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Schwarzschild deformed supergravity background: possible geometry origin of fermion generations and mass hierarchy

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Main results:

1) The certain conical singularity at the horizon of the Schwarzschild type deformed IIA supergravity 10-dimensional background leaves only three non-divergent fermion angular modes interpreted as three generations of the down-type quarks.

2) The resulting expressions for the quark masses are a geometry version of the Froggatt-Nielsen mechanism.

3) The compactification of extra 4- and 1-dimensional subspaces, gives the spectrum of Fermi fields which profiles in AdS_5 and corresponding Higgs generated masses in 4 dimensions depend on the eigenvalues of Dirac operator on the named compact subspaces.

As an unexpected bonus it is observed that Dirac equation received after compactification of the 4-dimensional base curiously coincides with the non-relativistic Schrödinger equation for an electron moving in a Coulomb field, where eigenvalues of Dirac operator at the compact space play the role of the electron's orbital momentum.

Up and *Down* quarks, charged leptons (electron, muon, τ-meson) and neutrinos are 4 families of elementary fermions in Standard Model (SM), each famtily includes 3 members (generations) similar in their interactions but essentially different in masses. Experimentally observed values of these masses remain unexplained for decades. And what may be the origin of 3 generations?

Models in extra dimensions are among the popular trends aimed at resolving this SM enigma. In particular, Dirac equations are considered on the background of slice of AdS_5 (Randall-Sundrum model) [1] - [3]. Then for fermion masses m_f in 4 dimensions it is obtained:

$$m_f \sim \epsilon^{2c_f - 1}, \quad c_f = \frac{M_f}{k}, \tag{1}$$

where M_f is the bulk mmass of corresponding fermion in 5D, k is curvature of AdS_5, $\epsilon = 10^{-16}$ is the Planck-TeV hierarchy parameter.

Thus to get the observed masses of quarks in the interval from 1 MeV to 200 GeV the special choice of values of parameters c_f in vicinity of 1/2 for every of 6 quarks is demanded. This fine-tuning of fermion bulk masses is an essential drawback of these models.

The present work is an attempt to find a geometric justification for three generations, for ratios of masses of different generations and for parameters c_f in (1). It generalizes and brings the supergravity basis to the approach of papers [4] - [7].

A toy model will be considered where the group structure and left-right asymmetri of the Standard Model is ignored, only one family is taken into account.

The background space-time is a familiar one satisfyibg the dynamics given by the reduced IIA supergravity Lagrangian modified by the addition of the Λ-term:

$$L_{IIA}^{(10)} = R^{(10)} - \frac{1}{2}(\nabla\phi)^2 - \frac{e^{\phi/2}}{2\cdot 24!}F_{(4)}^2 - 2\Lambda e^{-\phi/6}, \quad (2)$$

where $R^{(10)}$ is scalar curvature in 10 dimensions, ϕ - scalar field, $F_{(4)i_1,i_2,i_3,i_4}$ is 4-form.

The dynamic equations following from this Lagrangian admit the long time known [8] - [14] p-brane (fluxbrane) throat-like solution which also permits the Schwarzschild type Euclidean deformation. We write down this metric in Poincare-like coordinates convenient for studing Dirac equation on this background:

Background metric

$$ds^2_{(10)} = \frac{1}{(kz)^{9/4}}\left[\eta_{\mu\nu}dx^\mu dx^\nu + U(z)\left(\frac{T_\theta}{2\pi}\right)^2 d\theta^2 + \frac{dz^2}{U(z)} + \kappa^2 z^2 d\Omega^2_{(4)}\right], \quad (3)$$

where

$$U(z) = 1 - \left(\frac{z}{z_{IR}}\right)^6, \quad F_{(4)} = Q dy_1 \wedge ... \wedge dy_4 \quad (4)$$

$$M_{(4)} \times S^1 \times R \times S^4 \text{ Almost } AdS_6 \times S^4$$

$M_{(4)}$ is 4D Minkowski space-time; T_θ is period of compact coordinate S^1 $(0 < \theta < 2\pi)$; $d\Omega^2_{(4)}$ is volume element on sphere S^4 of unit radius, y_i are angles of this sphere $(i = 1, 2, 3, 4)$; constants k and κ are expressed through Q and Λ; z is isotropic coordinate along the throat. It changes in the interval

$$z_{UV} = \frac{1}{k} \leq z \leq z_{IR}. \quad (5)$$

defining the slice of the space-time (3).

In the vicinity of the Schwarzschild "horizon" change of coordinate $z = z_{IR}[1 - (3/2)(\tau/z_{IR})^2]$ transfotm metric (3) to

$$ds^2_{(10)} = \frac{1}{(kz_{IR})^{9/4}}\left[\eta_{\mu\nu}dx^\mu dx^\nu + \eta^2\tau^2\, d\theta^2 + d\tau^2 + \kappa^2 z^2_{IR} d\Omega^2_{(4)}\right], \quad (6)$$

with

$$U = 9\left(\frac{\tau}{z_{IR}}\right)^2. \qquad \eta = 3 \cdot \frac{T_\theta}{2\pi z_{IR}}. \quad (7)$$

In case $\eta \neq 1$ metric (6) presents conical singularity at $\tau = 0$ $(z = z_{IR})$. Dynamically this is possible if there is a codimension two plane at this point which dimensionless tention $\sigma = 2\pi(1 - \eta)$ [15] - [19]. η is one more free constant of the model, its value determines the permitted number of fermion generations.

Dirac equation

Dynamics of the zero mass spinor in 10D is given by the standard Action: Large

$$S_\Psi = \int \bar{\Psi}_{(32)}\tilde{\Gamma}^A D_A \Psi_{(32)}\sqrt{-g_{(10)}}d^{10}x. \tag{8}$$

Here $\tilde{\Gamma}^A$ are 32×32 gamma matrices in curved 10D space-time (see [21], [22]), D_A are covariant derivatives with account of spin connection.

32-component spinor living on 10D background (3) may be presented as a decomposition of 8-component spinors $\Psi^{(l)}_{(8)}(x^\mu, \theta, z)$ in 6D and eigenfunctions $\chi_{(l)}(y^i)$ of Dirac operator on sphere S^4 of unit radius, with eigenvalues $K^{(4)}_l = \pm(l+2)$ [20], here (l) enumerates main eigenvalues. Dependence of fields on θ is represented by a Fourier expansion over $e^{iq\theta}$ $(q = 0, 1, 2...)$.

8-component spinor is in turn a couple of 4-component spinors $\psi^\pm$, obeying ordinary Dirac equations $(\gamma^\mu\partial_\mu - m^\pm)\psi^\pm(x) = 0$. Each of them consist of the right and left Weyl spinors $(\psi^\pm = \psi^\pm_R + \psi^\pm_L)$ which profiles along the throat for every eigenvalues (l, q), $F^{\pm\, l,q}_{R,L}(z)$, are in the focus of our interest.

Omitting technical details and also the indices $(l, q$ of profiles $F^{\pm\,(l,q)}_{R,L}(z)$, we present the final Dirac equation for these profiles:

$$\left(\frac{d}{dz} + \frac{2\pi q}{T_\theta U} + \frac{c_l}{z\sqrt{U}}\right) F^-_R - \frac{m^-}{\sqrt{U}} F^-_L = 0, \quad U = 1 - \left(\frac{z}{z_{IR}}\right)^6, \tag{9}$$

$$\left(\frac{d}{dz} - \frac{2\pi q}{T_\theta U} - \frac{c_l}{z\sqrt{U}}\right) F^-_L + \frac{m^-}{\sqrt{U}} F^-_R = 0, \quad c_l = K^{(4)}_l/\kappa \tag{10}$$

$K^{(4)}_l = \pm(l+2)$, κ see in (3).

For $q = 0$, $U = 1$ these equations coincide with the similar equations in 5D models [1] - [3]; constants c_l determined by the geometry of the model are the analogy of constants $c_f = M_f/k$ of 5D models.

Three generation

For every component of spinor field Ψ_{32} that is for each mode q, l, $\pm$ and separately for R and L Weyl components coefficient in action (8) at the kinetic term $\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$ must be normalized to one. Thus for metric (3), with account of decomposition of $\Psi_{(32)}$ over eigenfunctions and their normalization $\int \bar{\chi}_l \chi_{l'} d\Omega_{(4)} = \delta_{l\,l'}$, $\int e^{i(q-q')\theta} d\theta = 2\pi\delta_{q\,q'}$ the following normalization condition of profiles $F^{\pm}_{R,L}(z)$ must be satisfied:

$$2\pi \int_{z_{UV}}^{z_{IR}} \frac{1}{\sqrt{U}} (F^{\pm}_{R,L})^2 dz = 1. \tag{11}$$

According to Dirac equations (9), (10) in the vicinity of the Schwarzsxhild IR end of the throat $F_{R,L} \sim \tau^{\pm q/\eta}$ and $dz/\sqrt{U} \sim d\tau$ (see (6), (7)) it is seen that integral (11) does not diverge if

$$\int_0 \tau^{\pm\frac{2q}{\eta}} d\tau < \infty. \tag{12}$$

For smooth IR end ($\eta = 1$) of metric (3) or (6) this integral is non-divergent only for one mode $q = 0$. In case

$$2 < \eta < 4 \tag{13}$$

integral (12) is finite for three modes $q = 0, \pm 1$ which may be interpreted as 3 fermion generations.

Thus, we reproduce here result of [6] but not for the extra 2-sphere as a background but for the Schwarzschild deformed supergravity background (3). Inequalities (13) mean that tension σ of the co-dimension two brane limiting the IR end of the throat (3) must be negative, $-6\pi < \sigma < -2\pi$. The surfaces with negative tension are callad O-planes or orientifolds [23], [24].

The case of non-chiral fermions. A curious parallel with quantum mechanics

In what follows we set $U = 1$ and consider $m^- \neq 0$ in Dirac equations (9), (10) for $F^-_{R,L}$. Then the second-order equations obtained from from this system look as:

$$\begin{aligned} \tfrac{d^2 F^-_R}{dz^2} - \left[\omega^2 + \tfrac{4\pi\, q\, c_l}{T_\theta\, z} + \tfrac{c_l(c_l+1)}{z^2}\right] F^-_R = 0, \\ \tfrac{d^2 F^-_L}{dz^2} - \left[\omega^2 + \tfrac{4\pi\, q\, c_l}{T_\theta\, z} + \tfrac{c_l(c_l-1)}{z^2}\right] F^-_L = 0, \end{aligned} \qquad \omega = \sqrt{\left(\frac{2\pi q}{T_\theta}\right)^2 - (m^-)^2}. \qquad (14)$$

It is curious to note that these equations coincide with the non-relativistic Schrödinger equation for an electron moving in a Coulomb field, where c_l in (14) play the role of the electron's orbital momentum. The solutions of these equations, as well as the corresponding energy spectrumn, are expressed through the degenerate hypergeometric functions [25]. In our case we speak about the spectrum of fermion mass m^-H received from (9), (10), (14) with imposing on $F_{R,L}(z)$ the well known boundary conditions (BC) required by the Hermiticity of Dirac operator:

$$F^-_L(z_{UV})\, F^-_R(z_{UV}) = F^-_L(z_{IR})\, F^-_R(z_{IR}) = 0. \qquad (15)$$

It may be shown that "normal" BC $F^-_L(z_{UV}) = F^-_L(z_{IR}) = 0$ (or $F^-_R(z_{UV}) = F^-_R(z_{IR}) = 0$) give a tower of Kaluza-Klein (KK) heavy modes with masses m^-_n, their lower value is of order $T_\theta^{-1} \approx z_{IR}^{-1}$. To escape the observationally forbidden phenomena, like flavor changing neutral currents generated by the exchange of these KK heavy modes, z_{IR}^{-1} must be above several TeV. Thus heavy modes can not be the low mass quarks or leptons.

Whereas imposing on the solutions of Eq-s (14) the so called "twisted" BC $F^-_L(z_{UV}) = F^-_R(z_{IR}) = 0$ [26] (see also [22], [27]) gives for every q, l the single mode with small mass $m^{tw}_{q,l} \ll z_{IR}^{-1}$. Unfortunately calculation shows that the values of these masses and the ratios of the masses of fermions of three generations ($q = 0, \pm$) strongly diverge from the observed values.

Thus, it is necessary to look at chiral fermions and the conventional Higgs mechanism for generating fermionic masses.

Higgs mechanism and zero modes profiles. Some predictions

For chiral fermions, that is for $m = 0$ in Eq-s (9), (10), and for $U = 1$ in these equations we have four zero modes profiles

$$F_R^- = C_R^- \, e^{-qt} \, t^{-c_l}, \qquad F_L^- = C_L^- \, e^{qt} \, t^{c_l},$$
$$F_R^+ = C_R^+ \, e^{qt} \, t^{c_l}, \qquad F_L^+ = C_L^+ \, e^{-qt} \, t^{-c_l}. \tag{16}$$

$$\epsilon^{8/9} \, \frac{3}{\eta} = t_{UV} < t = \frac{3}{\eta} \frac{z}{z_{IR}} < t_{IR} = \frac{3}{\eta}, \quad \epsilon = 10^{-16} \quad 2 < \eta < 4. \tag{17}$$

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Fermion mass Action with the Higgs field H looks as

$$S_H = \int \bar{\Psi}_{(32)} H \Psi_{(32)} \sqrt{-g_{(10)}} d^{10}x, \tag{18}$$

where vacuum average of the Higgs field H is located near the IR end of the throat. Then small fermion masses are acquired by the fields which profiles are small at the IR end, that is by the fields with profiles F_R^-, F_L^+ in (16).

Calculating coefficients C_R^-, C_L^+ in (16) and under the assumption $\langle H \rangle = Y \, \delta(z - z_{IR})/N$ (Y is dimensionless Yukawa coupling constant) we get finally

$$m_{l,q} = Y \, \frac{2c_l - 1}{z_{IR}} \left(\frac{z_{UV}}{z_{IR}} \right)^{2c_l - 1} e^{-q \cdot \frac{6}{\eta}}. \tag{19}$$

In this paper, three generations of one type of Standard Model fermions are considered. It is worth to try to identify three masses (19) obeying inequalities $m_{q=1,l=0} < m_{q=0,l=0} < m_{q=-1,l=0}$ with masses of three down type quarks d, s, b (at scale 2 GeV; errors are shown in brackets): $m_d = 4.67(32) MeV < m_s = 0.093(2) GeV < m_(b) = 4.18(2) GeV$ [**?**]. The observed ratios of these masses are equal to:

$$\frac{m_d}{m_s} = 5.0(7) \cdot 10^{-2}, \qquad \frac{m_s}{m_b} = 2.22(25) \cdot 10^{-2}, \tag{20}$$

whereas in the considered model these ratios, according to (19), are:

$$\frac{m_d}{m_s} = \frac{m_s}{m_b} = e^{-\frac{6}{\eta}}. \quad \frac{m_d}{m_s} = e^{-3} = 5 \cdot 10^{-2} \quad (\eta = 2). \tag{21}$$

Conclusion

The first obvious improvement of the considered model would be to drop the assumption of independence of Higgs field on the angular coordinate θ. Then Action (18) will generate mass matrix $m_{q,q'}$ with non-zero non-diagonal elements, $q \neq q'$; in particular, in this case the ratios of the eigenvalues of this mass matrix should not be equal to each other like in (21). In the models incorporating up and down quarks the knowledge of the $\hat{m}^{up}$ and $\hat{m}^{down}$ 3×3 mass matrices would allow us to calculate the Cabibbo-Kobayashi-Maskawa (CKM) matrix and hopefully to find the geometry origin of the so called "flavor puzzle".

Three main results of this paper may be outlined.

It is shown that Schwarzschild Euclidean deformation of the generalized IIA supergravity background with certain angle deficit factor $-6\pi < 2(1-\eta) < -2\pi$ of the conical singularity at the "horizon" leaves non-divergent three fermion angular modes interpreted as three generations of fermions of one and the same type. This result reproduces similar earlier results achieved perhaps in more artificial 6D models. Of course, the question remains about the mechanism for the appearance of a surface with negative tension, providing the necessary angular deficiy.

Proportion $m_q \sim e^{-6\,q/\eta}$ (19) is a geometry version of the Froggatt-Nielsen (FN) mechanism. Also for $\eta = 2+\epsilon$ in (21) ($\epsilon \ll 1$), the obtained ratio of masses of d and s quarks $m_d/m_s = e^{-3}$ is experimentally viable. However, it is necessary to emphasize that in the unrealistic model under consideration, which ignores the SM group nature, specific numerical predictions are hardly justified.

It is demonstrated that using 10 dimensional supergravity backgrounds the usually arbitrarily selected in the models of 5D warped compactifications fermions' bulk masses may be identified with the eigenvalues of the Dirac operator on a compact 4D subspace $K^{(4)}$.

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