Boundary CFT, Information Paradox and Entanglement Islands

Dmitry Ageev
Steklov Mathematical Institute, RAS
SIMC-MIAN

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Paper with I. Ya. Aref'eva and T. Rusalev : arXiv:2311.16244
and some work in progress

This talk is devoted to a recently discovered phenomena in (semi)classical gravity – entanglement islands. It two words – why are they important?

- Entanglement islands are the first "giant" clear manifestation of quantum nature of gravity which in principle should be observed in nature
- This phenomena helps us to resolve information paradox (at least on of manifestations of it).

There is a discussion whether islands exist[1]/help to resolve Page formulation of information paradox. In this talk we point out simple situation how the island mechanism fails – "blinking island effect".

[1] At least in higher-dimensional long range gravity there are counterarguments

Main quantum actor in our talk will be entanglement entropy – non-local quantum information measure, which can be computed in quantum systems/quantum field theory.
The original system is divided into two subsystems A and B, described by ρ_{tot}. Each is described by ρ_A and ρ_B – reduced density matrices obtained via taking partial trace

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B, \ \operatorname{Tr} \rho_{\text{tot}} = 1, \ \rho_A = \operatorname*{Tr}_{\mathcal{H}_B} \rho_{tot}, \ \rho_B = \operatorname*{Tr}_{\mathcal{H}_A} \rho_{tot}$$

Entanglement entropy

$$S(X) \equiv S(\rho_X) = -\operatorname{Tr} \rho_X \log \rho_X, \ \text{where } X = A, B$$

In quantum field theory the entanglement entropy is extremely hard to calculate. One of the rare examples where one can calculate the entanglement entropy is two-dimensional conformal field theories (for example 2d massless Dirac fermions). To do this usually the replica trick is used. The entanglement entropy can be represented as

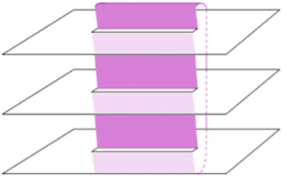
$$S(\rho_A) = -\operatorname*{Tr}_A \rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \operatorname{Tr}\left(\rho_A^n\right)$$

First, how to work with ρ_A?

From Euclidean path-integral point of view the description for ρ_A is the following – take geometry corresponding to ρ and add slit along A (with special boundary conditions).

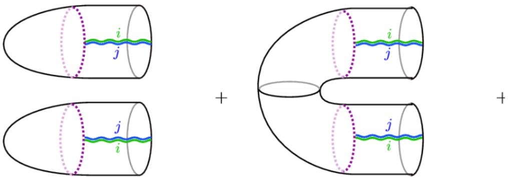
$$S(\rho_A) = -\operatorname*{Tr}_A \rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\,(\rho_A^n)$$

Calculation of anything with ρ_A^n and the quantity $\mathrm{Tr}\,(\rho_A^n)$ (Renyi entropy) is equivalent to deformation of the initial geometry to complicated Riemann surface.
Example : n=3 for A= interval of length ℓ



This geometry is called "Replica Geometry"

- If the geometry is fixed from the start and it IS nondynamical (i.e. is NOT fixed by any equations of motion like Einstein equations) – this is the end of the story.
- If the geometry IS dynamical (i.e. it is the solution of some gravity theory) one has the problems – new replicated solution has to be the solution of equations of motion as well.
- Among the "diagonal" solution we can have non-diagonal one – replica wormhole. Here it is one of the versions of replica wormholes depiction:



Hawking saddle   replica wormhole

As we already mentioned entanglement entropy is extremely hard to calculate in QFT. The explicit example is the entanglement entropy in two-dimensional conformal field theory (for example two-dimensional Dirac fermions). The entanglement entropy of interval of length ℓ is given by

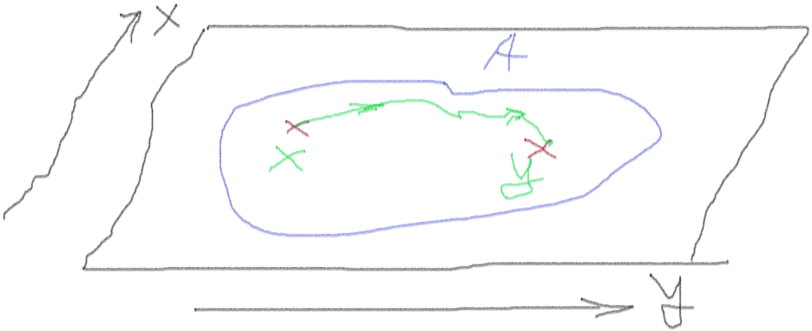
$$S(\ell) = \frac{c}{3} \log \frac{\ell}{\varepsilon}$$

remember that correlation function between two points $x_2 - x_1 = \ell$ is decreasing

$$G(\ell) = \frac{1}{\ell^{2\Delta}}$$

- Correlations function depends only on two points, quantum correlations weakens with distance
- Entanglement entropy is non-local, increasing and absorbing all quantum information in the interval

Two-dimensional system. Green functions is about creation particle in one point and destruction in another one. Entanglement entropy is about total amount of correlations in a large region in general.



- If you consider two-point correlation function on fixed background which is the solution of Einstein equations (no backreation) – it does not matter whether it is Einstein or Gauss-Bonnet (for example).
- The entanglement entropy takes care about it and the formula for entanglement entropy drastically changes.
- All this story in the end condeses in the "island formula". The statement – on the (semi)classical background described by some gravity theory the entanglement entropy is theory dependent and completely different from that one in non-dynamical geometry.

$$S(R) \simeq \min_{\mathcal{I}} \left\{ \frac{Area(\partial\mathcal{I})}{4G} + S_{matter}(R \cup \mathcal{I}) \right\}$$ (Almheiri et al. 19, Penington et al.'19)

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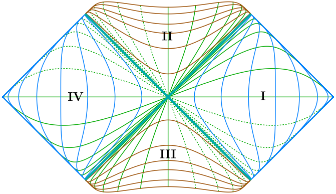
Remember that entanglement entropy $S(R) + S(I)$ is not equal to $S(R \cup I)$ –the latter is much more complicated to calculate

Eternal black hole – two causally disconnected, but entangled universes (left (IV) and right(I)).



The metric of the four-dimensional Schwarzschild black hole is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{r_h}{r}$$

where $r_h = 2GM$ denotes the black hole horizon, $d\Omega_2^2$ is the metric on S^2. Introducing Kruskal coordinates

$$U = -\frac{1}{\kappa_h}\, e^{-\kappa_h(t-r_*(r))}, \quad V = \frac{1}{\kappa_h}\, e^{\kappa_h(t+r_*(r))}$$

with the tortoise coordinate $r_*(r) = r + r_h \log|r - r_h|/r_h$ and the surface gravity $\kappa_h = 1/2r_h$, we can rewrite the metric in the form

Task: Calculate the entanglement entropy of Hawking radiation using the island formula for an 4d eternal Schwarzschild black hole (Hashimoto et al. '20)

$$S(R) \simeq \min_{\partial I}\left\{\operatorname*{ext}_{\partial I}\left[\frac{\text{Area}(\partial I)}{4G_{\rm N}} + S_{semi-class}(R \cup I)\right]\right\}$$

Problems:

1. The island formula has not been derived for the theory under consideration (derived rigorously only in JT gravity)
2. There are no convenient analytical formulas for the entanglement entropy $S_{semi-class}(R \cup I)$ in four-dimensional quantum field theory

Assumptions:

1. Assume that the island formula holds in 4D Einstein gravity
2. Reduce the problem to an effective two-dimensional field theory, preferably CFT_2, for which there are analytical formulas for the entanglement entropy (s-mode approximation)

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Comments on these assumptions:

1. In 2d JT gravity the existence of islands are proven. There is large discussion whether island formula and replica wormholes take place in higher-dimensional(!) gravity in the same way as in 2d (and whether they even exist). We do not have final arguments on both sides. This is open question.
2. Using s-mode approximation for quantum fields (i.e. going to two-dimensional background and 2d CFT formulas) gives essentially the same answer as if use some higher-dimensional formulas, for simplest setups and assuming islands presence (Bousso, Penington; arXiv:2312.03078)

For the metric of the form (BH metric written in Kruskal coordinates)

$$ds^2 = -e^{2\rho(r)} dUdV, \; e^{2\rho(r)} = \frac{r_h}{r} e^{-r/r_h}$$

the entanglement entropy of c copies of free massless Dirac fermions in curved spacetime (use Weyl invariance) (Casini et al. '05)

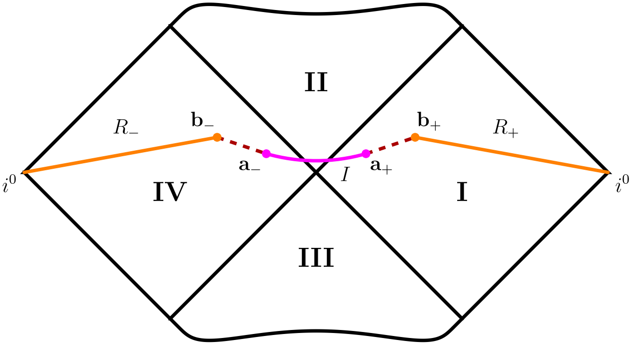
$$S_{semi-class}(R) = \frac{c}{3}\left(\sum_{i,j=1}^{N} \log\frac{d(\mathrm{x}_i,\mathrm{y}_j)}{\varepsilon} - \sum_{i<j}^{N} \log\frac{d(\mathrm{x}_i,\mathrm{x}_j)}{\varepsilon} - \sum_{i<j}^{N} \log\frac{d(\mathrm{y}_i,\mathrm{y}_j)}{\varepsilon}\right)$$

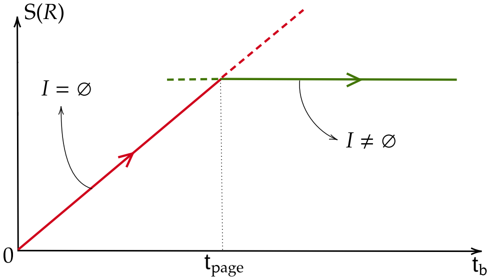
where $R = [x_1, y_1] \cup \ldots \cup [x_N, y_N]$, ε is UV cutoff and

$$d^2(\mathrm{x},\mathrm{y}) = [U(\mathrm{x}) - U(\mathrm{y})]\,[V(\mathrm{y}) - V(\mathrm{x})]\, e^{\rho(\mathrm{x})} e^{\rho(\mathrm{y})}, \quad x = \{r_x, t_x\}$$

How we one obtains this formula? Technically the entanglement entropy in 2d CFT and BCFT is essentialy given by special limit of primary operators n-point correlator.

Choose the infinite regions on both sides extending for some points $b_\pm$ to infinity (yellow). Assume that the island is located as shown in picture (magenta)





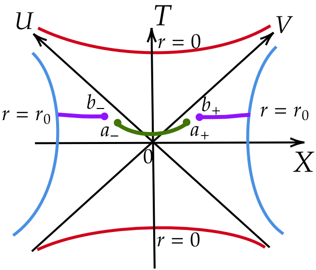
Information paradox: the entanglement entropy without islands (red) exhibits linear growth which is unbounded and after some time it will exceed the entropy of black hole. However after some time green line (EE with islands) stops the growth.

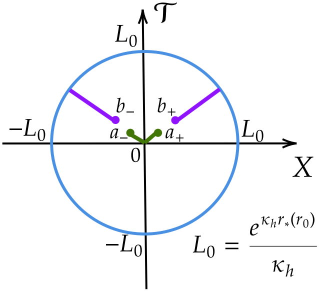
- $I = \varnothing: \; S(R) = \frac{c}{6} \log \left[\frac{16 r_h^2 (b - r_h)}{\varepsilon^2 b} \cosh^2 \frac{t_b}{r_h} \right] \underset{t_b \gg r_h}{\simeq} \frac{c\, t_b}{6 r_h} \quad \rightarrow$ growth
- $I \neq \varnothing: \; S(R) \simeq \frac{2\pi r_h^2}{G_N} + \frac{c}{6} \left[\log \left(\frac{16 r_h^3 (b - r_h)^2}{\varepsilon^4 b} \right) + \frac{b - r_h}{r_h} \right] \rightarrow$ constant

Black holes with reflecting boundaries (BH in cavity) and Hawking radiation in such black holes is the classical topic in the subject [Russo, Susskind, Thorlacius ('92), York ('85), Gross,Perry, Yaffe (82') etc.]. Why we would like to apply this setup

- Black hole in cavity is thermodynamically stable
- Regularization of effects related to infrared, divergences etc. For example in entanglement entropy there is IR mode which is typically omitted in calculations [D.S. Ageev, I.Ya. Aref'eva, A.I. Belokon, A.V. Ermakov, V.V. Pushkarev, T.A. Rusalev, arXiv:2209.00036]; [D.S. Ageev, I.Ya. Aref'eva, A.I. Belokon, V.V. Pushkarev, T.A. Rusalev arXiv:2304.12351]
- Probing more complicated dynamics i.e. interplay of reflected and straight radiation

To probe the radiation let us put the fixed reflecting boundary at some radius r_0 (on both sides of eternal black hole). Now Dirac fermions reflects from the boundary (i.e. one should proceed with BCFT calculations). Technically the problem reduces to calculation of correlators in the disc with curved metric (via mapping to upper-half plane where BCFT answers are known).

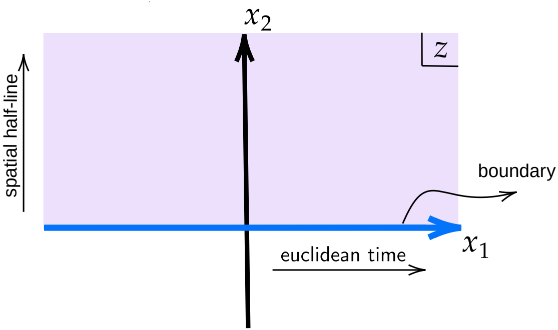




Remind, that the entanglement entropy in 2d CFT and BCFT is essentialy given by special primary operators correlator.

BCFT$_2$ on upper half-plane

Basic BCFT$_2$ geometry – Euclidean flat upper half-plane (UHP)



$$ds^2 = dx_1^2 + dx_2^2 = dz d\bar{z}, \quad z = x_1 + i x_2, \quad x_1 \in (-\infty, \infty), \quad x_2 \geq 0$$

Here x_1 is Euclidean time, x_2 is the spatial coordinate, $x_2 = 0$ is boundary

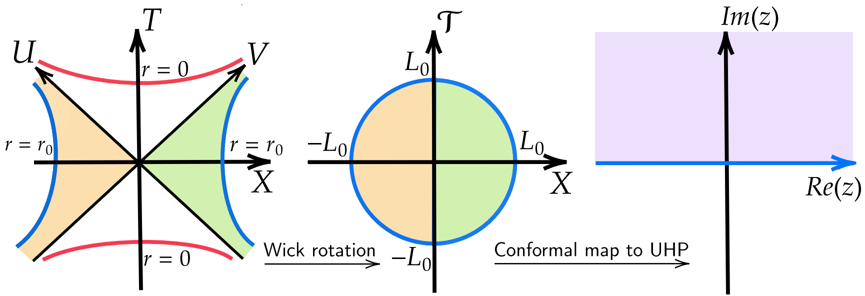
- Euclidean double-boundary geometry is the interior of the disc

$$\{X^2 + \mathcal{T}^2 \le L_0^2 \,|\, X, \mathcal{T} \in [-L_0, L_0]\}, \qquad L_0 = \frac{e^{\kappa_h r_*(r_0)}}{\kappa_h}$$

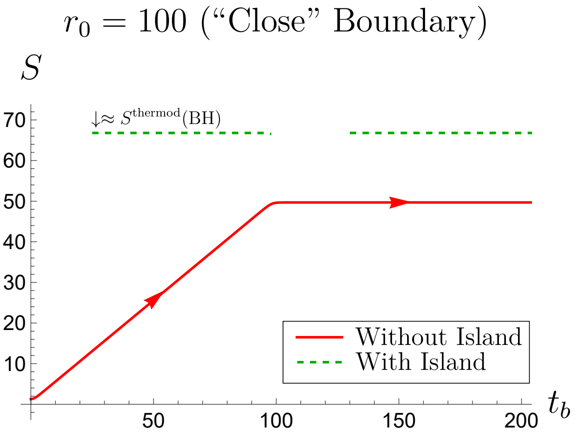
- Conformal map from disc to UHP is

$$z = i\,\frac{L_0 + w}{L_0 - w}$$

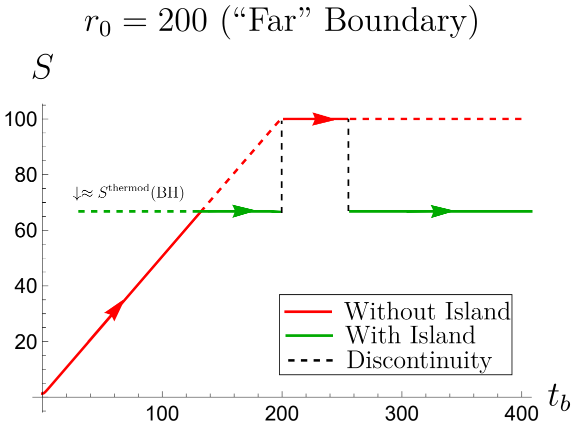
- Weyl transform to flat upper half-plane



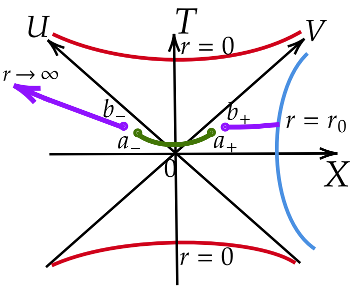
For close enough boundary we observe no information paradox at all. This is natural because the amount quanta of radiated at infinity is not enough to give large numbers – we see saturation at red line – below the green line which is BH entropy given by islands. Island is subdominant.

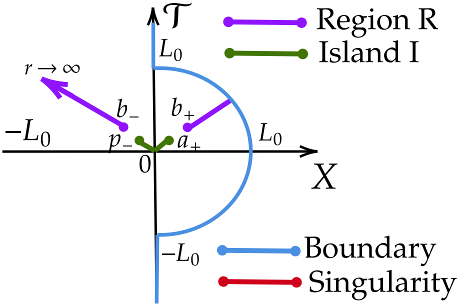


However for boundary located faraway enough we see the problems – first we have the growth via radiation. Then island mechanism start to work – green line at BH entropy scale. At some time we see how the island ceases to exist for a short time – "blink of island". Then everything is OK – green line. For the blinking time the entropy exceeds S_{BH}

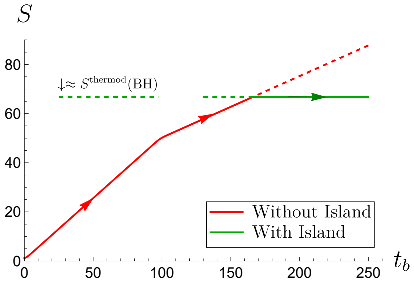


What will happen if we consider asymmetric state? I.e. we put the reflecting boundary only on the one side of the black hole. Two-sided black hole is believed to correspond to some kind of the thermofield double state (Israel '76). What is our universe is with boundary, and it cousin is free of it? Technically in Euclidean spacetime it is described by half-disc united with half-space and endowed by some non-trivial metric

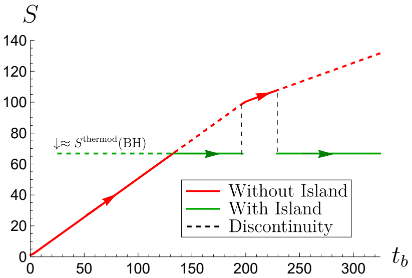




$r_0 = 100$ ("Close" Boundary)



$r_0 = 200$ ("Far" Boundary)



- The black hole boundaries introduces new effects to the radiation dynamics
- The interplay of old and young Hawking radiation quanta for Dirac massless fermions seem to make island mechanism as it is useless ot resolve Page formulation of information paradox.

Thank you for your attention!