

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu ,$$

We choose some natural Newman-Penrose (NP) null tetrad :

$$l = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\,\partial_t + C\,\partial_\varphi - \sqrt{A_2}\,\partial_r\right),\ m = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\,\partial_t + D\,\partial_\varphi - i\sqrt{B_2}\,\partial_y\right),$$

$$n = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\,\partial_t + C\,\partial_\varphi + \sqrt{A_2}\,\partial_r\right),\ \bar{m} = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\,\partial_t + D\,\partial_\varphi + i\sqrt{B_2}\,\partial_y\right).$$

$$(A_3 - B_3)C = \sqrt{A_3}(A_4 - B_4) + \sqrt{B_3}\sqrt{-\mathcal{P}},$$

$$(A_3 - B_3)\,D = \sqrt{B_3}(A_4 - B_4) + \sqrt{A_3}\sqrt{-\mathcal{P}}.$$

Using algebraic computing and hints from the known supergravity black hole solutions, one is led to consider the following constraints excluding two of ten BF functions:

$$A_4 = \sqrt{A_3 A_5}, \qquad B_4 = \sqrt{B_3 B_5}. \qquad \text{Then}$$

$$\sqrt{-\mathcal{P}} = \sqrt{A_3 B_5} - \sqrt{A_5 B_3}, \quad \Rightarrow \quad C = \sqrt{A_5}, \quad D = \sqrt{B_5}.$$

Recall the definitions of the NP projections of the covariant derivatives

$$D = l^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu,$$

and the action of D, Δ on the vectors l^μ, n^μ

$$Dl^\mu = (\epsilon + \bar{\epsilon}) l^\mu - \bar{\kappa} m^\mu - \kappa \bar{m}^\mu$$
$$\Delta n^\mu = -(\gamma + \bar{\gamma}) n^\mu + \nu m^\mu + \bar{\nu} \bar{m}^\mu.$$

Consider null congruences aligned with l^μ, n^μ. If $\kappa = 0 = \nu$ they are *geodesic*, with ϵ, γ being measure of non-affinity. Another important quantity of null congruences is *shear*, which is defined for them as

$$\sigma = -m^\mu \delta \, l_\mu, \qquad \bar{\lambda} = m^\mu \delta \, n_\mu$$

respectively. Calculating the spin coefficients for our tetrad, we find:

$$\kappa = \nu = 0, \qquad \sigma = \lambda = 0,$$

which means that both congruences are *geodesic* and *shearfree*. Other spin coefficients are generically non-zero and pairwise equal:

$$\mu = \rho, \quad \tau = \pi, \quad \epsilon = \gamma, \quad \alpha = \beta.$$

Such properties are typical for Petrov type D. To establish Petrov type in our case, we calculate the NP projections of the Weyl tensor:

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = -C_{\alpha\beta\gamma\delta} \left(l^\alpha n^\beta l^\gamma n^\delta - l^\alpha n^\beta m^\gamma \bar{m}^\delta \right) /2,$$
$$\Psi_3 = -C_{\alpha\beta\gamma\delta} n^\alpha l^\beta n^\gamma m^\delta,$$
$$\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta,$$

From the computer assisted calculations one finds that two are zero,

$$\Psi_0 = 0 = \Psi_4,$$

while the others are rather cumbersome in terms of BF coefficients. Still one can extract the following relation between the other two:

$$\Psi_1 = \Psi_3,$$

reflecting obvious symmetry of the tetrad under $A \leftrightarrow B$.

Vanishing of Ψ_0 and Ψ_4 means that the real vectors l^μ, n^μ are two distinct principal null directions of the constrained BF metric. Also this means that our tetrad is not canonical for determination of the Petrov type. We therefore proceed by computing the values of the quadratic and cubic curvature invariants of the Weyl tensor

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \qquad J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}.$$

As is known, in order for a metric to be algebraically special, the following relationship between invariants must be satisfied:

$$I^3 = 27J^2.$$

With our results one finds that the constrained BF metrics are algebraically special if

$$\Psi_1^2 = k\Psi_2^2, \quad \text{with} \quad k = 9/16, \ \text{or} \ 0.$$

Petrov types

- If this does not hold, the metric is of the Petrov type I.
- If holds with $k = 0$ and $\Psi_2 \neq 0$ then the metric type is D.
- Other algebraically special types are not possible , for example if one assumes type II, for which $\Psi_0 = 0 = \Psi_1$, one immediately finds that the Weyl tensor completely vanishes, i.e. the metric is of type O.
- In view of the Goldberg-Sachs theorem, for type I spacetime, admitting a null geodesic shear-free congruence, the Ricci tensor should be non-zero. This is the case for supergravity black holes.
- Thus our class I_B (eight A, B functions) consists of non-vacuum metrics, admitting a Killing tensor and a pair of null geodesic shear-free congruences. These properties are close to properties of D type, they will ensure separability of the Hamilton-Jacobi equation.
- Special feature of type I_B is that its algebraically special subsector is only type D

Klein-Gordon separability

Generic type I_B class of metrics still does not quarantee separability of the wave equations. Consider the Klein-Gordon equation for a real scalar field ϕ:

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = -\mu^2\phi.$$

Crucial for separability is the determinant of the metric, which with the first two constraints reads:

$$\sqrt{-g} = \frac{\Sigma^2}{\sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{A_5B_3}\right)}.$$

Consideration of the inverse metric clearly shows that the separability condition is

$$\sqrt{-g} = \Sigma.$$

This leads to the third restriction on the Benenti coefficient functions:

$$\Sigma = \sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{B_3A_5}\right),$$

which can be rewritten as $A_1 + B_1 = bA_{23} - aB_{23}$, introducing $A_{23} = \sqrt{A_2A_3}$, $B_{23} = \sqrt{B_2B_3}$ in the gauge $A_2A_5 = a^2$, $B_2B_5 = b^2$ (const).

Differentiating this with respect to the appropriate arguments, one can find useful differential relations following from the third constraint:

$$A_1' = bA_{23}', \qquad B_1' = aB_{23}'.$$

(Note, that in these relations primes can not be omitted!)
With the third constraint the separability of KG equation is easily shown:

$$\text{with} \qquad \phi(x^\mu) = \mathrm{e}^{-i\omega t + im\varphi} R(r) Y(y), \qquad \text{one gets :}$$

$$\frac{((A_2)'R)'}{R} + \frac{((B_2)'Y)'}{Y} + U(r) - V(y) = 0,$$

where primes denote derivatives with respect to r, y, and

$$U(r) = (\omega\sqrt{A_3} - m\sqrt{A_5})^2 - \mu^2 A_1,$$
$$V(y) = (\omega\sqrt{B_3} - m\sqrt{B_5})^2 + \mu^2 B_1.$$

This second constraint effectively reduces the number of arbitrary functions to seven, two of which A_2, B_2 can still be fixed using gauge freedom, so the number of essentially independent functions is five.

Carter's operator and commutativity

Klein-Gordon separability can be also explored with the help of Carter's second order differential operator associated with the Killing tensor:

$$\hat{K} = \nabla_\mu K^{\mu\nu} \nabla_\nu,$$

which must commute with D'Alembert operator. This commutator was elaborated by Carter:

$$[\Box, \hat{K}]\,\phi = \frac{4}{3} \nabla_\alpha (K_\sigma{}^{[\alpha} R^{\beta]\sigma}) \nabla_\beta\, \phi.$$

Projecting tensor at the right hand side onto the NP tetrad, we obtain

$$K_\sigma{}^{[\alpha} R^{\beta]\sigma} = 2(K_{ln} + K_{m\bar{m}})(n^\beta(\bar{m}^\alpha \Phi_{01} + m^\alpha \Phi_{10}) - n^\alpha(\bar{m}^\beta \Phi_{01} + m^\beta \Phi_{10}) + \\ + (l^\beta \bar{m}^\alpha - l^\alpha \bar{m}^\beta)\Phi_{12} + (l^\beta m^\alpha - l^\alpha m^\beta)\Phi_{21}).$$

So a sufficient condition for commutativity is vanishing of two Ricci scalars

$$\Phi_{01} = R_{\mu\nu} l^\mu m^\nu / 2 = \overline{\Phi_{10}}, \quad \Phi_{12} = R_{\mu\nu} n^\mu m^\nu / 2 = \overline{\Phi_{21}}.$$

Computing them with account for the third constraint and equating to zero one obtains:

$$aA''_{23} - bB''_{23} = 0,$$

where primes denotes derivatives over respective arguments. With account for previously found relations, this can be also rewritten as

$$a^2 A''_1 - b^2 B''_1 = 0.$$

Since one term is a function of r, while the other is a function of y, each of them must be constant. Other speaking A_1 and B_1 must be at most quadratic polynoms of respective arguments.

Separability of Einstein equations

After imposing three constraints on BF functions we can rewrite the spacetime metric as follows

$$ds^2 = \frac{A_2 B_2}{\Sigma}\left(\sqrt{B_5}\,dt - \sqrt{B_3}\,d\varphi\right)^2 - \frac{A_2 B_2}{\Sigma}\left(\sqrt{A_5}\,dt - \sqrt{A_3}\,d\varphi\right)^2 - \frac{\Sigma}{A_2}dr^2 - \frac{\Sigma}{B_2}dy^2,$$

where

$$\Sigma = \sqrt{A_2 B_2}\left(\sqrt{A_3 B_5} - \sqrt{A_5 B_3}\right).$$

This is exactly Carter's ansatz of 1968 up to a signature convention and notation. Carter showed that for these metrics the vacuum and electrovacuum (with the corresponding Maxwell form) Einstein equations are separable and lead to several families of solutions, among which were the Kerr and Kerr-Newman black holes belonging to the Petrov type D. Now we checked that our parameterization admits class I_B of general type I and thus is applicable to supergravity. Preliminary analysis shows that generic supargravity equations can be separated for type I_B indeed.

The Killing-Yano tensor $Y_{\mu\nu} = -Y_{\nu\mu}$ satisfying the equation

$$\nabla_{(\alpha} Y_{\mu)\nu} = 0,$$

can be regarded as a "square root" of the Killing tensor:

$$Y_{\mu}{}^{\alpha} Y_{\alpha\nu} = K_{\mu\nu}.$$

Since we know the Killing tensor independently of Petrov type of the metric, we may consider these equations as independent conditions which prescribe the metric to be of type D, and define the KY tensor itself. In terms of the constrained tetrad the Killing tensor has only two non-vanishing NP projections exactly as in the case of the type D :

$$K_{ln} = B_1, \qquad\qquad K_{m\bar{m}} = A_1.$$

So projecting KJ splitting of KT on the NP tetrad, one obtains:

$$Y^2_{ln} = B_1, \qquad\qquad Y^2_{m\bar{m}} = -A_1,$$

Extracting roots from these equations is somewhat subtle and demands further analysis. The result is

$$Y_{ln} = \sqrt{B_1}, \qquad Y_{m\bar{m}} = -i\sqrt{A_1}.$$

Now we have to satisfy the KY equation for consistency . Omitting details we arrive at the following relation for the metric to be D type:

$$A_1' = 2b\sqrt{A_1}, \qquad B_1' = 2a\sqrt{B_1};$$
$$A_{23}' = 2\sqrt{A_1}, \qquad B_{23}' = -2\sqrt{B_1}.$$

Intergration of this system provides generic form for some of the metric coefficients for type D BF sector:

$$A_1 = (br + c_1)^2, \qquad B_1 = (ay + d_1)^2,$$
$$A_{23} = br^2 + 2\,c_1 r + c_2, \qquad B_{23} = -(ay^2 + 2\,d_1 y + d_2),$$

where the constants c_1, d_1, c_2 and d_2 are subject to the following condition:

$$b\,c_2 + a\,d_2 = {c_1}^2 + {d_1}^2. \tag{1}$$

It remains to check that with these conditions the Weyl tensor projections $\Psi_{1,3}$ vanish. Taking into account three constraints, $\Psi_{1,3}$ can be obtained in the form

$$8\Sigma^3\Psi_{1,3} = -\sqrt{A_2 B_2}\left\{\Sigma(aA''_{23} - bB''_{23}) - ab({A'_{23}}^2 + {B'_{23}}^2)\right\},$$

Substituting here subsidiary conditions for type D obtained above, one finds that $\Psi_{1,3} = 0$ indeed.

Also note that in the static case, when $A_5 = 0 = B_3$, and hence $a = 0$, $B_{23} = 0$, we have $\Psi_{1,3} = 0$ for the full class I_B without imposing a condition of type D.

Kerr-Newman

Kerr-Newman metric is type D solution on $\mathcal{N}=2$ pure supergravity without scalar fields. In Boyer-Lindquist coordinates,

$$ds^2=\frac{\Delta_r}{\Sigma}(dt-a\sin^2\theta d\varphi)^2-\frac{1}{\Sigma}\sin^2\theta(adt-(r^2+a^2)d\varphi)^2-\frac{\Sigma}{\Delta_r}dr^2-\Sigma\, d\theta^2,$$
$$\Delta_r=r^2-2Mr+a^2+Q^2, \qquad \Sigma=r^2+a^2\cos\theta.$$

In BF form one obtains:

$$A_1=r^2, \qquad B_1=a^2y^2$$
$$A_3=\frac{(r^2+a^2)^2}{\Delta_r}, \qquad B_3=a^2(1-y^2),$$
$$A_5=\frac{a^2}{A_2}=\frac{a^2}{\Delta_r}, \qquad B_5=\frac{1}{B_2}=\frac{1}{1-y^2},$$

where $y=\cos\theta$. It is easy to verify that our conditions are met:

$$A_4^2=A_3A_5,\;\; B_4^2=B_3B_5,\;\; b=1,\;\; A_1+B_1=\sqrt{A_2B_2}\left(\sqrt{A_3B_5}-\sqrt{A_5B_3}\right),$$

as well as the type D conditions.

$\mathcal{N} = 4$ EMDA$_1$*solution*

$$ds^2 = \frac{\Delta_r - a^2 \sin^2\theta}{\Sigma}(dt - w d\varphi)^2 - \Sigma\left(\frac{dr^2}{\Delta_r} + d\theta^2 + \frac{\Delta_r \sin^2\theta}{\Delta_r - a^2\sin^2\theta} d\varphi^2\right),$$

$$\Delta_r = (r - r_0)(r - 2M) + a^2 - (N - N_-)^2,$$

$$\Sigma = r(r - r_-) + (a\cos\theta + N)^2 - N_-^2,$$

$$w = \frac{2}{a^2\sin^2\theta - \Delta_r}\left[N\Delta_r\cos\theta + a\sin^2\theta(M(r - r_-) + N(N - N_-))\right],$$

$$r_- = \frac{M|Q - iP|^2}{|M + iN|^2}, \qquad N_- = \frac{N|Q - iP|^2}{2|M + iN|^2},$$

Its BF form reads:

$$A_1 = r(r - r_-), \qquad B_1 = (a\,y + N)^2 - N_-^2,$$

$$A_3 = \frac{\left(r(r - r_-) + a^2 + N^2 - N_-^2\right)^2}{\Delta_r}, \qquad B_3 = \frac{\left[a(1 - y^2) - 2Ny\right]^2}{1 - y^2},$$

$$A_5 = \frac{a^2}{A_2} = \frac{a^2}{\Delta_r}, \qquad B_5 = \frac{1}{B_2} = \frac{1}{1 - y^2},$$

$$A_4^2 = A_3 A_5, \quad B_4^2 = B_3 B_5, \quad a = a, \quad b = 1, \quad A_1 + B_1 = \sqrt{A_2 B_2}\left(\sqrt{A_3 B_5} - \sqrt{A_5 B_3}\right)$$

Two non-zero NP projections of the Killing tensor are

$$K_{ln} = (ay + N)^2 - N_-^2, \qquad K_{m\bar{m}} = r(r - r_-).$$

The metric is a non-vacuum Petrov type I_B solution, with the following set of non-zero Weyl scalars:

$$\Psi_1 = \Psi_3 = \frac{a(4N_-^2 + r_-^2)\sin\theta\sqrt{\Delta_r}}{8\Sigma^3}, \qquad 12\Sigma^3\Psi_2 =$$

$$- 12(M - iN)(r + i(a\cos\theta + N))^3 + 6NN_-\big(2(r + i(N + a\cos\theta)) - r_-\big)^2 +$$
$$+ r_-^3(8M - r) + 8NN_-^3 - 4N_-^4 - 6ar_-\cos\theta(5M - 3iN)(a\cos\theta + 2N - 2ir) +$$
$$+ 2r_-\left(M\left(15(r + iN)^2 + 7N_-^2\right) + 18N^2 r + 9iN^3 + 3iN\left(N_-^2 - 3r^2\right) - 2N_-^2 r\right) +$$
$$+ 4N_-^2\left(2a^2 - a\cos\theta(3i(M + iN) + a\cos\theta) - 5Mr + 2N^2 - 3iN(M + r)\right) +$$
$$+ 4N_-^2 r^2 + r_-^2\left(2a^2 - 24iMN - 7N^2 + 2NN_- - N_-^2\right) -$$
$$- r_-^2\left(a\cos\theta(6i(4M - iN) + a\cos\theta) + 26Mr - 6iNr - r^2\right)$$

$$ds^2 = \frac{\Delta_r - a^2 \sin^2\theta}{\Sigma}(dt - w d\varphi)^2 - \Sigma\left(\frac{dr^2}{\Delta_r} + d\theta^2 + \frac{\Delta_r \sin^2\theta}{\Delta_r - a^2\sin^2\theta}d\varphi^2\right),$$

$$\Delta_r = r^2 - 2Mr + a^2,$$

$$w = -\frac{2Ma\omega\sin^2\theta}{\Delta_r - a^2\sin^2\theta}, \qquad \omega = ((\Pi_c - \Pi_s)r + 2M\Pi_s),$$

$$\Sigma^2 = \prod_{I=0}^{4}(r + 2Ms_I^2) + a^4\cos^4\theta +$$

$$+ 2a^2\cos^2\theta\left(r^2 + Mr\sum_{I=0}^{3}s_I^2 + 4M^2(\Pi_c - \Pi_s)\Pi_s - 2M^2\sum_{I<J<K}^{3}s_I^2 s_J^2 s_K^2\right),$$

and the products of charges are defined as follows

$$\Pi_s = \prod_{i=0}^{3}s_I = \prod_{I=0}^{3}\sinh\delta_I, \qquad \Pi_c = \prod_{I=0}^{3}\sqrt{1+s_I^2} = \prod_{I=0}^{3}\cosh\delta_I, \quad s_I^2 = \sinh^2\delta_I.$$

In the case of pairwise equality of charges $s_1 = s_3 = \mathcal{S}_1$, $s_2 = s_4 = \mathcal{S}_2$, we deal with the so-called two-charge solution. If $\mathcal{S}_2 = 0$, then the metric reduces to the Kerr-Sen solution. It is possible to transform the metric to the BF form only for the case of two-charge solution, for which

$$\Sigma_{2ch} = r^2 + a^2\cos^2\theta + 2Mr(\mathcal{S}_1^2 + \mathcal{S}_2^2) + 4M^2\mathcal{S}_1^2\mathcal{S}_2^2 = \\ = r^2 - 2Mr + a^2\cos^2\theta + 2Mw_{2ch}$$

$$A_1 = r^2 - 2Mr + 2Mw_{2ch}, \qquad B_1 = a^2y^2,$$

$$A_3 = \frac{(r^2 - 2Mr + a^2 + 2Mw_{2ch})^2}{\Delta_r}, \qquad B_3 = a^2(1-y^2),$$

$$A_5 = \frac{a^2}{A_2} = \frac{a^2}{\Delta_r}, \qquad B_5 = \frac{1}{B_2} = \frac{1}{(1-y^2)}.$$

$$\Psi_1 = \Psi_3 = \frac{aM^2\sin\theta(S_1^2 - S_2^2)^2\sqrt{\Delta_r}}{2\Sigma_{2ch}^3},$$

$$\frac{3\Sigma_{2ch}^3}{M}\Psi_2 = 12M^2\mathcal{S}_1^2\mathcal{S}_2^2(M - r - ia\cos\theta)(\mathcal{S}_1^2 + \mathcal{S}_2^2) - \\ - 3(r + ia\cos\theta)^3 - 3(r + ia\cos\theta)^2(M + r + ia\cos\theta)(\mathcal{S}_1^2 + \mathcal{S}_2^2) - \\ - 2M(r^2 + 2Mr - a^2 + 3iar\cos\theta + a^2y^2)(\mathcal{S}_1^4 + \mathcal{S}_2^4) + \\ + 4M\mathcal{S}_1^2\mathcal{S}_2^2(5a^2\cos^2\theta - a^2 + 4Mr - 5r^2 + 3iaM\cos\theta - 9iar\cos\theta)$$

Conclusions

- A parameterization of the metric admitting:
  - two commuting Killing vectors,
  - a second rank non-trivial Killing tensor,
  - two null geodesic shear-free congruences,
  - ensuring separability of the Klein-Gordon equation

  is found starting with Benenti-Francaviglia ansatz.
- The corresponding Petrov types are determined:
  - a sector I_B inside type I such that
  - its algebraically special subclass is only type D
- Compatibility with black hole solutions in extended four-dimensional supergravities is shown
- Separability of supergravity bosonic equations for the above class of metrics is conjectured