Banana diagrams as functions of geodesic distance

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Set up

- It is well known that Feynman diagrams in Minkowski space-time satisfy differential equations.
- Studying these differential equations provides information about the analytical structure of Feynman diagrams.

  For example, for the banana diagram:
- The position space differential equation for $G^2(x)$ is given by:

$$\left[x^{\mu}\partial_{\mu}\left(\Box + 4m^2\right) + (2d-2)\left(\Box + 2m^2\right)\right] G^2(x) = 0, \quad \text{for} \quad x > 0. \tag{1}$$

- The momentum space differential equation for

$$G^2(p) = \int d\alpha_1 d\alpha_2 \delta(1-\alpha_1-\alpha_2)\frac{(\alpha_1+\alpha_2)^{2-d}}{\left[\alpha_1\alpha_2\, p^2 - m^2(\alpha_1+\alpha_2)^2\right]^{\frac{4-d}{2}}}, \tag{2}$$

  satisfies Picard-Fuchs equations:

$$\left[p^2(p^2-4m^2)\partial_{p^2} - (d-4)p^2 - 4m^2\right] G^2(p) = 0. \tag{3}$$

- Can we generalize these approaches to curved space?

- To construct coordinate differential equations we should know how the 2^{nd} derivatives of the Green function can be expressed in terms of 1^{st} and 0^{th} derivatives.
- For example, in the general case we don't know how to do that:

$$\nabla_\mu \nabla_\nu G(x,y) = ???. \tag{4}$$

However, if the Green function is a function of the geodesic distance, then:

$$\nabla_\mu \nabla_\nu G(\sigma) = G'(\sigma) \nabla_\mu \nabla_\nu \sigma + G''(\sigma) \nabla_\mu \sigma \nabla_\nu \sigma. \tag{5}$$

Then from the Klein-Gordon equation:

$$\left(\triangle - m^2\right) G(\sigma) = G''(\sigma) \nabla_\mu \sigma \nabla^\mu \sigma + G'(\sigma) \triangle \sigma - m^2 G(\sigma) = 0, \tag{6}$$

and the fact that the vector $\nabla_\mu \sigma$ has unit length: $\nabla_\mu \sigma \nabla^\mu \sigma = 1$, we can obtain:

$$\nabla_\mu \nabla_\nu G(\sigma) = G'(\sigma) \nabla_\mu \nabla_\nu \sigma - \left(G'(\sigma) \triangle \sigma - m^2 G(\sigma)\right) \nabla_\mu \sigma \nabla_\nu \sigma. \tag{7}$$

- The Green function is a function of geodesic distance if:

$$\boxed{\triangle \sigma = p(\sigma)} \to \text{Coulomb's law depend on geodesic distance} \tag{8}$$

Harmonic space

Harmonic space

The definition of harmonic spaces is that the Laplacian of the geodesic distance in these spaces is a function of the geodesic distance:

$$\triangle \sigma = p(\sigma). \tag{9}$$

One of the key properties of harmonic spaces is that they are Einsteinian:

$$R_{\mu\nu} = \kappa g_{\mu\nu}. \tag{10}$$

The classification of harmonic spaces is not yet complete, and not all such spaces are known. A short list of harmonic spaces:

- Any space covered by $\mathbb{R}^n$
- Maximally symmetric space $\mathbb{S}^n$
- Real projective space $\mathbb{R}P^n$
- Complex projective space $\mathbb{C}P^n$
- Quaternionic projective space $\mathbb{H}P^n$
- Cayley projective plane $\mathbb{O}P^2$
- Complex Grassmannian $\mathbb{GR}(k, n)$

Simple harmonic space

- Among harmonic spaces, there is a specific category known as simple harmonic spaces (SH), for which the Laplacian of the geodesic distance appears the same as in d-dimensional flat space:

$$\boxed{\triangle\sigma = (d-1)\sigma^{-1}}. \tag{11}$$

- As an example of a simple harmonic space, one can consider the following space:

$$ds^2 = (x^2 dx^1 - x^1 dx^2)^2 + 2dx^1 dx^3 + 2dx^2 dx^4. \tag{12}$$

This space is Ricci flat ($R_{\mu\nu} = 0$), but it has a non-zero component of the curvature tensor ($R_{1212} = -3$). The geodesic distance has a form similar to the geodesic distance in Euclidean space:

$$\sigma(x,y) = \sqrt{(y_1 x_2 - y_2 x_1)^2 + 2(x_1 - y_1)(x_3 - y_3) + 2(x_2 - y_2)(x_4 - y_4)} = \tag{13}$$
$$= \sqrt{g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)},$$

satisfying relation (11).

Maximally Symmetric Space

- Maximally symmetric spaces have the following embedding:

$$z^2 + \eta_{\mu\nu} x^\mu x^\nu = r^2 \tag{14}$$

into a $d+1$ dimensional space with general signature $\eta_{\mu\nu} = \mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)$. The metric is given by:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{x^\alpha x^\beta \eta_{\alpha\mu}\eta_{\beta\nu}}{r^2 - \eta_{\mu\nu} x^\mu x^\nu}. \tag{15}$$

- The geodesic distance is defined by the angle between two points:

$$\sigma = r \arccos\left(\frac{z}{r}\right) = r \arccos\left(\frac{\sqrt{r^2 - \eta_{\mu\nu} x^\mu x^\nu}}{r}\right). \tag{16}$$

- The Laplacian of the geodesic distance is a function of the geodesic distance:

$$\boxed{\triangle\sigma = (d-1)\frac{1}{r}\cot\left(\frac{\sigma}{r}\right)}. \tag{17}$$

$\mathbb{CP}^n$

- The geodesic distance on $\mathbb{CP}^n$ is given by the Hermitian angle:

$$\sigma = \arccos\left(\sqrt{\frac{(1+z\cdot\bar{w})(1+\bar{z}\cdot w)}{(1+z\cdot\bar{z})(1+\bar{w}\cdot w)}}\right), \tag{18}$$

where $z_i = \frac{Z^i}{Z^0}$ are the affine coordinates.

- The Fubini-Study metric has a Kähler form:

$$g_{\mu\bar{\nu}} = \partial_\mu\bar{\partial}_{\bar{\nu}}\log\left(1+z\cdot\bar{z}\right). \tag{19}$$

- The Laplacian is given by:

$$\triangle = 2g^{\mu\bar{\nu}}\partial_\mu\bar{\partial}_{\bar{\nu}}. \tag{20}$$

From the following relation:

$$2\log(\cos(\sigma)) = \log(1+z\cdot\bar{w}) + \log(1+\bar{z}\cdot w) - \log(1+z\cdot\bar{z}) - \log(1+\bar{w}\cdot w) \tag{21}$$

we can obtain:

$$\triangle 2\log(\cos(\sigma)) = -2g^{\mu\bar{\nu}}\partial_\mu\bar{\partial}_{\bar{\nu}}\log(1+z\cdot\bar{z}) = -2\dim[\mathbb{CP}^n]. \tag{22}$$

- Hence, complex projective space is harmonic:

$$\boxed{\triangle\sigma = \left(\dim[\mathbb{CP}^n]-1\right)\cot(\sigma) - \tan(\sigma)}. \tag{23}$$

Grassmannian

- The Grassmannian $\mathbb{GR}(k,n)$ is a natural generalization of $\mathbb{CP}^n$, with the points labeling the k-planes. The Grassmannian can be parameterized by Pontrjagin coordinates $\left(Z_{ia}, \bar{Z}_{ia}\right)$, where the geodesic distance is given by:

$$\cos(\sigma) = \sqrt{\frac{\det\left(1_{ab} + Z_{ia}\bar{W}_{ib}\right)\det\left(1_{ab} + \bar{Z}_{ia}W_{ib}\right)}{\det\left(1_{ab} + Z_{ia}\bar{Z}_{ib}\right)\det\left(1_{ab} + W_{ia}\bar{W}_{ib}\right)}}. \tag{24}$$

- The metric has a Kähler form:

$$ds^2 = \log\det\left(1_{ab} + Z_{ia}\bar{Z}_{ib}\right) = \tag{25}$$
$$= \mathrm{Tr}\left[(1+Z\bar{Z})^{-1}dZd\bar{Z} - (1+Z\bar{Z})^{-1}Zd\bar{Z}(1+Z\bar{Z})^{-1}dZ\bar{Z}\right].$$

- The Laplacian has a simple form:

$$\triangle = 2g^{(ia)\ (\bar{j}b)}\partial_{ia}\bar{\partial}_{\bar{j}b}. \tag{26}$$

The Laplacian of the geodesic distance can be obtained from the relation:

$$\triangle 2\log(\cos(\sigma)) = -2g^{(ia)\ (\bar{j}b)}\partial_{ia}\bar{\partial}_{\bar{j}b}\log\det\left(1_{ab} + Z_{ia}\bar{Z}_{ib}\right) = -2\dim\left[\mathbb{GR}(k,n)\right]. \tag{27}$$

- Hence, the complex Grassmannian manifold is harmonic:

$$\boxed{\triangle\sigma = \left(\dim\left[\mathbb{GR}(k,n)\right] - 1\right)\cot(\sigma) - \tan(\sigma)}. \tag{28}$$

Λ formalism

Λ formalism

- Let us introduce the following linear differential operator:

$$\Lambda = f(\sigma)\partial_\sigma . \tag{29}$$

The function f will be defined below. Then using the following relation:

$$\Lambda^2 G = ff' G' + f^2 G'' , \tag{30}$$

one can rewrite the Klein-Gordon equation in terms of Λ as follows:

$$\Lambda^2 G - \left[f' - (\triangle\sigma) f \right] \Lambda G - (m^2 + \xi R) f^2 G = 0. \tag{31}$$

- We can choose f to cancel the linear term in Λ in the last equation:

$$f' = (\triangle\sigma) f = p(\sigma) f \implies f = e^{\int d\sigma \, \triangle\sigma} . \tag{32}$$

- Therefore, the differential equation can be written in a simple form:

$$\boxed{\left(\Lambda^2 - \lambda^2\right) G(\sigma) = 0}, \quad \text{and} \quad \lambda = \sqrt{m^2 + \xi R} e^{\int d\sigma \, \triangle\sigma} . \tag{33}$$

$\mathbb{M}$	$\dim(\mathbb{M})$	$\triangle\sigma$ or $p(\sigma)$	$f(\sigma)$
SH_n	n	$(n-1)\sigma^{-1}$	σ^{n-1}
$\mathbb{S}^n$, $\mathbb{RP}^n$	n	$(n-1)\mathrm{ctg}(\sigma)$	$\sin^{n-1}(\sigma)$
$\mathbb{CP}^n$	$2n$	$(2n-1)\mathrm{ctg}(\sigma) - \mathrm{tg}(\sigma)$	$\sin^{2n-1}(\sigma)\cos(\sigma)$
$\mathbb{GR}(k,n)$	$\dim = 2k(n-k)$	$(\dim -1)\mathrm{ctg}(\sigma) - \mathrm{tg}(\sigma)$	$\sin^{\dim -1}(\sigma)\cos(\sigma)$
$\mathbb{HP}^n$	$4n$	$(4n-1)\mathrm{ctg}(\sigma) - 3\mathrm{tg}(\sigma)$	$\sin^{4n-1}(\sigma)\cos^3(\sigma)$

Differential equation for banana diagram

The simplest example is G^2.
Then acting by Λ we get:

$$\Lambda(G^2) = 2G\Lambda G, \tag{34}$$

$$\Lambda^2(G^2) = 2\Lambda G\Lambda G + 2G\Lambda^2 G = 2\Lambda G\Lambda G + 2\lambda^2(G^2) \tag{35}$$

and

$$\Lambda^3(G^2) = 4\Lambda^2 G\Lambda G + 2\Lambda\lambda^2(G^2) = 2\lambda^2\Lambda(G^2) + 2\Lambda\lambda^2(G^2). \tag{36}$$

as result we get:

$$\left[\Lambda^3 - 2\Lambda\lambda^2 - 2\lambda^2\Lambda\right] G^2 = 0, \tag{37}$$

For G^3 we can get:

$$\left[\Lambda^4 - 3\Lambda^2\lambda^2 - 3\lambda^2\Lambda^2 - 4\Lambda\lambda^2\Lambda + 9\lambda^4\right] G^3 = 0 \tag{38}$$

- To derive the differential equation for the banana diagram $B_n = G^n$ let us introduce a set of differential operators $\{O_k\}$ that act on the function $B_n = G^n$ as follows:

$$O_k B_n = B_n^{(k)} (\Lambda G)^k, \tag{39}$$

- Applying the operator Λ to $O_{k-1}B_n$ we obtain the recurrence relation:

$$O_k = \Lambda O_{k-1} - (k-1)(n-k+2)\lambda^2 O_{k-2}. \tag{40}$$

- This relation allows us to determine an operator for any given number k. The chain ends at step $n+1$ where $O_{n+1}B_n = 0$ because $B_n^{(n+1)} = 0$.
  The recurrence relation (40) is similar to the recurrence relation for the determinant of a tridiagonal matrix. As a result:

$$\boxed{\det \begin{pmatrix} \Lambda & c_1\lambda & 0 & \dots & 0 & 0 \\ c_1\lambda & \Lambda & c_2\lambda & \dots & 0 & 0 \\ 0 & c_2\lambda & \Lambda & \dots & 0 & 0 \\ & & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \dots & \Lambda & c_n\lambda \\ 0 & 0 & 0 & \dots & c_n\lambda & \Lambda \end{pmatrix} G^n = 0}, \tag{41}$$

where $c_k = \sqrt{k(n-k+1)}$.

- For $n = 1$, the determinant gives the Klein-Gordon equation (33):

$$\hat{D}_1 = \det\begin{pmatrix} \Lambda & \lambda \\ \lambda & \Lambda \end{pmatrix} = \Lambda^2 - \lambda^2. \tag{42}$$

- For $n = 2$, one can obtain:

$$\hat{D}_2 = \det\begin{pmatrix} \Lambda & \sqrt{2}\lambda & 0 \\ \sqrt{2}\lambda & \Lambda & \sqrt{2}\lambda \\ 0 & \sqrt{2}\lambda & \Lambda \end{pmatrix} = \Lambda^3 - 2\Lambda\lambda^2 - 2\lambda^2\Lambda, \tag{43}$$

and so on.

PF for SH

- In Euclidean space, one can use momentum representation to derive the Feynman parameter representation of the Banana diagram:

$$\widetilde{B}_n(p^2) = \int_0^\infty \left[\prod_{i=1}^n d\alpha_i\right] \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \frac{U^{\frac{n(2-d)}{2}}(\alpha)}{\left(\alpha_1 \cdots \alpha_n p^2 - m^2 U(\alpha) \sum_{i=1}^n \alpha_i\right)^{1+\frac{(n-1)(2-d)}{2}}}, \quad (44)$$

where α is a Feynman parameter and $U(\alpha) = \prod_{i=1}^n \alpha_i \sum_{j=1}^n \alpha_j^{-1}$.

- In this case, banana integrals satisfy Picard-Fuchs equations :

$$\hat{PF}_{p^2} \cdot \widetilde{B}_n(p^2) = 0. \quad (45)$$

For example for $n = 2$:

$$\left[p^2(p^2 - 4m^2)\partial_{p^2} - (d-4)p^2 - 4m^2\right] \widetilde{B}_n(p^2) = 0. \quad (46)$$

- The Klein-Gordon equation in harmonic spaces is similar to the differential equation for the Green's function in Euclidean space with $m^2 \to m^2 + \xi R$:

$$(\triangle - m^2 - \xi R)G(\sigma) = G''(\sigma) + (d-1)\sigma^{-1}G'(\sigma) - (m^2 + \xi R)G(\sigma) = 0 \tag{47}$$

Therefore, the Green's function in simple harmonic space is given by:

$$G(\sigma) \sim \int_0^\infty ds K(s,\sigma), \tag{48}$$

the heat kernel is given by:

$$K(s,\sigma) = \sum_i e^{-s\lambda_i}\phi_i(x)\phi_i^*(y) \sim \frac{1}{s^{\frac{d}{2}}} e^{-\frac{\sigma^2}{4s} - (m^2+\xi R)s}, \tag{49}$$

- Now, let us consider multi-loop banana diagrams:

$$B_n = G^n(\sigma) \sim \int \prod_{k=1}^n \left[ds_k \sum_{i_k} e^{-s\lambda_{i_k}} \phi_{i_k}(x)\phi_{i_k}^*(y) \right] \tag{50}$$

$$\sim \int_0^\infty \left[\prod_{k=1}^n \frac{ds_k}{s_k^{\frac{d}{2}}} \right] e^{-\frac{\sigma^2}{4}\left(\Sigma_{k=1}^n \frac{1}{s_k}\right)} e^{-(m^2+\xi R)(\Sigma_{k=1}^n s_k)}. \tag{51}$$

How to compute the following expression?

$$\delta_{ij}\widetilde{B}_n(\lambda_i) = \int d^d x \int d^d y \; \phi_i(x)\phi_j^*(y)B_n(x,y) \tag{52}$$

- Exponential term can be wtritten as:

$$e^{-\frac{\sigma^2}{4\left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-1}}} \sim \left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-d/2} \sum_i e^{-\left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-1}(\lambda_i - m^2 - \xi R)} \phi_i(x)\phi_i^*(y). \quad (53)$$

- Therefore, we can rewrite the equation for the banana diagram in terms of mode expansion:

$$B_n(x,y) \sim \quad (54)$$

$$\sim \int_0^\infty \left[\prod_{k=1}^n \frac{ds_k}{s_k^{\frac{d}{2}}}\right] \left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-\frac{d}{2}} \sum_i e^{-\left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-1}(\lambda_i - m^2 - \xi R) - (m^2 + \xi R)(\sum_{k=1}^n s_k)} \phi_i(x)\phi_i^*(y).$$

- Hence we can obtain integrals in terms of the Feynman parameter:

$$\widetilde{B}_n(\lambda_i) \sim$$

$$\sim \int_0^\infty \left[\prod_{i=1}^n d\alpha_i\right] \frac{\delta\left(1 - \sum_{i=1}^n \alpha_i\right) U^{\frac{n(2-d)}{2}}(\alpha)}{\left(\alpha_1 \cdots \alpha_n(\lambda_i - m^2 - \xi R) - (m^2 + \xi R) U(\alpha) \sum_{i=1}^n \alpha_i\right)^{1 + \frac{(n-1)(2-d)}{2}}}$$

- As a result, the Feynman parameter representation of the banana diagram in a simple harmonic space is the same as in Euclidean space with replaced parameters: $p^2 \to \lambda_i - m^2 - \xi R$ and $m^2 \to m^2 + \xi R$. Consequently, the Picard-Fuchs equations are the same as in Euclidean space.

Thanks for your attention