Induced gravity and cosmological particle production

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Conformal transformation

Conformal transformation:

$$ds^2 = g_{\mu\nu}(x)dx^\mu\, dx^\nu = \Omega^2(x)\,\hat{g}_{\mu\nu}(x)dx^\mu\, dx^\nu,$$

$g_{\mu\nu}(x) =$ metric tensor, $\Omega(x) =$ conformal factor .

Inverse metric tensor $g^{\mu\nu}$: $g^{\mu\nu}\, g_{\nu\sigma} = \delta^\mu_\sigma$

$$g_{\mu\nu} = \Omega^2\,\widehat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{\Omega^2}\,\widehat{g}^{\,\mu\nu}, \quad \sqrt{-g} = \Omega^4\,\sqrt{-\widehat{g}}$$

Signature (+,-,-,-), 4 dim., geometric units c=$\hbar$=1.

Conformal invariance - fundamental symmetry (creation from "nothing") A. V. Vilenkin Phys. Lett. B 117 (1982)

$$(\,\ldots\,) = (\,\hat{}\;\hat{}\;\hat{}\,)$$

Riemannian geometry - main objects: metric $g_{\mu\nu}(x)$ tensor and connection $\Gamma^\lambda_{\mu\nu}(x)$.

Covariant derivatives:

Scalar $\varphi_{;\mu} = \varphi_{,\mu}$ ("," = partial, ";" = covariant)

Vector $l^\mu_{;\nu} = l^\mu_{,\nu} + \Gamma^\mu_{\nu\sigma}\, l^\sigma$ $(l_{\mu;\nu} = l_{\mu,\nu} - \Gamma^\sigma_{\nu\mu}\, l_\sigma)$

Weyl tensor

$$\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}, \quad g_{\mu\nu;\sigma} = 0, \quad \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda} \right)$$

Curvature tensor $R^{\mu}_{\ \nu\lambda\sigma}$

$$R^{\mu}_{\ \nu\lambda\sigma} = \frac{\partial \Gamma^{\mu}_{\nu\sigma}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\kappa\lambda} \Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\kappa} \Gamma^{\kappa}_{\nu\lambda}$$

Ricci tensor $R_{\mu\nu}$ $\qquad R_{\mu\nu} = R^{\lambda}_{\ \mu\lambda\nu}$

Curvature scalar R $\qquad R = R^{\lambda}_{\lambda} = g^{\lambda\sigma} R_{\lambda\sigma}$

Weyl tensor $C_{\mu\nu\lambda\sigma} =$ traceless part of $R_{\mu\nu\lambda\sigma}$

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{2} R_{\mu\lambda} g_{\nu\sigma} + \frac{1}{2} R_{\mu\sigma} g_{\nu\lambda} - \\ - \frac{1}{2} R_{\nu\sigma} g_{\mu\lambda} + \frac{1}{2} R_{\nu\lambda} g_{\mu\sigma} + \frac{1}{6} R \left(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\lambda\nu} \right)$$

$C^{\mu}_{\ \nu\lambda\sigma}$ - conformally invariant: $\qquad C^{\mu}_{\ \nu\lambda\sigma} = \hat{C}^{\mu}_{\ \nu\lambda\sigma}$

Action for the perfect fluid in Eulerian variables

The study of particle production processes in the presence of strong external fields plays an important role both in cosmology and in black hole physics. The most difficult task is to take into account the back reaction of these processes on the metric, since it includes not only the influence of the created particles, but also the contribution from the vacuum polarization.

The exact solution of the quantum problem requires boundary conditions, while the latter can be imposed only after solving field equations with the energy-momentum tensor obtained by appropriate averaging from the quantum problem. In order to avoid these obstacles we consider the modification of the action for an ideal fluid, because it describes the process of particle production phenomenologically at the classical level, but taking into account the back reaction.

Action for the perfect fluid in Eulerian variables

Model for particles = perfect fluid.

Euler's picture *J.R.Ray J.Math.Phys. 13 (1972)*

$$S_{\rm m} = -\int \varepsilon(X,n)\sqrt{-g}\, d^4x + \int \lambda_0 (u_\mu u^\mu - 1)\sqrt{-g}\, d^4x + \\ + \int \lambda_1 (n u^\mu)_{;\mu}\sqrt{-g}\, d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\, d^4x \qquad (1)$$

Dynamical variables:

n - particle number density,

u^μ - four-velocity,

X - trajectory enumeration,

$\lambda_0(x)$, $\lambda_1(x)$, $\lambda_2(x)$ - Lagrange multipliers.

$\varepsilon(n, X)$ - energy density

Hydrodynamic pressure p : $p = -\varepsilon + n\,\frac{d\varepsilon}{dn}$

The phenomenological description of particle creation

Constrains (variation of λ-s)

$u_\sigma u^\sigma - 1 = 0 \quad \Rightarrow \quad$ normalization of 4-velocity $\rightarrow \; u^\mu = \frac{dx^\mu}{ds}$

$(nu^\sigma)_{;\sigma} = 0 \quad \Rightarrow \quad$ particle number conservation

$X_{,\sigma} u^\sigma = 0 \quad \Rightarrow \quad X = const$ along trajectories

Hydrodynamical energy-momentum tensor

$$T^{\mu\nu} = (p + \varepsilon)\, u^\mu u^\nu - p\, g^{\mu\nu}$$

Modification due to particle creation (phenomenology)

V.A.Berezin Int.J.Mod.Phys. A 2 (1987)

$$(n\, u^\mu)_{;\mu} = \Phi$$

$$\Phi = \Phi(inv) = \text{creation law}$$

Particle source - scalar field φ.

Conformal invariance of the rate of particle production

Conformal transformation:

$$n = \frac{\hat{n}}{\Omega^3}, \quad u^\mu = \frac{\hat{u}^\mu}{\Omega}, \quad \varphi = \frac{\widehat{\varphi}}{\Omega}$$

$$\Phi(inv) - ?$$

Useful formula : $l^\mu_{;\mu} = \frac{\left(l^\mu \sqrt{-g}\right)_{,\mu}}{\sqrt{-g}} \quad \Rightarrow$

$$\left(n\, u^\mu\right)_{;\mu} \sqrt{-g} = \left(n\, u^\mu \sqrt{-g}\right)_{,\mu} = \left(\frac{\hat{n}}{\Omega^3}\, \frac{\hat{u}^\mu}{\Omega}\, \Omega^4 \sqrt{-\widehat{g}}\right)_{,\mu} =$$

$$= \left(\hat{n}\, \hat{u}^\mu \sqrt{-\hat{g}}\right)_{,\mu}$$ - conformal invariance!

$\Rightarrow \Phi \sqrt{-g}$ - conformally invariant!

Possible contributions:

$$\left.\begin{array}{r} geometry \rightarrow C^2 = C_{\mu\nu\lambda\sigma}\, C^{\mu\nu\lambda\sigma} \\ scalar\ field \rightarrow \varphi\Box\varphi - \frac{1}{6}\,\varphi^2\, R + \Lambda\,\varphi^4 \end{array}\right\} \Rightarrow$$

$$\Phi = \alpha\, C^2 + \beta\, \left(\varphi\Box\varphi - \frac{1}{6}\,\varphi^2\, R + \Lambda\,\varphi^4\right) \qquad \text{But!}$$

Particle creation law

Our particles are the on-shell quanta of the scalar field. Therefore, they also can produce "new" particles.

The rate of such production should depend on the number density of the "old" particles, i.e. it is some function of n.

The most natural choices are: $\varphi\, n$ and $n^{\frac{4}{3}}$, both of them, being multiplied by $\sqrt{-g}$, form conformal invariants.

Thus, our creation law becomes:

$$\Phi = \alpha\, C^2 + \beta\, \left(\varphi \Box \varphi - \frac{1}{6}\, \varphi^2\, R + \Lambda\, \varphi^4\right) + \gamma_1\, \varphi\, n + \gamma_2\, n^{\frac{4}{3}}.$$

The matter action integral S_m reads now as follows

$$S_{\rm m} = -\int \varepsilon(X, \varphi, n)\sqrt{-g}\, d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\, d^4x + \\ + \int \lambda_1 \left((nu^\mu)_{;\mu} - \Phi\right) \sqrt{-g}\, d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\, d^4x, \quad (2)$$

Note $\varepsilon = \varepsilon(X, \varphi, n)$.

If one demands that the gravity itself is conformal invariant, then one needs nothing more. This is because the Lagrange multiplier λ_1 is defined, actually, up to a constant. We adopt this point of view and get an example of Sakharov's induced gravity.

A. D. Sakharov Dokl. Akad. Nauk Ser. Fiz. 177 (1966), (1967) - english translation

Thus,

$$S_m = S_{tot} \, .$$

Evidently,

$$\frac{\delta S_{tot}}{\delta \Omega} = \frac{\delta S_m}{\delta \Omega} = 0,$$

on the solutions. Nevertheless, it is very instructive to make use of the specific structure of our matter action integral. It is not difficult to show that the only part of S_m that matters, is

$$\int \varepsilon(X, \varphi, n)\sqrt{-g}\, d^4x.$$

Remembering that $n = \frac{\hat{n}}{\Omega^3}, \quad \varphi = \frac{\widehat{\varphi}}{\Omega}, \quad \sqrt{-g} = \Omega^4 \sqrt{-\widehat{g}}$, one gets

$$\varphi \frac{\partial \varepsilon}{\partial \varphi} + 3n \frac{\partial \varepsilon}{\partial n} = 4\, \varepsilon, \tag{3}$$

with the solution

$$\varepsilon = F(x)\, \varphi^4,$$

where F is an arbitrary function of the variable $x = \frac{n}{\varphi^3}$.

Induced gravity

Two important examples:

1. dust matter - $p = 0 \quad \Rightarrow$

$$\varepsilon = \mu_0 \, n \, \varphi$$

2. radiation - $\varepsilon = 3p \quad \Rightarrow$

$$\varepsilon = \nu_0 \, n^{\frac{4}{3}},$$

   does not depend on φ .

Note the resemblance with two "hydrodynamical" terms in the creation law !

Equations of motion and constraints

Let us derive the (modified) hydrodynamical equations of motion and corresponding energy-momentum tesor.

$$S_{\rm m} = -\int \varepsilon(X,\varphi,n)\sqrt{-g}\, d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\, d^4x + \\ + \int \lambda_1 \left((n u^\mu)_{;\mu} - \gamma_1\, \varphi\, n - \gamma_2\, n^{\frac{4}{3}} + \ldots \right) \sqrt{-g}\, d^4x + \\ + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\, d^4x. \quad (4)$$

Dynamical variables are n, u^μ, φ and X:

$$\delta n : \quad -\frac{\partial \varepsilon}{\partial n} - \lambda_{1,\sigma}\, u^\sigma - \lambda_1\gamma_1\, \varphi - \frac{4}{3}\lambda_1\gamma_2\, n^{\frac{1}{3}} = 0,$$

$$\delta u^\mu : \quad \lambda_2\, X_{,\mu} + 2\lambda_0\, u_\mu - \lambda_{1,\mu}\, n = 0,$$

$$\delta X : \quad -\frac{\partial \varepsilon}{\partial X} - (\lambda_2\, u^\sigma)_{;\sigma} = 0,$$

Equations of motion and constraints

$$\delta\varphi: \quad \beta\left(\lambda_1\Box\varphi + \Box(\lambda_1\varphi) + 4\lambda_1\Lambda\varphi^3 - \frac{1}{3}\lambda_1\varphi R\right) + \gamma_1 n = -\frac{\partial\varepsilon}{\partial\varphi}.$$

Constraints:

$$\delta\lambda_0: \quad u_\sigma u^\sigma - 1 = 0,$$

$$\delta\lambda_1: \quad (nu^\sigma)_{;\sigma} = \Phi,$$

$$\delta\lambda_2: \quad X_{,\sigma} u^\sigma = 0.$$

2-nd equation - with $u^\mu +$ constraints $\Rightarrow$

$$2\lambda_0 = -n\frac{\partial\varepsilon}{\partial n} - \lambda_1\gamma_1\varphi n - \frac{4}{3}\lambda_1\gamma_2 n^{\frac{4}{3}}.$$

Energy-momentum tensor

$$S_m = -\frac{1}{2} \int T^{\mu\nu} \, \delta g_{\mu\nu} \sqrt{-g} \, d^4 x,$$

The hydrodynamical part of the energy-momentum tensor is the energy-momentum tensor of the perfect fluid plus contribution from the γ_1 and γ_2 terms:

$$T^{\mu\nu}[n] = \varepsilon \, g^{\mu\nu} - 2\lambda_0 \, u^\mu \, u^\nu + g^{\mu\nu} \left(n \, \lambda_{1,\sigma} \, u^\sigma + \lambda_1 \gamma_1 \, \varphi \, n + \lambda_1 \gamma_2 \, n^{\frac{4}{3}} \right) =$$
$$= \left(\varepsilon + p + \lambda_1 \gamma_1 \, \varphi \, n + \frac{4}{3} \lambda_1 \gamma_2 \, n^{\frac{4}{3}} \right) u^\mu \, u^\nu - g^{\mu\nu} \left(p + \frac{1}{3} \lambda_1 \gamma_2 \, n^{\frac{4}{3}} \right).$$

In homogeneous and isotropic cosmological models the Weyl tensor is identically zero, $\delta(C^2) \equiv 0$ as well, and one can safely put $C^2 = 0$ prior to the variation procedure. The result is:

$$T^{\mu\nu}[\varphi] = \lambda_1 \beta \Lambda \, \varphi^4 \, g^{\mu\nu} - \beta \, (\lambda_1 \varphi)_{,\sigma} \, \varphi^{,\sigma} \, g^{\mu\nu} + \beta \, (\lambda_1 \varphi)^{,\mu} \, \varphi^{,\nu} +$$
$$+ \beta \, (\lambda_1 \varphi)^{,\nu} \, \varphi^{,\mu} + \frac{\beta}{3} \left\{ \lambda_1 \varphi^2 \, G^{\mu\nu} - \left(\lambda_1 \varphi^2 \right)^{;\mu;\nu} + g^{\mu\nu} \, \Box \left(\lambda_1 \varphi^2 \right) \right\}. \quad (5)$$

Energy-momentum tensor

Because of the induced gravity:

$$T^{\mu\nu}[n] + T^{\mu\nu}[\varphi] = 0,$$

which in turn means that $T = T^{\mu\nu} g_{\mu\nu} = 0$. This condition is equivalent to the equation (3) obtained above:

$$T = \varepsilon - 3p + 4\beta\,\lambda_1\,\Lambda\,\varphi^4 - \frac{\beta}{3}\,\lambda_1\,\varphi^2\,R + \beta\,\varphi\,\Box\,(\lambda_1\varphi) + \beta\,\lambda_1\varphi\,\Box\varphi + \\ + \lambda_1\,\gamma_1\,\varphi n = \varepsilon - 3p - \varphi\,\frac{\partial\varepsilon}{\partial\varphi} = 0,$$

where in the second equality the equation of motion obtained by variation in φ was used.

"Gravitating mirages"

There appeared two accompanying persons:

- γ_1 - dust-like $(\lambda_1\, \gamma_1 \varphi > 0)$,
- γ_2 - radiation-like $(\lambda_1\, \gamma_2 > 0)$.

Very important note:

They are not the real one because the particle number density n refers to the real created particles whose equation of state can be arbitrary (whatever you like).

Thus, they are something like the echoes of the creation process itself. The best name - "gravitating mirages".

Cosmology = homogeneity + isotropy

Robertson-Walker line element:

$$ds^2 = dt^2 - a^2(t)dl^2 = a^2(\eta)\left(d\eta^2 - dl^2\right),$$

$$dl^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (k = 0, \pm 1)$$

t - cosmological time, η - conformal time

High symmetry $\Rightarrow$ all dynamic variables except the metric depend only on t:

- $u^\mu = \delta^\mu_0$,
- $T^0_0 = T^0_0(t), \quad T^1_1 = T^2_2 = T^3_3(t)$
- $F_{\mu\nu} \equiv 0$ for gauge fields
- $C^\mu_{\ \nu\lambda\sigma} \equiv 0$

Geometry: $R^0_0 = -3\,\frac{\ddot{a}}{a}, \quad R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$

The complete set of equations becomes:

$$T_0^0 = \varepsilon + \beta\,\lambda_1 \left\{ \Lambda\,\varphi^4 + \dot{\varphi}^2 + \varphi^2\,\frac{\dot{a}^2 + k}{a^2} + 2\varphi\dot{\varphi}\,\frac{\dot{a}}{a} \right\} + \\ + \beta\,\dot{\lambda}_1\,\varphi \left(\dot{\varphi} + \varphi\,\frac{\dot{a}}{a} \right) + \lambda_1\,\left(\gamma_1\,\varphi\,n + \gamma_2\,n^{\frac{4}{3}} \right) = 0, \quad (6)$$

$$\varphi\ddot{\lambda}_1 + \dot{\lambda}_1 \left(3\varphi\,\frac{\dot{a}}{a} + 2\dot{\varphi} \right) + \lambda_1 \left(2\ddot{\varphi} + 6\,\frac{\dot{a}}{a}\dot{\varphi} + 4\Lambda\,\varphi^3 - \frac{1}{3}\,\varphi\,R \right) + \lambda_1 \frac{\gamma_1}{\beta}\,n + \\ + \frac{1}{\beta}\frac{\partial\varepsilon}{\partial\varphi} = 0, \quad (7)$$

$$\Phi = \beta\,\varphi \left(\frac{1}{a^3}\frac{d}{dt}\left(a^3\,\dot{\varphi}\right) - \frac{1}{6}\,\varphi\,R + \Lambda\,\varphi^3 \right) + \gamma_1\,\varphi n + \gamma_2\,n^{\frac{4}{3}} = \frac{1}{a^3}\frac{d}{dt}\left(a^3\,n\right), \quad (8)$$

$$\frac{\partial\varepsilon}{\partial n} + \dot{\lambda}_1 + \lambda_1\,\gamma_1\,\varphi + \frac{4}{3}\,\lambda_1\,\gamma_2\,n^{\frac{1}{3}} = 0, \quad (9)$$

$$T = \varepsilon - 3p - \varphi\,\frac{\partial\varepsilon}{\partial\varphi} = 0 \;\Rightarrow\; \varepsilon = F(x)\varphi^4, \quad x = \frac{n}{\varphi^3}. \quad (10)$$

EOM for cosmology

If the function $\varphi(t)$ is invertible we can express $\lambda_1(t)$ and $\dot{\lambda}_1(t)$ as a functions of φ: $\lambda_1 = \lambda_1(t(\varphi))$, $\dot{\lambda}_1 = \dot{\lambda}_1(t(\varphi))$. Differentiating the equation (9) with respect to n, we obtain:

$$F'' + \lambda_1 \frac{4}{9}\gamma_2 \, x^{-\frac{2}{3}} = 0,$$

where the prime denotes the derivative with respect to x. Since according to the previous assumption λ_1 is a function of φ, there are only two options left: $\lambda_1 = const$ or $F'' = \gamma_2 = 0$.

Let's consider the first case, then it follows from the equation (9) that:

$$F(x) = -\lambda_1 \left(\gamma_1 x + \gamma_2 x^{\frac{4}{3}}\right) + \sigma,$$

where σ is some constant. An interesting observation is that the hydrodynamic part of the energy-momentum tensor is reduced to a quintessence-type term:

$$T^{\mu\nu}[n] = \sigma \, \varphi^4.$$

In the second case, both the energy density and the n-dependent part of the creation law contain only the dust component:

$$\varepsilon = \mu \, \varphi \, n + \sigma \, \varphi^4, \quad \gamma_2 = 0.$$

The gauge selection

The conformal invariance of the action and, as a consequence, the equations of motion allows us to arbitrarily choose the gauge.

If we found the specific solution for the set of dynamical variables $\{\widehat{a}, \widehat{n}, \widehat{\varphi}\}$, then the general solution is: $\{a, n, \varphi\}$, where $a = \widehat{a}\,\Omega,\ \widehat{n} = n\,\Omega^3,\ \widehat{\varphi} = \varphi\,\Omega$ with arbitrary smooth function $\Omega(t)$.

Below we will use two types of gauges. The first is $\hat{a} = 1$. The latter does not mean at all that the "real" Universe is static. It is chosen because in such a case the set of equations looks simplest.

The second is $\lambda_1\,\varphi^2 = const$. If one puts

$$\beta\,\lambda_1\,\varphi^2 = -\frac{3}{\kappa}, \quad \kappa = \frac{8\pi G}{c^4},$$

where c is the speed of light and G is the Newtonian constant, then General Relativity becomes restored in our cosmological field equations.

Surely, such a choice assumes, that $\lambda_1 \neq 0$ and $\varphi \neq 0$. The cases $\lambda_1 = 0$ or $\varphi = 0$ lead to the situation when the geometry $a(t)$ cannot be found by solving the field equations, what looks rather strange from the physical point of view, except for the vacuum solution ($n = 0$) when there is no observer to measure the scale factor.

In order to study the vacuum solution of our model, we consider the system (6–10) with $n = 0$. Moreover, we can use only three since not all of them are independent:

$$T_0^0 = F(0)\,\varphi^4 + \beta\,\lambda_1 \left\{ \Lambda\,\varphi^4 + \dot{\varphi}^2 + \varphi^2\,\frac{\dot{a}^2 + k}{a^2} + 2\varphi\dot{\varphi}\,\frac{\dot{a}}{a} \right\} +$$

$$+\beta\,\dot{\lambda}_1\,\varphi \left(\dot{\varphi} + \varphi\,\frac{\dot{a}}{a} \right) = 0,$$

$$\beta\,\varphi \left(\frac{1}{a^3}\frac{d}{dt}\left(a^3\,\dot{\varphi}\right) - \frac{1}{6}\,\varphi\,R + \Lambda\,\varphi^3 \right) = 0,$$

$$F'(0)\,\varphi + \dot{\lambda}_1 + \lambda_1\,\gamma_1\,\varphi = 0.$$

It is obvious that $\varphi = 0,\ \lambda_1 = const$ is a solution.

Vacuum solutions

When $\varphi \neq 0$ we can choose the gauge $\varphi = \varphi_0 = const$ with which it is easy to find the two remaining vacuum solutions:

- $$\lambda_1 = -\frac{F'(0)}{\gamma_1}, \quad \Lambda = \frac{2F(0)\gamma_1}{\beta\mu_1}, \quad \frac{\dot{a}^2 + k}{a^2} = -\frac{\Lambda}{2}\,\varphi_0^2 \Rightarrow$$

$$a(t) = a_0\, exp\left\{\pm\varphi_0\sqrt{-\frac{\Lambda}{2}}\,t\right\}, \quad k = 0, \quad a_0 = a(0),$$

$$a(t) = \frac{1}{\varphi_0}\sqrt{-\frac{2}{\Lambda}}\, ch\left\{\pm\varphi_0\sqrt{-\frac{\Lambda}{2}}\,t + arch\left(\varphi_0\sqrt{-\frac{\Lambda}{2}}\,a_0\right)\right\}, \quad k = 1,$$

$$a(t) = \frac{1}{\varphi_0}\sqrt{-\frac{2}{\Lambda}}\, sh\left\{\pm\varphi_0\sqrt{-\frac{\Lambda}{2}}\,t + arsh\left(\varphi_0\sqrt{-\frac{\Lambda}{2}}\,a_0\right)\right\}, \quad k = -1.$$

- $$a = \pm\frac{1}{\varphi_0}\sqrt{-\frac{k}{\Lambda}},\ F(0) = 0,\ \lambda_1(t) = -\frac{F'(0)}{\gamma_1} + \left(\lambda_1(0) + \frac{F'(0)}{\gamma_1}\right) e^{-\gamma_1\varphi_0\, t}$$

The gauge $\lambda_1 \varphi^2 = const$ allows us to draw analogy with $f(R)$ gravity. In order to show this, let's consider what the part of action $S[\varphi]$ looks like under the condition $\lambda_1 = -\frac{3}{\kappa\beta\varphi^2}$:

$$S[\varphi] = -\frac{1}{2\kappa}\int \left\{ R - \frac{6}{\varphi^2}\,\varphi_{,\mu}\,\varphi^{,\mu} - 6\Lambda\varphi^2 \right\} \sqrt{-g}\, d^4x. \qquad (11)$$

Using the replacement: $\widetilde{\varphi} = 2\sqrt{3}\, ln\varphi$ the action (11) can be brought to the form:

$$S[\varphi] = -\frac{1}{2\kappa}\int \left(R - \frac{1}{2}\, g^{\mu\nu}\, \widetilde{\varphi}_\mu \widetilde{\varphi}_\nu - \widetilde{V}(\widetilde{\varphi}) \right) \sqrt{-g}\, d^4x,$$

$$\widetilde{V}(\widetilde{\varphi}) = 6\Lambda\, e^{\frac{\widetilde{\varphi}}{\sqrt{3}}},$$

which is in turn equivalent to $f(R) \propto R^{\frac{3}{2}}$ gravity.

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